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NEVANLINNA THEORY AND PAINLEVÉ TRANSCENDENTS

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Consider the first Painlevé equation

$$w'' = 6w^2 + z \tag{I}$$

($' = d/dz$). All the solutions of (I) are transcendental and meromorphic in the whole complex plane (see [2], [6]). In this article we explain how the Nevanlinna theory is used in the study of Painlevé transcendents. In Section 1, we make a survey of basic facts in the Nevanlinna theory related to our purpose. For a detailed explanation about the Nevanlinna theory, the reader may consult [1], [3], [4]. In Section 2, the deficiency and the ramification index are examined for the first Painlevé transcendents. The final section is devoted to the proof of the finiteness of the growth order. Throughout this article, we use the notation below: for $\phi(r), \psi(r), r \in [r_0, +\infty)$,

(i) $\phi(r) \ll \psi(r)$, if $\phi(r) = O(\psi(r))$ as $r \to +\infty$;
(ii) $\phi(r) \asymp \psi(r)$, if $\phi(r) \ll \psi(r)$ and $\psi(r) \ll \phi(r)$ hold simultaneously.

1. Basic notation and facts in the Nevanlinna theory

Let $f(z)$ be an arbitrary meromorphic function in $\mathbf{C}$. For $r > 0$, denote by $n(r, f)$ the cardinal number of the poles of $f(z)$ in the disk $|z| \le r$, each counted according to its multiplicity. Then the *counting function* of $f(z)$ is defined by

$$N(r,f) = \int_0^r \frac{1}{\rho}\Big(n(\rho, f) - n(0, f)\Big)d\rho + n(0,f)\log r.$$

The *proximity function* of $f(z)$ is defined by

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\varphi})|d\varphi,$$

where $\log^+ x = \max\{\log x, 0\}$. Then we put

$$T(r,f) = m(r,f) + N(r,f),$$

which is called the *characteristic function* of $f(z)$. By definition, it is easy to see that, for meromorphic functions $f(z), g(z)$, formulas such as

$$\begin{aligned} &m(r,\alpha f+\beta g) \le m(r,f)+m(r,g)+O(1),\\ &m(r,fg) \le m(r,f)+m(r,g),\\ &T(r,\alpha f+\beta g) \le T(r,f)+T(r,g)+O(1),\\ &T(r,fg) \le T(r,f)+T(r,g) \end{aligned}$$

$(\alpha, \beta \in \mathbf{C})$ are valid. Let $\{a_i\}_{i=1}^k$ and $\{b_j\}_{j=1}^l$ be respectively the zeros and the poles of $g(z)$ in the disk $|z|<r$, each repeated according to its multiplicity. By the Jensen-Poisson formula, for every z satisfying $|z|<r$,

$$(1.1)\quad \log|g(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\varphi})| \cdot \frac{r^2-|z|^2}{|re^{i\varphi}-z|^2}d\varphi + \sum_{i=1}^{k}\log\left|\frac{r(z-a_i)}{r^2-\bar{a}_i z}\right| - \sum_{j=1}^{l}\log\left|\frac{r(z-b_j)}{r^2-\bar{b}_j z}\right|.$$

Substituting $g(z) = z^{-p}f(z) = c_p + O(z)$ $(c_p \neq 0,\ p \in \mathbf{Z})$ into (1.1) with $z=0$, we have

$$(1.2)\qquad m(r,f)+N(r,f) = m(r,1/f)+N(r,1/f)+\log|c_p|.$$

Replacing $f(z)$ by $f(z)-a$, $a \in \mathbf{C}$, we obtain the first main theorem.

Theorem 1.1. *For an arbitrary meromorphic function $f(z)$ and for an arbitrary $a \in \mathbf{C}$,*

$$T(r,1/(f-a)) = T(r,f)+O(1).$$

By the definition of $T(r,f)$, we have $T(r,e^z) \asymp r$, $T(r,\exp(z^2)) \asymp r^2$. Furthermore

Proposition 1.2. *A meromorphic function $f(z)$ satisfies $T(r,f) = O(\log r)$, if and only if $f(z)$ is a rational function.*

Proof. It is easy to see that an arbitrary rational function $f(z)$ satisfies $T(r,f) = O(\log r)$. To show the reverse, suppose that $T(r,f) = O(\log r)$. We have

$$\begin{aligned} n(r,f)-n(0,f) &\le \frac{1}{\log r}\int_r^{r^2}\frac{1}{\rho}(n(\rho,f)-n(0,f))d\rho \\ &\le \frac{N(r^2,f)}{\log r}+O(1) \le \frac{T(r^2,f)}{\log r}+O(1) = O(1); \end{aligned}$$

namely the number of poles is finite. Since $T(r,1/f) = T(r,f)+O(1) = O(\log r)$, the number of zeros is also finite. Then, $g(z) = f(z)\prod_i(z-a_i)^{-1}\prod_j(z-b_j)$ is entire and satisfies $g(z) \neq 0$, where $\{a_i\}$ and $\{b_j\}$ are respectively the zeros and the poles of $f(z)$. By (1.1), for $|z| \le r/2$,

$$\begin{aligned} \log|g(z)| &= \frac{1}{2\pi}\int_0^{2\pi}\log|g(re^{i\varphi})| \cdot \frac{r^2-|z|^2}{|re^{i\varphi}-z|^2}d\varphi \\ &\le \frac{2}{\pi}\int_0^{2\pi}\log|g(re^{i\varphi})|d\varphi \ll m(r,g) \ll T(r,f)+\log r \ll \log r, \end{aligned}$$

which implies that $g(z)$ is a polynomial. Hence $f(z)$ is a rational function. $\square$

The *growth order* of $f(z)$ is defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

For example, $\sigma(e^z) = 1$, $\sigma(\exp(z^2)) = 2$, and $\sigma(q) = 0$ for a rational function $q(z)$. From the identity due to H. Cartan ([1])

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} N(r, 1/(f - e^{i\varphi}))d\varphi + \log^+ |f(0)|,$$

we obtain

$$\frac{dT(r,f)}{d\log r} = \frac{1}{2\pi}\int_0^{2\pi} n(r, 1/(f - e^{i\varphi}))d\varphi,$$

provided that $|f(0)| \neq \infty$. Combining this with the relation $T(r,f) = T(r,1/f) + \log|c_p|$ (cf. (1.2)) in the complemental case $|f(0)| = \infty$ as well, we derive

Proposition 1.3. *The characteristic function $T(r,f)$ is increasing and convex with respect to $\log r$.*

Furthermore, we have

Proposition 1.4. *A meromorphic function $f(z)$ is transcendental if and only if $\log r/T(r,f) = o(1)$ as $r \to \infty$.*

Proof. Suppose that $f(z)$ is transcendental. We regard $T(r,f) = T_*(r)$ as a function of $\log r$. If $dT_*(r)/d\log r \le B$ for some $B > 0$, then $T_*(r) \le B\log r + O(1) \ll \log r$, which contradicts the supposition. Since $dT_*(r)/d\log r$ is increasing with respect to r (cf. Proposition 1.3), $dT_*(r)/d\log r \to \infty$ as $r \to \infty$. For any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that $dT_*(r)/d\log r \ge 1/\varepsilon$, $r \ge r_\varepsilon$ and hence $T_*(r) \ge \varepsilon^{-1}\log r + O(1)$, $r \ge r_\varepsilon$, which implies $\log r/T(r,f) \to 0$ as $r \to \infty$. $\square$

Put, for an arbitrary $a \in \mathbf{C}$,

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r, 1/(f-a))}{T(r,f)},$$

and

$$\delta(\infty,f) = \liminf_{r\to\infty} \frac{m(r,f)}{T(r,f)}.$$

These quantities are called the *deficiency* of a, and that of ∞, respectively.

Proposition 1.5. *Suppose that $f(z)$ is a transcendental meromorphic function. If $\delta(a,f) < 1$, then $f(z)$ admits infinitely many a-points ($a \in \mathbf{C} \cup \{\infty\}$). If $f(z)$ admits only finite number of a-points in $\mathbf{C}$, then $\delta(a,f) = 1$.*

Proof. For simplicity, we prove for the case where $a = \infty$. Suppose that $\delta(\infty,f) < \delta_0 < 1$ for some δ_0. Then there exists a sequence $\{r_\nu\}$ such that $m(r_\nu,f) < \delta_0 T(r_\nu,f)$ as $r_\nu \to \infty$. Observing that

$$N(r_\nu,f) = T(r_\nu,f) - m(r_\nu,f) > (1-\delta_0)T(r_\nu,f),$$

and that

$$N(r_\nu, f) = \int_0^{r_\nu} \frac{1}{\rho}(n(\rho, f) - n(0, f))d\rho + n(0, f)\log r_\nu \ll n(r_\nu, f)\log r_\nu,$$

we have

$$n(r_\nu, f) \gg (1 - \delta_0)T(r_\nu, f)/\log r_\nu \to \infty$$

as $r_\nu \to \infty$, because $f(z)$ is transcendental (Proposition 1.4). The second assertion is clear. □

To count the multiple poles, we consider $n_1(r, f) = \sum_\tau(\mu(\tau) - 1)$. Here $\mu(\tau)$ denotes the multiplicity of a pole τ and $\sum_\tau$ denotes the summation for all poles in the disk $|z| \le r$. The function

$$N_1(r, f) = \int_0^r \frac{1}{\rho}\Big(n_1(\rho, f) - n_1(0, f)\Big)d\rho + n_1(0, f)\log r$$

measures the frequency of the multiple poles. Then we consider, for $a \in \mathbf{C}$,

$$\vartheta(a, f) = \liminf_{r\to\infty} \frac{N_1(r, 1/(f-a))}{T(r, f)},$$

and

$$\vartheta(\infty, f) = \liminf_{r\to\infty} \frac{N_1(r, f)}{T(r, f)},$$

which are called the *ramification index* of a, and that of ∞, respectively. If all the a-points are simple, then $\vartheta(a, f) = 0$, and if they are double, then $\vartheta(a, f) \le 1/2$.

Let $\phi(r)$ be a function defined on the interval $[r_0, +\infty)$, $r_0 > 1$. We write

$$\phi(r) = S(r, f)$$

if $\phi(r) = o(T(r, f))$ as $r \to +\infty$ outside of a possible exceptional set of finite linear measure. Applying $(\partial/\partial x - i\partial/\partial y)$ to (1.1) (with $g = f$), we obtain

$$\frac{f'(z)}{f(z)} = \frac{1}{\pi}\int_0^{2\pi} \log|f(re^{i\varphi})| \cdot \frac{re^{i\varphi}}{(re^{i\varphi} - z)^2}d\varphi + \sum_{i=1}^{k}\Big(\frac{1}{z - a_i} + \frac{\bar{a}_i}{r^2 - \bar{a}_i z}\Big) - \sum_{j=1}^{l}\Big(\frac{1}{z - b_j} + \frac{\bar{b}_j}{r^2 - \bar{b}_j z}\Big)$$

for $|z| < r$. By this inequality, we estimate logarithmic derivatives (cf. [1]).

Proposition 1.6. *For a meromorphic function $f(z)$, and for an arbitrary $k \in \mathbf{N}$, $m(r, f^{(k)}/f) = S(r, f)$. In particular, if $\sigma(f) < +\infty$, then $m(r, f^{(k)}/f) = O(\log r)$.*

Remark 1.1. In Proposition 1.6, $S(r, f)$ can be replaced by $O\big(\log(rT(r, f))\big)$ *as $r \to +\infty$ outside of a possible exceptional set of finite linear measure.*

The second main theorem in the Nevanlinna theory, which is another basic result, is stated as follows (see [4; Theorem 2.5.1 and the proof]).

Theorem 1.7. *For an arbitrary non-constant meromorphic function $f(z)$ and for an arbitrary number of distinct points $a_1, ..., a_q \in \mathbf{C}, q \in \mathbf{N}$, we have*

$$m(r,f) + \sum_{j=1}^{q} m(r, 1/(f - a_j)) + N(r, 1/f') + N_1(r,f) \leq 2T(r,f) + S(r,f).$$

From this theorem, we immediately obtain

$$\delta(\infty, f) + \vartheta(\infty, f) + \sum_{j=1}^{q} (\delta(a_j, f) + \vartheta(a_j, f)) \leq 2.$$

This implies that, for each $n \in \mathbf{N}$, the number of the points $a \in \mathbf{C}$ satisfying $\delta(a,f) \geq 1/n$ (resp. $\vartheta(a,f) \geq 1/n$) does not exceed $2n$, and hence all the points with non-zero deficiency (resp. non-zero ramification index) constitute a countable set. Thus we have

Corollary 1.8. *For an arbitrary non-constant meromorphic function $f(z)$,*

$$\sum_{a \in \widehat{\mathbf{C}}} (\delta(a,f) + \vartheta(a,f)) \leq 2, \qquad \widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\},$$

where the summation ranges all the points in $\widehat{\mathbf{C}}$ such that $\delta(a,f) > 0$ or $\vartheta(a,f) > 0$.

The following lemma is due to J. Clunie (see [4; Lemma 2.4.2]).

Lemma 1.9. *Let f be a transcendental meromorphic function such that $f^{p+1} = Q(z,f)$, $p \in \mathbf{N}$, where $Q(z,u)$ is a polynomial in z, u and its derivatives. Suppose that the total degree of $Q(z,u)$ as a polynomial in u and its derivatives does not exceed p. Then $m(r,f) = S(r,f)$.*

Proof. Note that

$$m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\varphi})| d\varphi = \frac{1}{2\pi} \int_{\varphi \in F} \log^+ |f(re^{i\varphi})| d\varphi,$$
$$F = \{\varphi \in [0, 2\pi] \mid |f(re^{i\varphi})| \geq 1\}.$$

The polynomial $Q(z,u)$ is written in the form

$$Q(z,u) = \sum_{\iota_0 + \iota_1 + \cdots + \iota_\mu \leq p} q_\iota(z) u^{\iota_0} (u')^{\iota_1} \cdots (u^{(\mu)})^{\iota_\mu}, \quad q_\iota(z) \in \mathbf{C}[z],$$

$\iota = (\iota_0, \iota_1, ..., \iota_\mu)$, $\mu \in \mathbf{N}$. Hence, from $f^{p+1} = Q(z,f)$, we derive, for $\varphi \in F$,

$$|f(re^{i\varphi})| \leq \sum_{\iota} |q_\iota| |f'/f|^{\iota_1} \cdots |f^{(\mu)}/f|^{\iota_\mu},$$

which implies

$$m(r,f) \ll m(r, f'/f) + \cdots + m(r, f^{(\mu)}/f) + \log r \ll S(r,f).$$

□

The following lemma due to A. Z. Mohon'ko and V. D. Mohon'ko ([5], [4; Proposition 9.2.3]) gives an estimate for the proximity function of the reciprocal of $f(z) - c$.

Lemma 1.10. *Let $F(z,u)$ be a polynomial in z, u and its derivatives. Suppose that $u = f$ is a transcendental meromorphic function satisfying $F(z,f) = 0$, and that c is a complex number. If $F(z,c) \not\equiv 0$, then $m(r, 1/(f-c)) = S(r,f)$.*

Proof. Put $g = f - c$. Then, by supposition,

$$F(z,f) - F(z,c) = G(z,g) = \sum_{1 \le \iota_0+\iota_1+\cdots+\iota_\mu \le \gamma_0} h_\iota(z) g^{\iota_0} (g')^{\iota_1} \cdots (g^{(\mu)})^{\iota_\mu},$$

$h_\iota(z) \in \mathbf{C}[z]$, $\iota = (\iota_0, \iota_1, ..., \iota_\mu)$, $\mu \in \mathbf{N}$, $\gamma_0 = \deg G$. If $|g(re^{i\varphi})| \le 1$, then

$$|1/g(re^{i\varphi})| \le |F(z,c)|^{-1} \sum_\iota |h_\iota| |g'/g|^{\iota_1} \cdots |g^{(\mu)}/g|^{\iota_\mu},$$

from which the conclusion follows. □

Remark 1.2 In Lemma 1.9 or 1.10, $S(r,f)$ can be replaced by $O\big(\log(rT(r,f))\big)$ *as $r \to +\infty$ outside of a possible exceptional set of finite linear measure.* Furthermore, if $\sigma(f) < +\infty$, then $S(r,f)$ can be replaced by $O(\log r)$ (without an exceptional set) (cf. Remark 1.1 and Proposition 1.6).

From [4; Lemma 1.1.1], we have

Lemma 1.11. *Let $\phi(r)$ and $\psi(r)$ be monotone increasing functions on $(0, +\infty)$ satisfying $\phi(r) \le \psi(r)$ outside of an exceptional set of finite linear measure. Then, there exists a number $r_0 > 0$ such that $\phi(r) \le \psi(2r)$ on $(r_0, +\infty)$.*

2. Deficiency and ramification index for solutions of (I)

Every solution of (I) is transcendental. Indeed, if (I) admits a rational solution expressed in the form $z^m \sum_{j \ge 0} c_j z^{-j}$ ($c_0 \ne 0, m \in \mathbf{Z}$) around $z = \infty$, then substitution of this into (I) yields $m = 1/2$, which is a contradiction. Let $w(z)$ be an arbitrary transcendental meromorphic solution of (I). Then we have ([7], [8], [10])

Theorem 2.1. *For every $a \in \mathbf{C}$, $\delta(a,w) = 0$; and $\delta(\infty,w) = 0$.*

Theorem 2.2. *For every $a \in \mathbf{C}$, $\vartheta(a,w) \le 1/6$; and $\vartheta(\infty,w) \le 1/2$.*

For every $a \in \mathbf{C}$, $w \equiv a$ is not a solution of (I). Applying Lemma 1.10 to $w(z)$, we have $m(r, 1/(w-a)) = S(r,w)$. By Lemma 1.9, we also have $m(r,w) = S(r,w)$. This estimate and Proposition 1.6 yields that $m(r,w') \le m(r,w) + m(r,w'/w) = S(r,w)$. Thus we have the lemma below, from which Theorem 2.1 immediately follows.

Lemma 2.3. *For every $a \in \mathbf{C}$, $m(r, 1/(w-a)) = S(r,w)$, $m(r,w) = S(r,w)$, and $m(r,w') = S(r,w)$.*

Substituting the Laurent series expansion around a movable pole into (I), we have

Lemma 2.4. *Around each movable pole $z = c_\infty$,*

$$w(z) = (z - c_\infty)^{-2} + O(z - c_\infty).$$

Proof of Theorem 2.2. From (I), we obtain

$$\Psi'(z) = -2w(z), \tag{2.1}$$

$$\Psi(z) = w'(z)^2 - 4w(z)^3 - 2zw(z). \tag{2.2}$$

For $a \in \mathbf{C}$, consider the set

$$A = \{z \mid w(z) = a, \ w'(z) = 0\}.$$

We may suppose the cardinal number of A is ∞; otherwise $\vartheta(a, w) = 0$. If $z^* \in A, w''(z^*) = 0$, then $z^* = -6a^2$. Now choose a point $z_0 \in A \setminus \{-6a^2\}$. Then, from (2.1), we have

$$\Psi(z) - \Psi(z_0) = -2 \int_{z_0}^{z} w(t)dt,$$

and hence, by (2.2),

$$G(z) = w'(z)^2 - 4(w(z)^3 - a^3) - 2z(w(z) - a) \tag{2.3}$$

$$= 2a(z - z_0) - 2 \int_{z_0}^{z} w(t)dt. \tag{2.4}$$

Furthermore,

$$G'(z) = 2(a - w(z)), \quad G''(z) = -2w'(z), \quad G^{(3)}(z) = -2w''(z) = -2(6w(z)^2 + z).$$

Hence, for every $\sigma \in A \setminus \{-6a^2\}$,

$$G(\sigma) = G'(\sigma) = G''(\sigma) = 0, \quad G^{(3)}(\sigma) = -2(6a^2 + \sigma) \neq 0,$$

namely σ is a triple zero of $G(z)$. This fact means that

$$N_1(r, 1/(w - a)) \leq \frac{1}{3} N(r, 1/G) + O(\log r) \leq \frac{1}{3} T(r, G) + O(\log r). \tag{2.5}$$

By Lemma 2.3, $m(r, G) \ll m(r, w') + m(r, w) = S(r, w)$. Substituting the expression of Lemma 2.4 into (2.4), we have $N(r, G) = (1/2)N(r, w) = (1/2)T(r, w) + S(r, w)$. Hence $T(r, G) = (1/2)T(r, w) + S(r, w)$. Combining this with (2.5), we obtain

$$N_1(r, 1/(w - a)) \leq \frac{1}{6} T(r, w) + S(r, w), \tag{2.6}$$

from which $\vartheta(a, w) \leq 1/6$ immediately follows. Since every pole of $w(z)$ is double,

$$N_1(r, w) = \frac{1}{2} N(r, w) = \frac{1}{2}(T(r, w) + S(r, w)). \tag{2.7}$$

Hence $\vartheta(\infty, w) \leq 1/2$, which completes the proof. □

3. Finiteness of the growth order

The following result ([9]) indicates that the order of $w(z)$ is finite.

Theorem 3.1. *For an arbitrary solution $w(z)$ of* (I), *we have $T(r,w) = O(r^C)$, where C is a positive number independent of $w(z)$ and the coefficient α.*

By Remarks 1.1 and 1.2, once Theorem 3.1 is established, the notation $S(r,w)$ in the results for solutions of (I) is replaced by $O(\log r)$. For example, we have

Corollary 3.2. *Let $w(z)$ be an arbitrary solution of* (I). *Then,*
(i) $m(r,w) = O(\log r)$, $m(r,1/(w-a)) = O(\log r)$ *for every* $a \in \mathbf{C}$;
(ii) $N_1(r,1/(w-a)) \le (1/6)T(r,w) + O(\log r)$ *for every* $a \in \mathbf{C}$;
(iii) $N_1(r,w) = (1/2)T(r,w) + O(\log r)$ *and* $\vartheta(\infty,w) = 1/2$.

We prove Theorem 3.1 for solutions of (I).

3.1. Basic lemmas. Let $w(z)$ be an arbitrary solution of (I). Put

$$\theta = 2^{-4}. \tag{3.1}$$

We begin with the following lemma, which is proved by a modification of Hukuhara's argument ([6]).

Lemma 3.3. *Let a be a point satisfying $|a| > 5$. If $|w(a)| \le \theta^2|a|^{1/2}/6$, then*
(i) *$w(z)$ is analytic and bounded for $|z-a| < \delta_a$,*
(ii) *$|w(z)| \ge \theta^2|a|^{1/2}/5$ for $(5/6)\delta_a \le |z-a| \le \delta_a$,*
where

$$\theta|a|^{-1/4}\min\{1, \theta|a|^{3/4}/|w'(a)|\} < \delta_a \le 3\theta|a|^{-1/4}. \tag{3.2}$$

Proof. We put $z = a + \rho t$, $\rho = a^{-1/4}$, $w(z) = w(a+\rho t) = \theta a^{1/2} v(t)$ in (I). Then (I) becomes

$$\ddot{v}(t) = 6\theta v(t)^2 + \theta^{-1}(1 + \rho^5 t) \tag{3.3}$$

($\dot{} = d/dt$). Integrating both sides twice, we have

$$v(t) = v(0) + \dot{v}(0)t + \theta^{-1}t^2/2 + g(t), \tag{3.4}$$
$$g(t) = \frac{1}{6}\theta^{-1}\rho^5 t^3 + 6\theta \int_0^t \int_0^\tau v(s)^2 ds d\tau,$$

where

$$v(0) = \theta^{-1}a^{-1/2}w(a), \qquad \dot{v}(0) = \theta^{-1}a^{-3/4}w'(a).$$

By supposition,

$$|v(0)| \le \theta^{-1}|a|^{-1/2}|w(a)| \le \theta/6. \tag{3.5}$$

(1) *Case* $|\dot{v}(0)| \le 1$. We put

$$\eta_0 = \sup\{\eta | M(\eta) \le 8\theta\}, \quad M(\eta) = \max\{|v(t)| \,|\, |t| \le \eta\}.$$

By (3.5), $\eta_0 > 0$. Suppose that $\eta_0 < 3\theta$. Since $|a| > 5$, by (3.1), we observe that, for $|t| \le \eta_0$,

$$(3.6)\quad |g(t)| \le \frac{1}{6}\theta^{-1}|\rho|^5|t|^3 + 6\theta \int_0^t \int_0^\tau |v(s)|^2|ds||d\tau|$$
$$\le \theta^{-1}|\rho|^5(3\theta)^3/6 + 6\theta(8\theta)^2(3\theta)^2/2 < \theta/4.$$

Hence, from (3.4), (3.5) it follows that, for $|t| \le \eta_0$,

$$(3.7)\quad |v(t)| \le |v(0)| + |t| + \theta^{-1}|t|^2/2 + \theta/4 \le (1/6 + 3 + 9/2 + 1/4)\theta < 7.92\theta,$$

which contradicts the definition of η_0. This implies that $\eta_0 \ge 3\theta$, and that (3.6) is valid for $|t| \le 3\theta$. Moreover, by (3.4), if $2.5\theta \le |t| \le 3\theta$, then

$$|v(t)| \ge \theta^{-1}|t|^2/2 - |v(0)| - |t| - |g(t)| \ge (2.5^2/2 - 1/6 - 2.5 - 1/4)\theta > \theta/5.$$

Therefore, $|w(z)| \ge \theta^2|a|^{1/2}/5$ for $(5/6)\delta_a \le |z-a| \le \delta_a$ with $\delta_a = 3\theta|a|^{-1/4}$.

(2) *Case* $|\dot v(0)| = \kappa > 1$. Put

$$\eta_1 = \sup\{\eta | M(\eta) \le 5\theta\},$$

and suppose that $\eta_1 < (2/\kappa)\theta$. Then, by (3.1), for $|t| \le \eta_1$,

$$(3.8)\qquad |g(t)| \le \theta^{-1}|\rho|^5(2\theta)^3/6 + 6\theta(5\theta)^2(2\theta)^2/2 < \theta/24$$

(cf. (3.6)). By (3.4), (3.5) and this inequality, for $|t| \le \eta_1$,

$$(3.9)\quad |v(t)| \le |v(0)| + \kappa|t| + \theta^{-1}|t|^2/2 + \theta/24 \le (1/6 + 2 + 2^2/2 + 1/24)\theta < 4.3\theta,$$

which contradicts the definition of η_1. This implies $\eta_1 \ge (2/\kappa)\theta$, and hence (3.9) is valid for $|t| \le (2/\kappa)\theta$. For $(0.8/\kappa)\theta \le |t| \le (1.2/\kappa)\theta$, we have

$$|v(t)| \ge \kappa|t| - \theta^{-1}|t|^2/2 - |v(0)| - |g(t)| \ge (0.8 - 0.8^2/2 - 1/6 - 1/24)\theta > \theta/5.$$

Hence $|w(z)| \ge \theta^2|a|^{1/2}/5$ in $(5/6)\delta_a \le |z-a| \le \delta_a$ with $\delta_a = (1.2/\kappa)\theta|a|^{-1/4} = 1.2\theta|a|^{-1/4}(\theta|a|^{3/4}/|w'(a)|)$, which completes the proof. □

Lemma 3.4. *Under the same supposition as in Lemma* 3.3, *if* $|w(a)| \le \theta^2|a|^{1/2}/6$, *then*

(ii′) $|w(z)| > \theta^2|z|^{1/2}/5.5$ *for* $(5/6)\delta_a \le |z-a| \le \delta_a$.

Proof. By (3.1), (3.2) and the supposition $|a| > 5$, we have $|z| \ge |a| - \delta_a > 4$ and $||z|^{1/2} - |a|^{1/2}| \le |z-a|/(|z|^{1/2} + |a|^{1/2}) \le \delta_a/4 \le \theta < 1/10$. Hence, by Lemma 3.3,(ii), for $(5/6)\delta_a \le |z-a| \le \delta_a$,

$$|w(z)| - \theta^2|z|^{1/2}/5.5 \ge |w(z)| - \theta^2|a|^{1/2}/5.5 - \theta^2||z|^{1/2} - |a|^{1/2}|/5.5$$
$$\ge \theta^2(1/5 - 1/5.5)|a|^{1/2} - \theta^2||z|^{1/2} - |a|^{1/2}|/5.5 \ge (\sqrt{5}/55 - 1/55)\theta^2 > 0,$$

which completes the proof. □

3.2. Path. By Lemma 3.4, we construct the path as follows:

Lemma 3.5. *Let σ be an arbitrary pole of $w(z)$ satisfying $|\sigma| > 10$, and R_0 a number satisfying $5 < R_0 < 6$. Then there exists a curve $\Gamma(\sigma) : z = \phi(x)$, $0 \le x \le x_\sigma$ such that*
(1) $|\phi(0)| = R_0$, $\phi(x_\sigma) = \sigma$;
(2) *x is the length of $\Gamma(\sigma)$ from $\phi(0)$ to $\phi(x)$;*
(3) *$|\phi(x)|$ is monotone increasing on $[0, x_\sigma]$;*
(4) *$|dz| \le (6/\sqrt{11})d|z|$ along $\Gamma(\sigma)$;*
(5) *$|w(z)| \ge 2^{-11}|z|^{1/2}$ along $\Gamma(\sigma)$.*

Proof. For the simplicity of the description, we treat the case where $\arg\sigma = 0$. For σ in the generic position, we can show this lemma by the same argument. Consider the segment $S_0 = [R_0, \sigma] \subset \mathbf{R}$. Start from $z = R_0$, and proceed along S_0. If $|w(z)| > \theta^2|z|^{1/2}/6$ on S_0, then we put $\Gamma(\sigma) = S_0$. Suppose that a point $a \in S_0$ satisfies $|w(a)| \le \theta^2|a|^{1/2}/6$ and $|w(z)| > \theta^2|z|^{1/2}/6$ for $R_0 \le z < a$. Draw the semi-circle $C_a : |z - a| = \delta_a$, $\mathrm{Re}\, z \ge 0$ (cf. Lemma 3.3) which crosses $\mathbf{R}$ at a_- and a_+ $(a_- < a_+)$. Note that $a_+ \in S_0$, because the pole σ does not belong to the interior of C_a. Let a_-^*, a_+^* $(\mathrm{Re}\, a_-^* < \mathrm{Re}\, a_+^*)$ be the points on the semi-circle $C_a^* : |z - a| = (5/6)\delta_a$, $\mathrm{Re}\, z \ge 0$ such that the segments $[a_-, a_-^*]$ and $[a_+^*, a_+]$ come in contact with C_a^*. Replace the segment $[a_-, a_+]$ by the curve $\gamma(a)$ which consists of the segments $[a_-, a_-^*]$, $[a_+^*, a_+]$ and the shorter arc $(a_-^*, a_+^*)^\frown \subset C_a^*$. Then we get a new curve $\Gamma_1 = \big((S_0 \setminus [a_-, a_+]) \cup \gamma(a)\big) \cap \{z \,|\, |z| \ge R_0\}$. By Lemmas 3.3 and 3.4, and by a geometric consideration, we have, on Γ_1,

$$|w(z)| \ge \theta^2|z|^{1/2}/6 > 2^{-11}|z|^{1/2} \tag{3.10}$$

and

$$|dz| \le (6/\sqrt{11})d|z|. \tag{3.11}$$

Start again from $z = a_+$. Suppose that we first meet a point $b \in \Gamma_1$, $b > a_+$ such that $|w(b)| = \theta^2|b|^{1/2}/6$. (If such a point does not exist, then we put $\Gamma(\sigma) = \Gamma_1$.) By the same argument as above, we obtain the curve $\gamma(b)$, which crosses Γ_1 at b'_-, b_+ $(\mathrm{Im}\, b'_- \ge 0$, $b_+ \in S_0$, $\mathrm{Re}\, b'_- < b_+)$. Replacing the part Γ_1 from b'_- to b_+ by that of $\gamma(b)$ from b'_- to b_+, we get a path Γ_2. On it, (3.10) and (3.11) are valid. Start from $z = b_+$, and continue this procedure. As will be shown below, after repeating this procedure finitely many times, we arrive at the pole σ. Thus we get the path $\Gamma(\sigma)$ with the properties (1) through (5). To show the finiteness, suppose the contrary that there exists a sequence $\{a(\nu)\}_{\nu=0}^\infty \subset S_0$ satisfying $\sum_{\nu=0}^\infty \delta_{a(\nu)} \le 1$ and $|w(a(\nu))| \le \theta^2|\sigma|^{1/2}/6$. Hence, by (3.2), we may choose a subsequence $\{a(\nu_j)\}_{j=0}^\infty$ satisfying $a(\nu_j) \to a_* \in S_0$, $w(a(\nu_j)) \to w_* \ne \infty$, $w'(a(\nu_j)) \to \infty$ as $j \to \infty$, which implies $w(a_*) = w_* \ne \infty$, $w'(a_*) = \infty$. This is a contradiction. Thus the lemma is proved. □

3.3. Auxiliary function. Consider the auxiliary function

$$\Phi(z) = w'(z)^2 + \frac{w'(z)}{w(z)} - 4w(z)^3 - 2zw(z), \tag{3.12}$$

which was used in the proof of the Painlevé property ([2]). Using the relation

$$(w'(z)^2)' = \left(4w(z)^3 + 2zw(z)\right)' - 2w(z)$$

obtained from (I), we have the relation

$$\Phi'(z) + \frac{\Phi(z)}{w(z)^2} = -\frac{z}{w(z)} + \frac{w'(z)}{w(z)^3}. \tag{3.13}$$

Solving (3.13), we have

Lemma 3.6. *For an arbitrary path $\gamma(z_0, z)$ starting from z_0 and ending at z, if $w(t) \neq 0$ on $\gamma(z_0, z)$, then*

(3.14)
$$\Phi(z) = E(z_0, z)^{-1}\left[\Phi(z_0) - \frac{E(z_0,z)}{2w(z)^2} + \frac{1}{2w(z_0)^2} - \int_{\gamma(z_0,z)} \frac{E(z_0,t)}{2w(t)^4}(2tw(t)^3 - 1)dt\right]$$

with $E(z_0, t) = \exp\left(\int_{\gamma(z_0,t)} w(\tau)^{-2}d\tau\right)$, $t \in \gamma(z_0, z)$. Here $\gamma(z_0, t) \subset \gamma(z_0, z)$ is the part of $\gamma(z_0, z)$ from z_0 to t.

3.4. Completion of the proof. Take the circle $|z| = R_0$ $(5 < R_0 < 6)$ on which $\Phi(z) \neq \infty$. Let σ be an arbitrary pole of $w(z)$ such that $|\sigma| > 10$, and $U(\sigma)$ a domain defined by

$$U(\sigma) = \{z \mid |z - \sigma| < \eta(\sigma)\}, \quad \eta(\sigma) = \sup\{\eta \le 1 \mid |w(z)| > 2|z|^{1/2} \text{ in } |z - \sigma| < \eta\}.$$

Then we have

Lemma 3.7. *In $U(\sigma)$, $|\Phi(z)| \le K_0|z|^{\Delta}$. Here K_0 is a positive number independent of σ, and $\Delta \ge 3/2$ a number independent of $w(z)$ and σ.*

Proof. Recall the path $\Gamma(\sigma)$ given in Lemma 3.5 starting from $z_0(\sigma)$, $|z_0(\sigma)| = R_0$. Then, $|w(t)| \ge 2^{-11}|t|^{1/2}$, $|dt| \le (6/\sqrt{11})d|t|$ along $\Gamma(\sigma)$. From these facts, for $t \in \Gamma(\sigma)$, it follows that

$$|E(z_0(\sigma), t)^{\pm 1}| \le \exp\left(\int_{\Gamma(\sigma,t)} \frac{|d\tau|}{|w(\tau)|^2}\right) \le \exp\left(\frac{2^{22}\cdot 6}{\sqrt{11}}\int_{R_0}^{|t|}\frac{d|\tau|}{|\tau|}\right) = O(t^{\Delta'}),$$

$\Delta' = 2^{23}\cdot 3/\sqrt{11}$, where $\Gamma(\sigma, t) \subset \Gamma(\sigma)$ denotes the part of $\Gamma(\sigma)$ from $z_0(\sigma)$ to t. Moreover, $1/w(t) = O(t^{-1/2})$ along $\Gamma(\sigma)$. Using Lemma 3.6 and these estimates, and observing that $|\Phi(z_0(\sigma))| \le M_0$, we have $\Phi(\sigma) = O(\sigma^{2\Delta'+3/2})$, where $M_0 = \max\{|\Phi(z)| \mid |z| = R_0\}$. In $U(\sigma)$, applying Lemma 3.6 with $z_0 = \sigma$, $\gamma(z_0, z) = [\sigma, z] \subset U(\sigma)$, we obtain $\Phi(z) = O(z^{2\Delta'+3/2})$ in $U(\sigma)$. This completes the proof. □

Now we are ready to prove the theorem. Put $w(z) = u(z)^{-2}$, $z = \sigma + \sigma^{-\Delta/6}s$ in (3.12). Then it is written in the form

$$4u'(z)^2 - 2u(z)^5u'(z) - 4 - 2zu(z)^4 - u(z)^6\Phi(z) = 0.$$

Hence $v(s) = u(\sigma + \sigma^{-\Delta/6}s)$ satisfies

$$(3.15) \qquad (dv/ds)(s) = \sigma^{-\Delta/6}(1 + h(s, v(s))), \quad |h(s, v(s))| < 1/2, \quad v(0) = 0,$$

as long as

$$(3.16) \qquad |z^{\Delta/6}u(z)| = \left|(\sigma + \sigma^{-\Delta/6}s)^{\Delta/6}\right||v(s)| < \varepsilon_0,$$

and $z \in U(\sigma)$ (cf. Lemma 3.7), where $\varepsilon_0 = \varepsilon_0(K_0)$ is a sufficiently small positive constant independent of σ. If z satisfies (3.16), then $|w(z)| > \varepsilon_0^{-2}|z|^{\Delta/3} \geq \varepsilon_0^{-2}|z|^{1/2}$, and hence $z \in U(\sigma)$. Put

$$(3.17) \qquad \eta_* = \sup\{\eta \mid (3.16) \text{ is valid for } |s| < \eta\}.$$

Suppose that $\eta_* < \varepsilon_0/4$. Then, integrating (3.15), we have

$$(3.18) \qquad |s|/2 \leq |\sigma^{\Delta/6}||v(s)| \leq 3|s|/2 \leq 3\varepsilon_0/8$$

for $|s| \leq \eta_* < \varepsilon_0/4$, which implies

$$\left|(\sigma + \sigma^{-\Delta/6}s)^{\Delta/6}\right||v(s)| \leq |\sigma^{\Delta/6}||v(s)|(1 + 1/L)^{\Delta/6} \leq \varepsilon_0/2$$

for $|s| \leq \eta_*$ and for $|\sigma| \geq L$, where L is sufficiently large. For $|\sigma| \geq L$, this contradicts (3.17), which implies $\eta_* \geq \varepsilon_0/4$. Therefore, for $|\sigma| \geq L$, (3.18) is valid for $|s| < \varepsilon_0/4$, and $w(z)$ is analytic for $0 < |z - \sigma| < (\varepsilon_0/4)|\sigma|^{-\Delta/6}$. Thus we have

Lemma 3.8. *For every pole σ of $w(z)$ satisfying $|\sigma| > L$ (> 10), $w(z)$ is analytic in the domain $0 < |z - \sigma| < (\varepsilon_0/4)|\sigma|^{-\Delta/6}$.*

For each pole σ, $|\sigma| > L$, we allocate the disk $U_*(\sigma) : |z - \sigma| < (\varepsilon_0/8)|\sigma|^{-\Delta/6}$. Then, for arbitrary distinct poles σ_1, σ_2, we have $U_*(\sigma_1) \cap U_*(\sigma_2) = \emptyset$. Hence the cardinal number of the poles in the disk $|z| < r$ does not exceed $O(r^{2+\Delta/3})$. Since $m(r, w) = S(r, w)$, using Lemma 1.11, we have $T(r, w) = O(N(2r, w)) = O(r^{2+\Delta/3})$, which completes the proof for (I).

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