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Limit computation of some discontinuous functions

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1 Introduction

Computability on the continuum has been investigated over the years, but we are interested in the mathematical approach such as the one by Pour-El and Richards [2]. In their approach, classical mathematics is accepted. The point is to see how computable objects and operators look like in ordinary mathematics.

Computabilities of real numbers and continuous (real) functions were defined by Grzegorczyk. A real number is computable if it can be effectively represented by a rational sequence. A continuous function is computable if it preserves sequential computability and it is effectively uniformly continuous.

We do, however, compute and draw graphs of discontinuous function. We can let Mathematica draw graphs of such functions. The problem arising in so doing is the computation of the value at a jump point. This is because it is not in general decidable if a real number is a jump point, that is, $a = 0$ is not decidable even for a computable a.

One approach to this problem was proposed by Pour-El and Richards. It was a functional approach, that is, a function is regarded as computable if it can be effectively approximated by rational coefficient polynomials with respect to the norm of a function space, such as a Banach space or a Frechet space.

In such a case, a function is regarded as computable as a point in a space. This is sufficient in order to draw a rough graph of the function, but does not supply us with information on computation of individual values.

One way of computing the value at a jump point is to do it in terms of *limiting recursive functional.* Another is to change the topology of the domain.

Here we will report the former method.

Of course, there are many other ways of dealing with the computability of discontinuous functions, but we will concentrate on our treatment.

For a quick reference of Pour-El and Richards approach, [7] is available.

In Section 2, a counterexample of a computable sequence of real numbers whose values of the integer part function do not form a computable sequence. In Section 3, the limiting recursive functional is defined according to Gold's idea. In Section 4, examples of discontinuous functions which can be computed in terms of limiting recursive functionals. This report is concluded by listing some problems yet to be worked out (Section 5).

Below, the counter-example in Section 2 and Sections 3 and 4 are direct quotations from [5].

2 Counte-rexample

Let us take up the integer part function $[x]$ as a simple example of a discontinuous function. In trying to compute the value of this function at a (computable) point x, the problem arises if x is in fact an integer but the information of x does not yield that fact within any finite steps. Needless to say that the value $[x]$ is computable for any x. The point is that there is no effective method to evaluate $[x]$ from the information x has. This can be assured by showing that there is a computable sequence of real numbers, say $\{x_n\}$ for which $\{[x_n]\}$ is not a computable sequence. This is the reason why we have to develop a theory of computation for some discontinuous functions.

Let a be a recursive injection whose range is not recursive. Let $\{x_n\}$ be the computable sequence of reals in Example 4, Chapter 0 of [2]:

$$x_n = \left\{ \begin{array}{ll} \frac{1}{2^m} & \text{if } n = a(m) \text{ for some } m \\ 0 & \text{otherwise} \end{array} \right\}$$

$y_n := 1 - x_n$ $\{y_n\}$ is computable.

$$y_n = \left\{ \begin{array}{ll} 1 - \frac{1}{2^m} & \text{if } n = a(m) \text{ for some } m \\ 1 & \text{otherwise} \end{array} \right\}$$

This implies that

$$[y_n] = \left\{ \begin{array}{ll} 0 & \text{if } n = a(m) \text{ for some } m \\ 1 & \text{otherwise} \end{array} \right\}$$

Suppose $\{[y_n]\}$ were a computable sequence. Then, the range of a would be recursive, yielding a contradiction. So, $\{[y_n]\}$ cannot be a computable sequence.

So, the Gaußian function does not necessarily preserve computability of a sequence of reals.

3 Limiting recursive functionals

The definition of limiting recursive functionals is due to Gold[1].

Definition 3.1 (Gold) (1) (Limiting recursive functional) Σ: a set of total functions

A *partial* functional, $F(\phi_1, \cdots, \phi_r, p_1, \cdots, p_s)$, $r, s \geq 0$ is *limiting recursive on* Σ if there is a number-theoretic, total recursive function $g(z_1, \cdots, z_r, p_1, \cdots, p_s, n)$ satisfying

$$F(\phi_1, \cdots, \phi_r, p_1, \cdots, p_s) = \lim_n g(\check{\phi}_1(n), \cdots, \check{\phi}_r(n), p_1, \cdots, p_s, n)$$

for all r-tuples $\langle \phi_1, \cdots, \phi_r \rangle$ of functions in Σ and all $p_1, \cdots, p_s \in \mathbf{N}$,

$\tilde{\phi}(n) = \langle \phi(0, p_1, \cdots, p_s), \cdots, \phi(n, p_1, \cdots, p_s) \rangle$

F is ***limiting recursive*** if Σ is the class of all total recursive functions.

Example 1 (Least value functional) Let ϕ denote a number-theoretic function with $s+1$ arguments, $s \geq 0$. The least value property

$$\exists m \forall n \ (\phi(n, p_1, \cdots, p_s) \geq \phi(m, p_1, \cdots, p_s))$$

holds for ϕ. Let $L(\phi, p_1, \cdots, p_s)$ denote this least value. L is limiting recursive, and will be called the ***least value functional.***

Example 2 (Limit functional) Let ϕ denote a number-theoretic function, and let Lim be a partial functional such that $Lim(\phi, p_1, \cdots, p_s) = \lim_n \phi(n, p_1, \cdots, p_s)$. Then Lim is limiting recursive.

4 Examples of limit computation

This section is directly quoted from Yasugi-Brattka-Washihara [5].

Proposition 4.1 (A computation for $[x]$) $[x]$ can be computed using the least value functional L.

Proof Let $\langle \{r_n\}, \alpha \rangle$ be a representation of x.

1. Find some n such that $n < x < n+2$.

2. Define a recursive sequence (of rationals) $\{N_p\}$ such that $N_p = n+1$ if $r_{\alpha(p)} \geq (n+1) - \frac{1}{2^p}$; $N_p = n$ if $r_{\alpha(p)} < (n+1) - \frac{1}{2^p}$.

3. If $l = L(\{N_p\}) = n+1$, then put $\beta(p) = 1$ for all p; if $l = L(\{N_p\}) = n$, then put $\beta(p) = \min \{q \in \mathbf{N} : r_{\alpha(q)} < n + 1 - \frac{1}{2^q}\}$.

4. Output $\langle \{N_p\}, \beta \rangle$. This represents $[x]$.

Theorem 1 (Sequential computation of $[x]$) The Gaußian function has a sequential computation using the least value functional. Namely, for any computable sequence of real numbers, the sequence of values $\{[x_i]\}$ can be computed effectively in terms of the least value functional L.

Theorem 2 (Upper semi-computability, lower semi-computability and L) Each upper (lower) semi-computable function can be computed in terms of the functional L.

Example 3 $h(x) := x - [x]$. We may assume $r_m > x$. The n satisfying $n < x < n+2$ can be found.

$$R(p) \equiv (r_{\alpha(p)} < (n+1) - \frac{1}{2^p})$$

$$N_q = \begin{cases} n, & \text{if } \exists p \le q \ R(p) \\ n+1 & \text{if } \forall p \le q \ \neg R(p) \end{cases}$$

$$s_q = \begin{cases} r_q - n, & \text{if } \exists p \le q \ R(p) \\ r_q - (n+1) & \text{if } \forall p \le q \ \neg R(p) \end{cases}$$

If $L(\{N_q\}) = n$, then $\beta(p) = \max(\alpha(p), p_0)$, where $p_0 = \mu(r_{\alpha(p)} < (n+1) - \frac{1}{2^p})$; if $L(\{N_p\}) = n+1$, then $\beta(p) = \alpha(p)$. $\langle\{s_q\}, \beta\rangle$ represents h.

Example 5 $\sigma(x) = 1(x \in (0, \infty)), = \frac{1}{2}(x = 0), = 0(x \in (-\infty, 0))$. σ is neither upper nor lower semi-computable.

$$M_p = \begin{cases} 2, & \text{if } r_{\alpha(p)} > \frac{1}{2^p} \\ 1, & \text{if } |r_{\alpha(p)}| \le \frac{1}{2^p} \\ 0 & \text{if } r_{\alpha(p)} < -\frac{1}{2^p} \end{cases}$$

$\{M_p\}$ is recursive.

$$s_p = \frac{1}{2} M_p$$

$$\sigma(x) = \lim_p s_p$$

$Lim(\{M_p\}) = 2$ for Case 1, $Lim(\{M_p\}) = 1$ for Case 2, and $Lim(\{M_p\}) = 0$ for Case 3.

The modulus of convergence γ: If $Lim(\{M_p\}) = 2$ or $Lim(\{M_p\}) = 0$, then $\gamma(p) = \min\{q \in \mathbf{N} : |r_{\alpha(q)}| > \frac{1}{2^q}\}$.

If $Lim(\{M_p\}) = 1$, then $\gamma(p) = 1$.

$\langle\{s_p\}, \gamma\rangle$ represents σ

Example 7 $\tau(x) = \tan x$ if $\frac{2n+1}{2}\pi < x < \frac{2n+3}{2}\pi$; $\tau(x) = 0$ if $x = \frac{2n+1}{2}\pi$ for all n. τ is unbounded on appropriately large compact sets.

An integer n satisfying $\frac{2n+1}{2}\pi < x < \frac{2n+5}{2}\pi$ can be found.

$$N_p = \begin{cases} n, & \text{if } r_{\alpha(p)} < \frac{2n+3}{2}\pi - \frac{1}{2^p} \\ n+2, & \text{if } r_{\alpha(p)} > \frac{2n+3}{2}\pi + \frac{1}{2^p} \\ n+1, & \text{if } \frac{2n+3}{2}\pi - \frac{1}{2^p} \le r_{\alpha(p)} \le \frac{2n+3}{2}\pi + \frac{1}{2^p} \end{cases}$$

$$t_q = \begin{cases} \tan r_q, & \text{if } N_q = n \text{ or } N_q = n+2 \\ 0, & \text{if } N_q = n+1 \end{cases}$$

$\{t_q\}$ converges to $\tau(x)$. The modulus of convergence can be defined similarly to that of Example 5 according as $Lim(\{N_q\}) = n, n+2$ or $n+1$.

5 Problems

In the end, we will list some open problems concerning the computation of functions.

1. The upper semi-computable function can be computed in terms of a limiting recursive functional (Theorem 2). In the existing proof of this theorem,

the least value functional applied to the construction of a rational sequence approximating the value of the function.

Can we prove this with a limiting recursive functional appliled only to modulus of convergence?

2. Interrelation between application of a limiting recursive functional and treatment by way of a function space.

3. Preservation (or non-preservation) of computability in metrization (by the general construction) of an effective uniform space (cf. [3], [6]).

4. Preservation of computability in metrization of a Frechet space by d_3? (See [4].)

The third metric d_3 is defined by means of the neighborhood system.

Put $U_n = \{x : p_n(x) < 1/2^n\}$ and denote the set of all finite binary fractions in the interval $(0, 1)$ by D. For an element $r \in D$ represented by

$$r = 1/2^{i_1} + 1/2^{i_2} + \ldots + 1/2^{i_\ell} \quad (1 \leq i_1 < i_2 < \ldots < i_\ell),$$

put

$$A(r) = U_{i_1} + U_{i_2} + \cdots + U_{i_\ell}$$

and define ϕ by

$$\phi(x) = \begin{cases} \inf\{r \in D : x \in A(r)\}, & \text{if} \quad x \in A(r) \quad \text{for some} \quad r, \\ 1, & \text{otherwise}, \end{cases}$$

where $U + V = \{x + y : x \in U, y \in V\}$. By means of this ϕ, we obtain the third metric

$$d_3(x, y) = \phi(x - y).$$

Does this metric preserves computability? The following partial solution: A sufficient condition:

Put $U(n, r) = \{x : p_n(x) < r\}$. Then, the inclusion

$$A(r) \subset U(i_1, r)$$

holds for any binary fraction r as above.

The inverse inclusion

$$(*) \qquad U(i_1, r) \subset A(r)$$

holds, for instance, in $C(\mathbf{R})$, but does not hold in $C^\infty[0, 1]$. Under the condition $(*)$, the metric d_3 preserves computability.

Effective convergence with resepct to d_3 is equivalent to effective convergence with respect to semi-norm effective convergence.

5. Interrelation between application of a limiting recursive functional and the uniform space method (cf. [5], [3]).

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