数理解析研究所講究録 1290 巻 2002 年 112-120



# On the first homology of the group of equivariant Lipschit homeomorphisms of the plane with circle action

信州大学・理学部 阿部 孝順 (Kōjun Abe)
Department of Mathematical Sciences,Shinshu University,
京都産業大学・理学部 福井 和彦 (Kazuhiko Fukui)
Department of Mathematics, Kyoto Sangyo University,
山形大学・工学部 三浦 毅 (Takeshi Miura)
Department of Basic Technology, Applied Mathematics and Physics,
Yamagata University,

## §1. Introduction and statement of the result

Let $L_G(M)$ denote the group of equivariant Lipschitz homeomorphisms of a $G$-manifold $M$ which are isotopic to the identity through equivariant Lipschitz homeomorphisms with compact supports. In the previous papers [AF3],[AF4], we treated the subgroup $\mathcal{H}_{LIP,G}(M)$ of $L_G(M)$ whose elements are isotopic to the identity with respect to the compact open Lipschitz topology, and proved that $\mathcal{H}_{LIP,G}(M)$ is perfect when $M$ is a principal $G$-manifold or $M$ is a smooth $G$-manifold for a finite group $G$.

In this paper we consider the case of the complex plain $\mathbf{C}$ with canonical $U(1)$-action. We shall prove that the group $L_{U(1)}(\mathbf{C})$ is not perfect by calculating the the first homology group $H_1(L_{U(1)}(\mathbf{C}))$ which is defined as the quotient of $L_{U(1)}(\mathbf{C})$ by its commutator subgroup.

Let $\mathcal{C}(\mathbf{R})$ be the set of real valued functions $f$ on $(0,1]$ such that there exists a positive number $M$ satisfying

$$|f(x)-f(y)| \leq \frac{M}{x}(y-x) \quad \text{for } 0<x\leq y\leq 1.$$

Then $\mathcal{C}(\mathbf{R})$ is a vector space over $\mathbf{R}$. Let $\mathcal{C}_0(\mathbf{R})$ denote the subspace of those $f \in \mathcal{C}(\mathbf{R})$ with $f$ bounded on $(0,1]$. Then we shall prove the following.

**Theorem 1**

$$H_1(L_{U(1)}(\mathbf{C})) \cong \mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R}).$$

Here the isomorphism is induced from the map assigning each $h \in L_{U(1)}(\mathbf{C})$ a function $\hat{a}_h \in \mathcal{C}(\mathbf{R})$ which stand for the degree of rotation of $h$ as the point tend to zero (see §2). We note that the group $\mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R})$ is fairly large group since it contains linearly independent family of elements parameterized by $(0,1]$.

The situation is quite different in smooth category. Let $D_{U(1)}(\mathbf{C})$ denote the group of equivariant diffeomorphism group of $\mathbf{C}$ which are equivariantly diffeomorphic to the identity through compact supports. By [AF2], Theorem 3.2, we have that there exists an isomorphism $H_1(D_{U(1)}(\mathbf{C})) \cong \mathbf{R} \times \mathbf{U(1)}$ induced from the map assigning each $h \in D_{U(1)}(\mathbf{C})$ the differential of $h$ at 0. Then it follows from Theorem 1 that the group $D_{U(1)}(\mathbf{C})$ is contained in the commutator subgroup of $L_{U(1)}(D)$, which implies that the first homology group of $D_{U(1)}(\mathbf{C})$ detect absolutely different geometric property.

## §2. Orbit preserving equivariant Lipschitz homeomorphisms

Let $D$ denote the unit disc in $C$ and $L_{U(1)}(D)$ denote the group of $U(1)$-equivariant Lipschitz homeomorphisms of $D$ which are isotopic to the identity through $U(1)$-equivariant homeomorphisms with identity on the boundary $\partial D$. Since $U(1)$ acts freely except for the origin, by combining Theorem 5.1 with Corollary 5.5 in [AF3], the group $H_1(L_{U(1)}(\mathbf{C}))$ is isomorphic to $H_1(L_{U(1)}(D))$.

Let $L([0,1])$ denote the group of Lipschitz homeomorphisms of the unit interval $[0,1]$ which are isotopic to the identity through Lipschitz homeomorphisms. Then we have a group homomorphism $P: L_{U(1)}(D) \to L([0,1])$ given by

$$P(h)(x) = |h(x)| \quad \text{for} \quad h \in L_{U(1)}(D),\ x \in [0,1].$$

There exists a right inverse $\Psi: L([0,1]) \to L_{U(1)}(D)$ of $P$ defined by

$$\Psi(f)(xz) = f(x)z \quad \text{for} \quad f \in L([0,1]),\ x \in [0,1],\ z \in U(1).$$

Note that the kernel $KerP$ of $P$ coincides with the set of those $h \in L_{U(1)}(D)$ which are orbit preserving. Next we shall investigate the relation between the groups $KerP$ and $\mathcal{C}(\mathbf{R})$.

For $h \in KerP$, let $a_h : (0,1] \to U(1)$ be the map satisfying

$$h(xz) = xza_h(x) \quad \text{for} \quad x \in (0,1],\ z \in U(1).$$

Now we investigate the properties of those maps $a_h$. For a map $\alpha : (0,1] \to U(1) \subset \mathbf{C}$, we define maps $\bar{\alpha} : [0,1] \to D$ and $F_\alpha : D \to D$ as follows.

$$\bar{\alpha}(x) = \begin{cases} x\alpha(x) & (0 < x \leq 1) \\ 0 & (x = 0) \end{cases},$$

$$F_\alpha(xz) = z\bar{\alpha}(x) \qquad (0 \leq x \leq 1,\ z \in U(1)).$$

**Lemma 2** *The following conditions* (1), (2) *and* (3) *are equivalent.*
(1) *There exists a positive number* $K$ *such that*

$$|\alpha(x) - \alpha(y)| \leq \frac{K}{x}(y - x) \quad \textit{for} \quad 0 < x \leq y \leq 1.$$

(2) $\bar{\alpha}$ *is a Lipschitz map.*
(3) $F_\alpha$ *is a Lipschitz map.*

*Proof.* First assume the condition (1). Then, for $0 < x \leq y \leq 1$, we have

$$|\bar{\alpha}(x) - \bar{\alpha}(y)| \leq x|\alpha(x) - \alpha(y)| + |\alpha(y)||x - y| \leq (K + 1)|x - y|.$$

Since $|\bar{\alpha}(x)| \leq x$ for $0 < x \leq 1$, the condition (2) is satisfied.

Secondly assume the condition (2). Then, for $0 < x \leq y \leq 1,\ z_1, z_2 \in U(1)$,

$$\begin{aligned} |F_\alpha(xz_1) - F_\alpha(yz_2)| &\leq |z_1(\bar{\alpha}(x) - \bar{\alpha}(y)| + |(z_1 - z_2)\bar{\alpha}(y)| \\ &\leq M(|x - y| + |z_1(y - x) + (z_1 x - z_2 y)|) \\ &\leq 3M|xz_1 - yz_2|, \end{aligned}$$

where $M$ is a Lipschitz constant of $\bar{\alpha}$. Since $|F_\alpha(xz)| \leq M|xz|$, the condition (3) is satisfied.

Finally assume the condition (3). Then, for $0 < x \leq y \leq 1$, we have

$$\begin{aligned} |\alpha(x) - \alpha(y)| &\leq \frac{1}{x}(|x\alpha(x) - y\alpha(y)| + |(y - x)\alpha(y)|) \\ &= \frac{1}{x}(|F_\alpha(x) - F_\alpha(y)| + |y - x|) \leq \frac{L + 1}{x}(y - x), \end{aligned}$$

where $L$ is a Lipschitz constant of $F_\alpha$. Thus the condition (1) is satisfied and Lemma 2 follows.

Let $E : \mathbf{R} \to U(1)$ denote the exponential map given by $E(x) = e^{\sqrt{-1}x}$. Let $h \in KerP$. Since $h$ is identity on $\partial D$, $a_h(1) = 1$. Let $\hat{a}_h : (0,1] \to \mathbf{R}$ be the lifting of $a_h$ for $E$ with $\hat{a}_h(1) = 0$. Then $E \circ \hat{a}_h = a_h$.

**Lemma 3** *$\hat{a}_h$ is contained in $\mathcal{C}(\mathbf{R})$. Conversely if $\hat{\alpha} \in \mathcal{C}(\mathbf{R})$, then $E \circ \hat{\alpha}$ satisfies the condition* (1) *in* Lemma 2.

*Proof* By Lemma 2, there exists a positive number $K$ such that

$$|a_h(x) - a_h(y)| \leq \frac{K}{x}(y - x) \quad for \quad 0 < x \leq y \leq 1.$$

Note that, for each $x, y \in (0, 1]$ with $x < y$, the restriction $a_h\,|_{[x,y]}$ is Lipschitz. Then we can choose an increasing series of points $x = x_0 < x_1 < \cdots < x_{n-1} < x_n = y$ such that

$$|a_h(x_{i-1}) - a_h(x_i)| \leq \sqrt{3} \quad (i = 1, ..., n).$$

It follows that

$$|\hat{a}_h(x_{i-1}) - \hat{a}_h(x_i)| \leq \frac{2\pi}{3} \quad (i = 1, ..., n).$$

Then we have

$$\begin{aligned} |a_h(x_{i-1}) - a_h(x_i)| &= |e^{\sqrt{-1}\,\hat{a}(x_{i-1})} - e^{\sqrt{-1}\,\hat{a}(x_i)}| \\ &= 2\left|\sin\frac{\hat{a}_h(x_{i-1}) - \hat{a}_h(x_i)}{2}\right| \\ &= \left|\cos\frac{\theta\,(\hat{a}_h(x_{i-1}) - \hat{a}_h(x_i))}{2}\right| |\hat{a}_h(x_{i-1}) - \hat{a}_h(x_i)|, \end{aligned}$$

for some $0 < \theta < 1$. Thus

$$|\hat{a}_h(x_{i-1}) - \hat{a}_h(x_i)| \leq 2\,|a_h(x_{i-1}) - a_h(x_i)| \leq \frac{2K}{x_{i-1}}|x_{i-1} - x_i|.$$

Therefore we have

$$|\hat{a}_h(x) - \hat{a}_h(y)| \leq \sum_{i=1}^{n} \frac{2K}{x_{i-1}}|x_{i-1} - x_i| \leq \frac{2K}{x}(y - x),$$

and then we have that $\hat{a}_h \in \mathcal{C}(\mathbf{R})$.

Since

$$|E(x) - E(y)| = |e^{\sqrt{-1}x} - e^{\sqrt{-1}y}| \leq (y - x) \quad \text{for} \quad 0 < x \leq y \leq 1,$$

it is clear that, for each $\hat{\alpha} \in \mathcal{C}(\mathbf{R})$, $E \circ \hat{\alpha}$ satisfies the condition (1) in Lemma 2. This completes the proof of Lemma 3.

## §3. Basic homomorphisms

By Lemma 3 we can define a homomorphism

$$T: \; Ker P \to \mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R}), \qquad T(h) = \hat{a}_h \mod \mathcal{C}_0(\mathbf{R}).$$

Now we have a map

$$\Theta: \; L_{U(1)}(D) \to L([0,1]) \times \mathcal{C}/\mathcal{C}_0$$

defined by

$$\Theta(h) = (P(h), \; T(\Psi(P(h))^{-1} \circ h)).$$

**Proposition 4** *$\Theta$ is an onto group homomorphism.*

*Proof.* First we prove that $\Theta$ is a group homomorphism. For each $h \in L_{U(1)}(D)$, we set $\tilde{h} = \Psi(P(h))^{-1} \circ h$. Let $h_i \in L_{U(1)}(D)$ $(i = 1,2)$. Since $P$ is a group homomorphism, in order to prove $\Theta$ a group homomorphism it is sufficient to prove that

$$\hat{a}_{\widetilde{h_1 \circ h_2}} = \hat{a}_{\tilde{h}_1} + \hat{a}_{\tilde{h}_2} \quad \mod \; \mathcal{C}_0(\mathbf{R}).$$

If $0 < x \le 1$, $z \in U(1)$, then

$$h_i(xz) = P(h_i)(x)\, z a_{\tilde{h}_i}(x)^{-1} \quad (i = 1,2),$$

and

$$(h_1 \circ h_2)(xz) = P(h_1 \circ h_2)(x)\, z a_{\widetilde{h_1 \circ h_2}}(x)^{-1}.$$

On the other hand we have

$$(h_1 \circ h_2)(xz) = P(h_1 \circ h_2)(x)\, z a_{\tilde{h}_2}(x)^{-1} a_{\tilde{h}_1}(P(h_2)(x))^{-1}.$$

Then

$$a_{\widetilde{h_1 \circ h_2}} = (a_{\tilde{h}_1} \circ P(h_2)) \cdot a_{\tilde{h}_2}.$$

Thus

$$\hat{a}_{\widetilde{h_1 \circ h_2}} = \hat{a}_{\tilde{h}_1} \circ P(h_2) + \hat{a}_{\tilde{h}_2}.$$

Let $M$ and $M'$ be Lipschitz constants of $P(h_2)$ and $P(h_2)^{-1}$, respectively. Let $x \in (0,1]$. For the case $x \leq P(h_2)(x)$, by Lemma 3 there exists a positive number $K$ such that

$$|\hat{a}_{\tilde{h}_1}(P(h_2)(x)) - \hat{a}_{\tilde{h}_1}(x)| \leq \frac{K}{x}|P(h_2)(x) - x| \leq K(M+1).$$

By definition $x \leq M' P(h_2)(x)$. Then, for the case $P(h_2)(x) < x$, we have

$$|\hat{a}_{\tilde{h}_1}(P(h_2)(x)) - \hat{a}_{\tilde{h}_1}(x)| \leq \frac{K}{P(h_2)(x)}|P(h_2)(x) - x| \leq K(1+M').$$

Then we have

$$\hat{a}_{\tilde{h}_1} \circ P(h_2) - \hat{a}_{\tilde{h}_1} \in \mathcal{C}_0(\mathbf{R}).$$

Thus

$$\hat{a}_{\widetilde{h_1 \circ h_2}} = \hat{a}_{\tilde{h}_1} + \hat{a}_{\tilde{h}_2} \quad \mod \ \mathcal{C}_0(\mathbf{R}).$$

Therefore $\Theta$ is a group homomorphism.

Let $f \in L([0,1])$, $\hat{\alpha} \in \mathcal{C}(\mathbf{R})$. Combining Lemma 2 with Lemma 3, we have that $F_{E\circ\hat{\alpha}} \in KerP$. Set

$$h(xz) = f(x)F_{E\circ\hat{\alpha}}(xz) \quad \text{for} \quad 0 \leq x \leq 1, \ z \in U(1).$$

Then we see that $h \in L_{U(1)}(D)$ and $\Theta(h) = (f, \ \hat{\alpha} \mod \mathcal{C}_0(\mathbf{R}))$. Thus $\Theta$ is onto. This completes the proof of Proposition 4.

## §4 Proof of main theorem

**Proposition 5** *$Ker\ \Theta$ is contained in the commutator subgroup of $L_{U(1)}(D)$.*

*Proof.* If $h \in Ker\ \Theta$, then $h \in KerP$ and $\hat{a}_h \in \mathcal{C}_0(\mathbf{R})$. Thus, for any positive number $\varepsilon$, there exists an integer $n$ such that $\left|\frac{\hat{a}_h(x)}{n}\right| \leq \varepsilon$ for $0 < x \leq 1$ and

$$\left|\frac{\hat{a}_h(x)}{n} - \frac{\hat{a}_h(y)}{n}\right| \leq \frac{\varepsilon}{x}(y-x) \quad \text{for} \quad 0 < x \leq y \leq 1.$$

Note that $a_h = E(n\hat{a}_h) = E(\hat{a}_h)^n$. Then, for a sufficiently small positive number $\varepsilon$, we can assume that $|\hat{a}_h(x)| \leq \varepsilon$ for $0 < x \leq 1$ and

$$|\hat{a}_h(x) - \hat{a}_h(y)| \leq \frac{\varepsilon}{x}(y-x) \quad \text{for} \quad 0 < x \leq y \leq 1.$$

Let $v$ be a real valued smooth monotone increasing function on $(0,1]$ such that

$$v(x)=\begin{cases}\log x & (0<x\leq 1/2),\\ 0 & (3/4\leq x\leq 1).\end{cases}$$

Then it is easy to see $v\in\mathcal{C}(\mathbf{R})$. Let $f$ be a real valued function on $[0,1]$ defined by

$$f(x)=\begin{cases}xe^{\hat{a}_h(x)} & (0<x\leq 1),\\ 0 & (x=0).\end{cases}$$

Note that $f(1)=1$. We shall prove that $f\in L([0,1])$ for sufficiently small $\varepsilon$. If $0<x\leq y\leq 1$, then we have

$$\begin{aligned}&|(f(y)-y)-(f(x)-x)|\\ =\ & |(y-x)(e^{\hat{a}_h(y)}-1)+x(e^{\hat{a}_h(y)}-e^{\hat{a}_h(x)})|\\ \leq\ & (y-x)|e^{|\hat{a}_h(y)|}-1|+x|\hat{a}_h(y)-\hat{a}_h(x)|e^{\hat{a}_h(x)+\theta(\hat{a}_h(y)-\hat{a}_h(x))}\\ \leq\ & ((e^{\varepsilon}-1)+\varepsilon e^{3\varepsilon})(y-x),\end{aligned}$$

for some $0<\theta<1$. Here we take the positive number $\varepsilon$ satisfying

$$(e^{\varepsilon}-1)+\varepsilon e^{3\varepsilon}<1.$$

Then it follows from [AF3], Lemma 4.1 that the function $f$ is a Lipschitz homeomorphism of $[0,1]$ which is isotopic to the identity through Lipschitz homeomorphisms.

If $0<x\leq\dfrac{1}{2e^{\varepsilon}}$, then we have

$$v(f(x))-v(x)=\log(xe^{\hat{a}_h(x)})-\log x=\hat{a}_h(x).$$

Then, for $0<x\leq\dfrac{1}{2e^{\varepsilon}}$, $z\in U(1)$ we have

$$\begin{aligned}(F_{Eov}^{-1}\circ\Psi(f)^{-1}\circ F_{Eov}^{-1}\circ\Psi(f))(xz) &= (F_{Eov}^{-1}\circ\Psi(f)^{-1}\circ F_{Eov}^{-1})(f(x)z)\\ &= (F_{Eov}^{-1}\circ\Psi(f)^{-1})(f(x)ze^{\sqrt{-1}\,v(f(x))})\\ &= F_{Eov}^{-1}(xze^{\sqrt{-1}\,v(f(x))})\\ &= xze^{\sqrt{-1}\,v(f(x))}e^{-\sqrt{-1}\,v(x)}\\ &= h(xz)\end{aligned}$$

Set

$$g=h\circ\Psi(f)^{-1}\circ F_{Eov}^{-1}\circ\Psi(f)\circ F_{Eov}.$$

$$g(xz) = xz \quad \text{for} \quad 0 \le x \le \frac{1}{2e^{\varepsilon}}, \ z \in U(1).$$

Thus the support of $g$ is contained in $D \setminus \{0\}$. From [AF3], Theorem 5.1, $g$ is contained in the commutator subgroup of $L_{U(1)}(D)$. Hence h is also contained in the commutator subgroup. This completes the proof of Proposition 5.

*Proof of Theorem 1.* Let $\iota : Ker\Theta \to L_{U(1)}(D)$ denote the inclusion. By Proposition 4 we have the following exact sequence.

$$\begin{aligned} Ker\Theta/[Ker\Theta, L_{U(1)}(D)] &\overset{\iota_*}{\to} H_1(L_{U(1)}(D)) \\ &\overset{\Theta_*}{\to} H_1(L([0,1]) \times \mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R})) \to 1. \end{aligned}$$

Since $\iota_* = 0$ by Proposition 5, $\Theta_*$ is isomorphic. By [TS], [AF4], the group $L([0,1])$ is perfect. Thus we have

$$H_1(L_{U(1)}(D)) \cong \mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R}).$$

**Remark.** Let $v_c$ $(0 < c \le 1)$ be real valued smooth functions on $(0,1]$ such that

$$v_c(x) = \begin{cases} (-\log x)^c & (0 < x \le 1/2), \\ 0 & (3/4 \le x \le 1). \end{cases}$$

Then $v_c \in \mathcal{C}(\mathbf{R})$. Thus the group $\mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R})$ contains linearly independent families $\{v_c \mod \mathcal{C}_0 \ ; \ 0 < c \le 1\}$.

# References


[AF1] K. Abe and K. Fukui, *On commutators of equivariant diffeomorphisms*, Proc. Japan Acad., 54 (1978), 52-54.

[AF2] K. Abe and K. Fukui, *On the structure of the group of equivariant diffeomorphisms of G-manifolds with codimension one orbit*, Topology, 40 (2001), 1325-1337.

[AF3] K. Abe and K. Fukui, *On the structure of the group of Lipschitz homeomorphisms and its subgroups*, J. Math. Soc. Japan, 53 (2001), 501-511.

[AF4] K. Abe and K. Fukui, *On the structure of the group of Lipschitz homeomorphisms and its subgroups II*, preprint.

[B1] A. Banyaga, *On the structure of the group of equivariant diffeomorphisms*, Topology, 16(1977), 279-283.

[F] K. Fukui, *Homologies of the group of* $Diff^{\infty}(R^n, 0)$ *and its subgroups*, J. Math. Kyoto Univ., 20(1980), 475-487.

[TS] T. Tsuboi, *On the perfectness of groups of diffeomorphisms of the interval tangent to the identity at the endpoints*, preprint.

[TH] W. Thurston, *Foliations and group of diffeomorphisms*, Bull. Amer. Math. Soc., 80(1974), 304-307.