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A SURVEY ON CHARACTERIZATION OF NUCLEAR C*-ALGEBRAS

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1. NUCLEARITY AND INJECTIVITY

We assume the reader to have basic knowledge on tensor products of C^*-algebras. For this, the appendix T of [We] is readable. See also [Ta1].

A C^*-algebra A is said to be *nuclear* if one has $A\otimes_{\max}B=A\otimes_{\min}B$ for any B. Here, by $A\otimes_{\max}B=A\otimes_{\min}B$, we mean that the canonical quotient map Q from $A\otimes_{\max}B$ onto $A\otimes_{\min}B$ is $*$-isomorphic. (I am not sure whether it can happen that $A\otimes_{\max}B$ and $A\otimes_{\min}B$ are $*$-isomorphic without Q being injective.) If $\varphi\colon A_1\to A_2$ is a cp (completely positive) contraction, then the map $\varphi\otimes\mathrm{id}_B\colon A_1\otimes_{\mathrm{alg}}B\to A_2\otimes_{\mathrm{alg}}B$ extends to a cp contractions $\varphi\otimes_{\min}\mathrm{id}_B\colon A_1\otimes_{\min}B\to A_2\otimes_{\min}B$ and $\varphi\otimes_{\max}\mathrm{id}_B\colon A_1\otimes_{\max}B\to A_2\otimes_{\max}B$. This fact follows from the Stinespring representation theorem. We usually omit the subscript of $\varphi\otimes\mathrm{id}_B$. The second dual A^{**} of a C^*-algebra A is a von Neumann algebra. A von Neumann algebra $M\subset\mathbb{B}(\mathcal{H})$ is said to be *injective* if there is a cp projection φ from $\mathbb{B}(\mathcal{H})$ onto M. This property does not depends on a choice of faithful normal representations of M. A cp projection is often called a *conditional expectation* because of the following fact. Let A be a C^*-subalgebra of B and φ be a cp contraction from B into $\mathbb{B}(\mathcal{H})$. If $\varphi|_A$ is multiplicative, then φ is automatically an A-bimodule map, i.e., $\varphi(axb)=\varphi(a)\varphi(x)\varphi(b)$ for $a,b\in A$ and $x\in B$. This follows from the Stinespring representation theorem; $\varphi(xy)-\varphi(x)\varphi(y)=[V^*\pi(x)(1-VV^*)^{1/2}][(1-VV^*)^{1/2}\pi(y)V]=XY$ and $XX^*=\varphi(xx^*)-\varphi(x)\varphi(x)^*$, etc. See [Ch] for the detail. A C^*-algebra A is said to have the CPAP (completely positive approximation property) if there is a net of finite rank cp contractions θ_i on A which converges to id_A pointwisely, i.e., $\lim_i\|a-\theta_i(a)\|=0$ for all $a\in A$. We often require that θ_i factors through a full matrix algebra, i.e., there are $n=n(i)\in\mathbb{N}$ and cp contractions $\sigma_i\colon A\to\mathbb{M}_n$ and $\rho_i\colon\mathbb{M}_n\to A$ such that $\rho_i\sigma_i=\theta_i$. These two CPAP's are equivalent as we will see in Theorem 1. The corresponding notion for von Neumann algebras is semidiscreteness. A von Neumann algebra M is said to be *semidiscrete* if there is a net of normal finite rank ucp (unital completely positive) maps θ_i on A which converges to id_M in the point-σ-weak topology, i.e., σ-weak-$\lim_i\theta_i(a)=a$ for all $a\in M$. We often require that θ_i factors through a full matrix algebra. Since these properties which we will deal with are all stable under unitization, we will assume that all C^*-algebras are unital from now on.

The following theorem is fundamental in the study of nuclear C^*-algebras. The part (i)$\Rightarrow$(ii) is due to [EL] and the converse (ii)$\Rightarrow$(i) is due to [CE1]. The part (i)$\Leftrightarrow$(iii) is due to [CE2] and [Ki1]. We will only prove (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i). Direct proofs of (i)$\Rightarrow$(iii) are found in [CE2] and [Ki1], the latter of which is almost same as the proof of (ii)$\Rightarrow$(iii) in Theorem 2. The implication (iv)$\Rightarrow$(ii) is due to [Co2] and [BP]. The converse implication (i)$\Rightarrow$(iv) is a deep result of Haagerup [Ha1] which uses Connes' celebrated theorem [Co1] (Theorem 2 below). We will prove later a poor man's version of this implication.

Theorem 1. *For a C^*-algebra A, the following are equivalent.*

(i). *The C^*-algebra A is nuclear.*
(ii). *The second dual A^{**} is injective.*
(iii). *The C^*-algebra A has the CPAP.*
(iv). *The Banach algebra A is amenable.*

Proof. (i)$\Rightarrow$(ii). We follow [La] for the proof. Let $A^{**} \subset \mathbb{B}(\mathcal{H})$ be a faithful normal representation. Since A is nuclear, the representation

$$\pi\colon A \otimes_{\min} A' \ni \sum_k a_k \otimes x_k \longmapsto \sum_k a_k x_k \in \mathbb{B}(\mathcal{H})$$

is continuous. Let $\Phi\colon A \otimes_{\min} \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ be a ucp extension of π and let $\varphi\colon \mathbb{B}(\mathcal{H}) \ni x \mapsto \Phi(1 \otimes x) \in \mathbb{B}(\mathcal{H})$. Since Φ is an $(A \otimes_{\min} A')$-bimodule map, φ is a conditional expectation from $\mathbb{B}(\mathcal{H})$ onto A'. This shows A', and a fortiori $A'' = A^{**}$, is injective.

(ii)$\Rightarrow$(iii). It follows from Theorem 2 below that injectivity is equivalent to semidiscreteness. We will prove that A has the CPAP provided that A^{**} is semidiscrete.

By semidiscreteness, the identity map $\mathrm{id}_{A^{**}}$ on A^{**} is approximated, in the point-weak* topology, by finite rank ucp maps which factor through full matrix algebras. Since a ucp map from a full matrix algebra $\mathbb{M}_n$ into A^{**} is approximated, in the point-weak* topology, by a ucp map from $\mathbb{M}_n$ into A (observe that a map φ from $\mathbb{M}_n$ into a C^*-algebra B is cp if and only if $[\varphi(e_{ij})]_{i,j} \in \mathbb{M}_n(B)$ is positive), the identity map id_A on A is approximated, in the point-weak topology, by finite rank ucp maps which factor through full matrix algebras. Since the point-weak closure of a convex set of bounded linear maps on a Banach space coincides with the point-norm closure, this completes the proof.

(iii)$\Rightarrow$(i). Let $\{\varphi_i\}_i$ be a net of finite rank ucp maps on A which converges to id_A pointwisely. Let B be a C^*-algebra, $Q\colon A \otimes_{\max} B \to A \otimes_{\min} B$ be the canonical quotient and take $x \in \ker Q$. We have that $(\varphi_i \otimes_{\min} \mathrm{id}_B)Q = Q(\varphi_i \otimes_{\max} \mathrm{id}_B)$ since both maps are continuous and coincide on $A \otimes_{\mathrm{alg}} B$. This implies that $Q(\varphi_i \otimes_{\max} \mathrm{id}_B)(x) = 0$. Since φ_i is of finite rank, we have $(\varphi_i \otimes_{\max} \mathrm{id}_B)(x) \in A \otimes_{\mathrm{alg}} B$. It follows that $(\varphi_i \otimes_{\max} \mathrm{id}_B)(x) = 0$. Since the ucp maps $\varphi_i \otimes_{\max} \mathrm{id}_B$ converges to $\mathrm{id}_{A \otimes_{\max} B}$, we have $x = 0$. This shows $A \otimes_{\max} B = A \otimes_{\min} B$. □

The following is a celebrated theorem of Connes [Co1]. The part (iii)⇔(ii)⇒(i) is due to [EL]. We will only prove the equivalence among (i), (ii) and (iii). See [Ha2] and [Po] for simple proofs of (i)⇒(iv).

Theorem 2. *For a von Neumann algebra $M \subset \mathbb{B}(\mathcal{H})$, the following are equivalent.*

(i). *The von Neumann algebra M is injective.*
(ii). *The representation $\pi\colon M \otimes_{\min} M' \ni \sum a_k \otimes b_k \mapsto \sum a_k b_k \in \mathbb{B}(\mathcal{H})$ is continuous.*
(iii). *The von Neumann algebra M is semidiscrete.*
(iv). *The von Neumann algebra M is AFD.*

Proof. (i)⇒(ii). We follow [Wa1] for the proof. We first assume that M is finite with a normal faithful tracial state τ and let ψ be a conditional expectation from $\mathbb{B}(\mathcal{H})$ onto M. To prove continuity of π, we may assume that $M \subset \mathbb{B}(\mathcal{H})$ is a standard representation. Then, continuity of π follows by applying Theorem 3 below to the hypertrace $\tau\psi$ for M. Now, it is not hard to show continuity of π for a semifinite injective von Neumann algebra. The general case then follows from the Takesaki duality theorem [Ta2]. See [Wa1] for the detail.

(ii)⇒(iii). We follow [Ki1] for the proof. Let Ω be the convex set of all (not necessarily contractive) cp maps θ on M of the form $\theta = \rho\sigma$ where σ is a ucp map from A into a full matrix algebra $\mathbb{M}_n$ and ρ is a (not necessarily contractive) cp map from the full matrix algebra $\mathbb{M}_n$ into M. It suffices to show that the identity map id_M is in the point-σ-weak closure of Ω since the point-σ-weak closure of a convex set coincides with the point-σ-strong closure (hence we can perturb them to unital ones). To prove it, we give ourselves normal states $f_1, \ldots, f_n$ on M, $a_1, \ldots, a_n \in M$ and $\varepsilon > 0$. We have to find $\theta \in \Omega$ with $|f_k(a_k) - f_k(\theta(a_k))| < \varepsilon$ for all $k = 1, \ldots, n$. Let $f = n^{-1}\sum_k f_k$ and $(\pi_f, \mathcal{H}_f, \xi_f)$ be the GNS triplet. It follows that there are $x_1, \ldots, x_n \in \pi_f(M)'$ such that $f_k(a) = (\pi_f(a)x_k\xi_f, \xi_f)$ for $a \in M$. Let ω be a state on $M \otimes_{\min} \pi_f(M)'$ given by $\omega(a \otimes x) = (\pi_f(a)x\xi_f, \xi_f)$. This ω is well-defined by the assumption (ii). We approximate ω by a vector state from $\mathcal{H} \otimes \mathcal{H}_f$; $\exists \eta_1, \ldots, \eta_l \in \mathcal{H} \otimes_{\mathrm{alg}} \pi_f(M)\xi_f$ with $|\omega(a_k \otimes x_k) - \sum_{j=1}^{l}((a_k \otimes x_k)\eta_j, \eta_j)| < \varepsilon$ for all $k = 1, \ldots, n$ For each j, fix a representation $\eta_j = \sum_{p=1}^{m_j} \zeta_{j,p} \otimes \pi_f(b_{j,p})\xi_f$ with $\{\zeta_{j,p}\}_{p=1}^{m_j}$ orthonormal. It follows that the map $\sigma_j\colon M \to M_{m_j}$ defined by $\sigma_j(a) = [(a\zeta_{j,q} \mid \zeta_{j,p})]_{p,q}$ is ucp and the map $\rho_j\colon M_{m_j} \to M$ defined by $\rho_j([\alpha_{pq}]_{p,q}) = \sum_{p,q} \alpha_{pq} b_{j,p}^* b_{j,q}$ is cp. Moreover, we see that $\theta = \sum_{j=1}^{l} \rho_j\sigma_j$ in Ω satisfies $(\pi_f(\theta(a))x\xi_f, \xi_f) = \sum_{j=1}^{k}((a \otimes x)\eta_j, \eta_j)$ for any $a \in A$ and $x \in M'$. Therefore, we have

$$|f_k(a_k) - f_k(\theta(a_k))| = |\omega(a_k \otimes x_k) - \sum_{j=1}^{k}((a_k \otimes x_k)\eta_j, \eta_j)| < \varepsilon$$

for all $k = 1, \ldots, n$ and we are done.

(iii)⇒(ii). See the proof of (iii)⇒(i) in Theorem 1.

(ii)⇒(i). See the proof of (i)⇒(ii) in Theorem 1. □

Let A be a C^*-subalgebra in $\mathbb{B}(\mathcal{H})$. A state φ on $\mathbb{B}(\mathcal{H})$ is called a *hypertrace* for A if it satisfies $\varphi(ax) = \varphi(xa)$ for any $a \in A$ and any $x \in \mathbb{B}(\mathcal{H})$. The following theorem of Kirchberg [Ki3] generalizes Connes' result [Co1] on II_1-factors.

Theorem 3. *For a tracial state τ on a C^*-subalgebra A in $\mathbb{B}(\mathcal{H})$, TFAE.*

(i). *The trace τ extends to a hypertrace φ on $\mathbb{B}(\mathcal{H})$.*

(ii). *There is a net of ucp maps $\theta_i\colon A \to \mathbb{M}_{n(i)}$ such that $\tau(a) = \lim_i \mathrm{tr}_{n(i)}(\theta_i(a))$ and $\lim_i \mathrm{tr}_{n(i)}(\theta_i(ab^*) - \theta_i(a)\theta_i(b^*)) = 0$ for any a, b in A.*

(iii). *The functional $\sigma\colon A \otimes_{\min} \overline{A} \ni \sum_k a_k \otimes \overline{b_k} \mapsto \tau(\sum_k a_k b_k^*) \in \mathbb{C}$ is continuous.*

(iv). *The representation $\pi\colon A \otimes_{\min} \overline{A} \ni \sum_k a_k \otimes \overline{b_k} \mapsto \sum_k \pi_\tau(a_k)\pi_\tau^c(\overline{b_k}) \in \mathbb{B}(\mathcal{H}_\tau)$ is continuous, where $(\pi_\tau, \mathcal{H}_\tau, \xi_\tau)$ is the GNS-triplet for τ and the representation $\pi_\tau^c\colon \overline{A} \to \mathbb{B}(\mathcal{H})$ is given by $\pi_\tau^c(\overline{b})\pi_\tau(a)\xi_\tau = \pi_\tau(ab^*)\xi_\tau$ for $a \in A$ and $\overline{b} \in \overline{A}$.*

Proof. (i)$\Rightarrow$(ii). The following proof is taken from [Ha2]. To prove (ii), we give ourselves a finite set E of unitaries in A and $\varepsilon > 0$. We approximate φ by $\mathrm{Tr}(h\,\cdot)$, where h is a positive trace class operator on $\mathcal{H}$ with $\mathrm{Tr}(h) = 1$. By a standard approximation argument, we may assume that we have $|\mathrm{Tr}(hu) - \tau(u)| < \varepsilon$ and $\|h - uhu^*\|_{1,\mathrm{Tr}} < \varepsilon$ for $u \in E$, and that h is of finite rank and has no irrational eigenvalues; let $p_1/q, \ldots, p_m/q$ ($p_1, \ldots, p_m, q \in \mathbb{N}$) be the non-zero eigenvalues of h with the corresponding eigenvectors $\zeta_1, \ldots, \zeta_m \in \mathcal{H}$.

We denote by $\mathbb{C}^d$ the d-dimensional Hilbert space with a distinguished basis $\{\delta_i\}_{i=1}^d$. Put $p = \max\{p_1, \ldots, p_m\}$ and define isometries $V_k\colon \mathbb{C}^{p_k} \to \mathcal{H} \otimes \mathbb{C}^p$ by $V_k\delta_i = \zeta_k \otimes \delta_i$. Finally let $V\colon \bigoplus_{k=1}^m \mathbb{C}^{p_k} \to \mathcal{H} \otimes \mathbb{C}^p$ be the concatenation of V_k's. Identifying $\mathbb{M}_q$ with $\mathbb{B}(\bigoplus_{k=1}^m \mathbb{C}^{p_k})$, we define a ucp map $\theta\colon A \to \mathbb{M}_q$ by $\theta(a) = V^*(a \otimes 1)V$. It follows that we have $\mathrm{tr}_q(\theta(a)) = \mathrm{Tr}(ha)$ for any $a \in A$, and denoting $u_{k,l} = (u\zeta_l \mid \zeta_k)$, we have

$$
\begin{aligned}
\mathrm{tr}_q(\theta(uu^*) - \theta(u)\theta(u^*)) &= \sum\nolimits_{k,l} |u_{k,l}|^2 (p_k - \min\{p_k, p_l\})/q \\
&\le \Big(\sum_{k,l} |u_{k,l}|^2 (p_k^{1/2} + p_l^{1/2})^2/q\Big)^{1/2} \Big(\sum_{k,l} |u_{k,l}|^2 (p_k^{1/2} - p_l^{1/2})^2/q\Big)^{1/2} \\
&= \|h^{1/2}u + uh^{1/2}\|_{2,\mathrm{Tr}} \|h^{1/2}u - uh^{1/2}\|_{2,\mathrm{Tr}} \\
&\le 2\|hu - uh\|_{1,\mathrm{Tr}} < 2\varepsilon
\end{aligned}
$$

for $u \in E$, where we have used the Powers-Størmer inequality [PS] in the last line.

(ii)$\Rightarrow$(iii). The net of states $\sigma_i\colon A \otimes_{\min} \overline{A} \ni \sum_k a_k \otimes \overline{b_k} \mapsto \mathrm{tr}_{n(i)}(\sum_k \theta_i(a_k)\theta_i(b_k^*)) \in \mathbb{C}$ is well-defined and converges to the functional σ.

(iii)$\Rightarrow$(iv). This immediately follows from the fact that ξ_τ is cyclic for $\pi(A \otimes_{\mathrm{alg}} \overline{A})$ and the corresponding vector state (which is σ) is continuous on $A \otimes_{\min} \overline{A}$.

(iv)$\Rightarrow$(i). Let $\Psi\colon \mathbb{B}(\mathcal{H}) \otimes_{\min} \overline{A} \to \mathbb{B}(\mathcal{H}_\tau)$ be a ucp extension of π and let $\psi\colon \mathbb{B}(\mathcal{H}) \ni x \mapsto \Psi(x \otimes 1) \in \mathbb{B}(\mathcal{H}_\tau)$. Since Ψ is an $(A \otimes_{\min} \overline{A})$-bimodule map, we have that $\psi|_A = \pi_\tau$ and $\psi(\mathbb{B}(\mathcal{H})) \subset \pi_\tau(A)''$. It follows that the desired hypertrace extension φ of τ is given by $\varphi(x) = (\psi(x)\xi_\tau \mid \xi_\tau)$. □

2. A POOR MAN'S 'NUCLEARITY⇔AMENABILITY'

We prove a version of 'nuclearity⇔amenability'. (Sadly, it is still complicated.) Let A be a C^*-algebra. The Haagerup norm for $T \in A \otimes_{\mathrm{alg}} A$ is defined as

$$\|T\|_h = \inf\{\|\sum a_k a_k^*\|^{1/2}\|\sum b_k^* b_k\|^{1/2} : T = \sum_{k=1}^{n} a_k \otimes b_k\}$$

Regarding $\sum a_k b_k$ as a product of $[a_1, \dots, a_n] \in \mathbb{M}_{1,n}(A)$ and $[b_1, \dots, b_n]^T \in \mathbb{M}_{n,1}(A)$, we see that the product map $p\colon A \otimes_{\mathrm{alg}} A \ni \sum_k a_k \otimes b_k \mapsto \sum_k a_k b_k \in A$ is contractive w.r.t. the Haagerup norm. If $A \subset \mathbb{B}(\mathcal{H})$ and $T = \sum_k a_k \otimes b_k \in A \otimes_{\mathrm{alg}} A$, then we put $\Phi_T\colon \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ by $\Phi_T(x) = \sum_k a_k x b_k$. Again, it can be seen that $\|\Phi_T\|_{\mathrm{cb}} \le \|T\|_h$. We say a unital C^*-algebra A is *amenable w.r.t. the Haagerup tensor product* if there is a net $\{T_i\}_{i\in I}$ in $A \otimes_{\mathrm{alg}} A$ satisfying that

(i). $\sup_i \|T_i\|_h < \infty$,
(ii). $p(T_i) = 1$ for all $i \in I$,
(iii). $\lim_i \|x \cdot T_i - T_i \cdot x\|_h = 0$ for all $x \in A$.

It is clear that amenability w.r.t. the Haagerup tensor product is formally weaker than the usual amenability. There are natural equivalent definitions of this concept via virtual diagonal and via cohomology. Consult [Ru] for this matter.

Theorem 4. *A C^*-algebra is nuclear if and only if it is amenable w.r.t. the Haagerup tensor product.*

Passing from this theorem to Theorem 1 seems require a serious tool such as non-commutative Grothendieck inequality. Consult [Ef] and [Ru].

We first show the 'if' part. The following proof is taken from [BP]. Take a faithful normal representation $A^{**} \subset \mathbb{B}(\mathcal{H})$ and let $\{T_i\}_{i\in I}$ in $A \otimes_{\mathrm{alg}} A$ be as above. Then, the point-weak* cluster point of the net $\{\Phi_{T_i}\}$ is a quasi-expectation from $\mathbb{B}(\mathcal{H})$ onto A'. It follows that A', and a fortiori $A'' = A^{**}$, is injective. See [BP] [Ru] for the proof.

For the proof of the 'only if' part, we need the following ingredients; a theorem of Kirchberg [Ki2] saying that a separable nuclear C^*-algebra is a subquotient of the CAR-algebra (see [Wa2] for a simple proof), and Kasparov's Stinespring theorem [Ka]. The following result, inspired from [KS], is sufficient for the 'only if' part.

Lemma 5. *Let A be a unital nuclear C^*-algebra, F be a finite set of unitaries in A, which are in the connected component of the identity and let $\varepsilon > 0$. Then, there are $n \in \mathbb{N}$ and a finite subset G in $\mathbb{M}_{1,n}(A)$ such that $xx^* = 1$ for $x \in G$ and for any $u \in F$, there is a bijection f of G onto G with $\|ux - f(x)\| < \varepsilon$ for $x \in G$.*

Here, for $x = [x_1, \dots, x_n] \in \mathbb{M}_{1,n}(A)$ and $u \in A$, we define $xx^* = \sum x_k x_k^*$ and $ux = [ux_1, \dots, ux_n] \in \mathbb{M}_{1,n}(A)$.

Proof. We may assume that A is separable. By Kirchberg's theorem [Ki2], there is a ucp map φ from the CAR-algebra B onto A, whose restriction $\varphi|_{\tilde{A}}$ to some unital C^*-subalgebra $\tilde{A}$ in B becomes a surjective $*$-homomorphism onto A.

We give ourselves a finite set F of unitaries in A, which are in the connected component of the identity, and $\varepsilon > 0$. Lifting each element in F, we find a finite set $\tilde{F}$ of unitaries in $\tilde{A}$. Since B is AF, there is a finite set $\tilde{G}$ of unitaries in B such that for any $\tilde{u} \in \tilde{F}$, there is a bijection $\tilde{f}$ of $\tilde{G}$ onto $\tilde{G}$ with $\|\tilde{u}w - \tilde{f}(w)\| < \varepsilon/2$ for $w \in \tilde{G}$. By Kasparov's Stinespring theorem [Ka], there is a unital representation π of B on the Hilbert A-module $\mathcal{H}_A = \ell_2 \otimes A$ such that $\varphi(b) = \langle \zeta, \pi(b)\zeta \rangle$ for $b \in B$, where $\zeta = (1, 0, 0, \dots) \in \mathcal{H}_A$. We observe that $\pi(\tilde{u}^*)\zeta = (u^*, 0, 0, \dots) = \zeta u^*$ for $\tilde{u} \in \tilde{F}$ since $\varphi(\tilde{u}^*) = u^*$ is a unitary and $\|\pi(\tilde{u}^*)\| = 1$. For each $w \in \tilde{G}$, we define $w_1, w_2, \dots \in A$ by $\pi(w^*)\zeta = (w_1^*, w_2^*, \dots) \in \mathcal{H}_A$. It follows that $\pi((\tilde{u}w)^*)\zeta = ((uw_1)^*, (uw_2)^*, \dots)$. Take $n \in \mathbb{N}$ to be large enough so that $\| \sum_{k \geq n} w_k w_k^* \| < \varepsilon/8$ for all $w \in \tilde{G}$ and put $\widehat{w} = [w_1, \dots, w_{n-1}, (\sum_{k \geq n} w_k w_k^*)^{1/2}] \in \mathbb{M}_{1,n}(A)$. The set $G = \{\widehat{w} : w \in G\} \subset \mathbb{M}_{1,n}(A)$ and the bijection f on G induced from $\tilde{f}$ are what we desired. □

3. A non-operator algebraist's non-amenability of $\mathbb{B}(\ell_2)$

We present here a proof of the fact that $\mathbb{B}(\ell_2)$ (or any von Neumann algebra which is not subhomogeneous) is not amenable. This proof was suggested by G. Pisier.

Theorem 6. *The Banach algebra $\mathbb{B}(\ell_2)$ is not amenable.*

Actually we will show that there is no net $\{T_i\}_i$ in $\mathbb{B}(\ell_2) \otimes_{\text{alg}} \mathbb{B}(\ell_2)$ satisfying the condition (ii) and (iii) in Section 2. In stead of operator algebra theory, we need the following ingredients; Kazhdan's property (T) for, say, $SL(3, \mathbb{Z})$ and operator inequalities. A discrete group Γ is said to have *Kazhdan's property* (T) if for any finite subset E of generators in Γ, there are a constant $\kappa > 0$ and a decreasing function $f \colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$ such that the following is true: if π is a unitary representation on a Hilbert space $\mathcal{H}$ and $\xi \in \mathcal{H}$ is a unit vector with $\varepsilon = \max_{s \in E} \|\pi(s)\xi - \xi\| < \kappa$, then there is a unit vector $\eta \in \mathcal{H}$ with $\|\xi - \eta\| < f(\varepsilon)$ such that $\pi(s)\eta = \eta$ for all $s \in \Gamma$. It is well-known that the group $SL(3, \mathbb{Z})$ has Kazhdan's property (T). We refer the reader to [HV] for the information of Kazhdan's property (T). For any trace class operators h and k on a Hilbert space, the Powers-Størmer inequality [PS] says that $\|h^{1/2} - k^{1/2}\|_2^2 \leq \|h - k\|_1$ and Kosaki's inequality [Ko] says that $\| \, |h| - |k| \, \|_1 \leq (2\|h + k\|_1 \|h - k\|_1)^{1/2}$. The proof of these inequalities for matrices (which we will need) are rather elementary (cf. [Mc]).

Lemma 7. *Let E be a finite set of generators of Γ. Then, there are a constant $\delta > 0$ and a decreasing function $c \colon (0, \delta) \to (0, 1)$ with $\lim_{\varepsilon \to 0} c(\varepsilon) = 0$ which satisfy the following property. If $\pi \colon \Gamma \to \mathbb{M}_n$ is an irreducible representation and*

$$T = \sum_{i=1}^{r} a_i \otimes b_i \in \mathbb{M}_{n,\infty} \otimes \mathbb{M}_{\infty,n}$$

is such that $\sum_{i=1}^{r} a_i b_i = I_n$ and $\varepsilon = \max_{s \in E} \|\pi(s) \cdot T - T \cdot \pi(s)\|_h < \delta$, then we have $r \geq (1 - c(\varepsilon))n$.

Proof. Let $\{\delta_k\}_{k=1}^{\infty}$ be an orthonormal basis of ℓ_2 and $\{e_k\}$ be the corresponding orthogonal projections. Let $T(k) = \sum_{i=1}^{r} a_i e_k b_i \in \mathbb{M}_n$. Since $\sum_{i=1}^{r} a_i b_i = I_n$, we have that $\sum_{k=1}^{\infty} \|T(k)\|_{1,\mathrm{Tr}} \geq n$. On the other hand, we claim that

$$\forall s \in E \quad \sum_{k=1}^{\infty} \|\pi(s)T(k)\pi(s)^* - T(k)\|_{1,\mathrm{Tr}} < \varepsilon n.$$

Indeed, this follows from a standard homogeneity trick and the following inequality; for $x \in \mathbb{M}_{n,\infty}$ and $y \in \mathbb{M}_{\infty,n}$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} \|xe_k y\|_{1,\mathrm{Tr}} &= \sum_{k=1}^{\infty} \|x\delta_k\|_{\ell_2} \|y^*\delta_k\|_{\ell_2} \\ &\leq \frac{1}{2}\sum_{k=1}^{\infty}(\|x\delta_k\|_{\ell_2}^2 + \|y^*\delta_k\|_{\ell_2}^2) = \frac{1}{2}\mathrm{Tr}_n(xx^* + y^*y).\end{aligned}$$

These inequalities implies the existence of $k_0 \in \mathbb{N}$ such that

$$\forall s \in E \quad \|\pi(s)T(k_0)\pi(s)^* - T(p_0)\|_{1,\mathrm{Tr}} < |E|\,\varepsilon\, \|T(k_0)\|_{1,\mathrm{Tr}}.$$

We put $h = T(k_0)/\|T(k_0)\|_{1,\mathrm{Tr}}$. It follows from Kosaki's inequality that

$$\|\pi(s)|h|\pi(s)^* - |h|\,\|_{1,\mathrm{Tr}} \leq (2\|\pi(s)h\pi(s)^* + h\|_{1,\mathrm{Tr}}\|\pi(s)h\pi(s)^* - h\|_{1,\mathrm{Tr}})^{1/2} < (4|E|\,\varepsilon)^{1/2}.$$

Combined with Powers-Størmer inequality, this implies

$$\forall s \in E \quad \|\pi(s)|h|^{1/2}\pi(s)^* - |h|^{1/2}\|_{2,\mathrm{Tr}} < (4|E|\,\varepsilon)^{1/4}.$$

If $\varepsilon > 0$ is small enough, then it follows Schur's lemma that for $c(\varepsilon) = f((4|E|\,\varepsilon)^{1/4})$ (here f is as in the above definition of Kazhdan's property (T)), we have

$$\|\,|h|^{1/2} - n^{-1/2}I_n\|_{2,\mathrm{Tr}}^2 < c(\varepsilon)$$

since fixed vectors for the representation $\mathrm{Ad}\,\pi$ of Γ on the Hilbert-Schmidt class S_2 is a multiple of identity. Since $\mathrm{rank}\,|h|^{1/2} = \mathrm{rank}\,h \leq r$, we have $r \geq (1 - c(\varepsilon))n$. This completes the proof. □

Proof of Theorem 6. Let $\pi_k\colon SL(3,Z) \to \mathbb{M}_{n(k)}$ be a sequence of finite dimensional irreducible representations such that $n(k) \to \infty$ as $k \to \infty$. We identify ℓ_2 with $\bigoplus_{k=1}^{\infty} \ell_2^{n(k)}$ and denote by P_k the orthogonal projection from ℓ_2 onto $\ell_2^{n(k)}$. Let $\pi(s) = \bigoplus_{s=1}^{\infty} \pi_n(s) \in \mathbb{B}(\ell_2)$ for $s \in SL(3,\mathbb{Z})$ and let $\delta > 0$ be as in Lemma 7. To show $\mathbb{B}(\ell_2)$ is not amenable by reductio ad absurdum, suppose that there is $T = \sum_{i=1}^{r} a_i \otimes b_i \in \mathbb{B}(\ell_2) \otimes \mathbb{B}(\ell_2)$ such that $\sum_{i=1}^{r} a_i b_i = I_{\ell_2}$ and $\varepsilon = \max_{s\in E} \|\pi(s)\cdot T - T\cdot\pi(s)\|_h < \delta$. Applying Lemma 7 to $P_k \cdot T \cdot P_k \in \mathbb{M}_{n(k),\infty} \otimes \mathbb{M}_{\infty,n(k)}$, we obtain $r > (1 - c(\varepsilon))n(k)$ for all k. This is absurd. □

4. Exactness

A C^*-algebra A is said to be *exact* if

$$0 \to A \otimes_{\min} J \to A \otimes_{\min} B \to A \otimes_{\min} B/J \to 0$$

is exact for any C^*-algebra B and any closed 2-sided ideal J in B. As we will see in the proof of Theorem 8, it suffices to check exactness of the above sequence only for either $J = \bigoplus_{n=1}^{\infty} \mathbb{M}_n \lhd B = \prod_{n=1}^{\infty} \mathbb{M}_n$ or $J = \mathbb{K}(\ell_2) \lhd B = \mathbb{B}(\ell_2)$. Kirchberg [Ki2] showed that exactness is characterized by the following property, known as *nuclear embeddability.*

Theorem 8. *Let $A \subset \mathbb{B}(\mathcal{H})$ be an exact C^*-algebra and let $(P_n)_{n=1}^{\infty}$ be an increasing sequence of projections on $\mathcal{H}$, which converges strongly to the identity on $\mathcal{H}$. We denote by $\varphi_n \colon \mathbb{B}(\mathcal{H}) \to \mathbb{B}(P_n\mathcal{H})$ the compression. Then, there is a net of ucp maps $\theta_i \colon \mathbb{B}(P_{n(i)}\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ such that $\lim_i \|\theta_i\varphi_{n(i)}(a) - a\| = 0$ for all $a \in A$.*

Proof. The following proof is taken from [Pi]. Given a finite dimensional subspace $E \subset A$ and $\varepsilon > 0$, we will find $n \in \mathbb{N}$ and a ucp map $\theta \colon \mathbb{B}(P_n\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ such that $\|\theta\varphi_n|_E - \mathrm{id}_E\|_{\mathrm{cb}} < \varepsilon$. It is easy to see that $\varphi_n|_E$ is one-to-one for n large enough. We claim that $\lim_{n\to\infty} \|(\varphi_n|_E)^{-1} \colon \varphi_n(E) \to E\|_{\mathrm{cb}} = 1$. Since φ_n factors through φ_m when $m > n$, the sequence $\|(\varphi_n|_E)^{-1}\|_{\mathrm{cb}}$ is monotonically decreasing and the limit $c \geq 1$ exists. For each n, we take $x_n \in E \otimes \mathbb{M}_{k(n)}$ with $\|x_n\| = 1$ and $\|(\varphi_n \otimes \mathrm{id}_{k(n)})(x_n)\| \leq c^{-1} + n^{-1}$, and let $x = (x_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty}(E \otimes \mathbb{M}_{k(n)}) = E \otimes M \subset A \otimes M$, where $M = \prod_{n=1}^{\infty} \mathbb{M}_{k(n)}$ is the ℓ_∞-direct product. (That $\prod_{n=1}^{\infty}(E \otimes \mathbb{M}_{k(n)}) = E \otimes M$ is because E is finite dimensional.) Let $J = \bigoplus_{n=1}^{\infty} \mathbb{M}_{k(n)}$ be the c_0-direct product, which is an ideal in M. Since $\|x_n\| = 1$ for all n, we have $\|x + A \otimes J\|_{A\otimes M/A\otimes J} = 1$. On the other hand, if M/J is faithfully represented on $\mathcal{K}$, then we have

$$\begin{aligned}
\|(\mathrm{id}_A \otimes Q)(x)\|_{A\otimes M/J} &= \|(\mathrm{id}_A \otimes Q)(x)\|_{\mathbb{B}(\mathcal{H}\otimes\mathcal{K})} \\
&= \lim_{n\to\infty} \|(\varphi_n \otimes \mathrm{id}_{M/J})(\mathrm{id}_A \otimes Q)(x)\|_{\mathbb{B}(P_n\mathcal{H}\otimes\mathcal{K})} \\
&= \lim_{n\to\infty} \|(\mathrm{id}_{\mathbb{B}(P_n\mathcal{H})} \otimes Q)((\varphi_n \otimes \mathrm{id}_M)(x))\|_{\mathbb{B}(P_n\mathcal{H}\otimes\mathcal{K})} \\
&\leq \lim_{n\to\infty} \limsup_{m\to\infty} \|(\varphi_n \otimes \mathrm{id}_{\mathbb{M}_{k(m)}})(x_m)\|_{\mathbb{B}(P_n\mathcal{H})\otimes\mathbb{M}_{k(m)}} \\
&\leq \limsup_{m\to\infty} \|(\varphi_m \otimes \mathrm{id}_{\mathbb{M}_{k(m)}})(x_m)\|_{\mathbb{B}(P_m\mathcal{H})\otimes\mathbb{M}_{k(m)}} \\
&\leq c^{-1}
\end{aligned}$$

Hence, by the assumption on exactness, we have $1 \leq c^{-1}$ and obtain the claim.

It follows that $\|(\varphi_n|_E)^{-1}\|_{\mathrm{cb}} \leq 1 + \varepsilon$ for sufficiently large n. Using the injectivity of $\mathbb{B}(\mathcal{H})$, we extend $(\varphi_n|_E)^{-1}$ to a self-adjoint map $\psi \colon \mathbb{B}(P_n\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ with the same cb-norm. Since ψ is unital self-adjoint, we find a ucp $\theta \colon \mathbb{B}(P_n\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ with $\|\theta - \psi\|_{\mathrm{cb}} < \varepsilon$ by Lemma 9 below. This completes the proof. □

Lemma 9 (Kirchberg). *If $\psi \colon A \to \mathbb{B}(\mathcal{H})$ is a unital self-adjoint map, then there is a ucp map $\theta \colon A \to \mathbb{B}(\mathcal{H})$ such that $\|\theta - \psi\|_{\mathrm{cb}} \leq \|\psi\|_{\mathrm{cb}} - 1$.*

Proof. Use Wittstock decomposition theorem or Stinespring theorem for cb maps, or consult Wassermann's monograph [Wa2]. □

The converse of the above theorem due to Wassermann states that

Theorem 10. *Let $A \subset \mathbb{B}(\mathcal{H})$ be a separable C^*-algebra such that there is a sequence of finite rank maps $\varphi_n \colon A \to \mathbb{B}(\mathcal{H})$ such that*

$$\lim_n \|(\varphi_n \otimes \mathrm{id}_{\mathbb{B}(\ell_2)})(x) - x\|_{\mathbb{B}(\mathcal{H})\otimes\mathbb{B}(\ell_2)} = 0$$

for any $x \in A \otimes \mathbb{B}(\ell_2)$, (e.g. a sequence of finite rank ucp maps $\varphi_n \colon A \to \mathbb{B}(\mathcal{H})$ such that $\forall a \in A$ $\lim_n \|\varphi_n(a) - a\| = 0$) then A is exact.

Proof. We give ourselves a C^*-algebra B, an ideal $J \lhd B$ and $x \in \ker(\mathrm{id}_A \otimes Q)$. We may assume that $B \subset \mathbb{B}(\ell_2)$. It follows that

$$(\mathrm{id}_{\mathbb{B}(\mathcal{H})} \otimes Q)(\varphi_n \otimes \mathrm{id}_{\mathbb{B}(\ell_2)})(x) = (\varphi_n \otimes \mathrm{id}_{\mathbb{B}(\ell_2)})(\mathrm{id}_A \otimes Q)(x) = 0.$$

Since φ_n is of finite rank, we have $(\varphi_n \otimes \mathrm{id}_{\mathbb{B}(\ell_2)})(x) \in A \otimes J$. Taking limit, we have $x \in A \otimes J$. This completes the proof. □

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