数理解析研究所講究録 1293 巻 2002 年 98-109

Riesz decomposition and limits at infinity for p-precise functions on a half space

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1 Introduction

Let u be a nonnegative superharmonic function on $D=\{x=(x_1,\ldots,x_{n-1},x_n)\in\mathbf{R}^n;x_n>0\}$, where $n\geqq 2$. Then it is known (cf. Lelong-Ferrand [6]) that u is uniquely decomposed as

$$u(x)=ax_n+\int_D G(x,y)d\mu(y)+\int_{\partial D}P(x,y)d\nu(y),$$

where a is a nonnegative number, μ (resp. ν) is a nonnegative measure on D (resp. ∂D), G is the Green function for D and P is the Poisson kernel for D. The first author showed in [9] that if $0\leqq\beta\leqq 1$, $1-n\leqq\gamma<1$ and $\int_D y_n^\gamma d\mu(y)+\int_{\partial D}|y|^{\gamma-1}d\nu(y)<\infty$, then

$$\lim_{|x|\to\infty,x\in D-E'} x_n^{-\beta}|x|^{n+\gamma-2+\beta}[u(x)-ax_n]=0$$

with a suitable exceptional set $E'\subset D$. For related results, we also refer the reader to Essén-Jackson [3, Theorem 4.6], Aikawa [1] and Miyamoto-Yoshida [8].

Our main aim in this paper is to establish the analogue of these results for locally p-precise functions u in D satisfying

$$\int_D|\nabla u(x)|^p x_n^\gamma dx<\infty, \tag{1}$$

where ∇ denotes the gradient, $1<p<\infty$ and $-1<\gamma<p-1$ (see Ohtsuka [15] and Ziemer [17] for locally p-precise functions).

2 Fine limits at infinity

Denote by $\mathbf{D}^{p,\gamma}$ the space of all locally p-precise functions on D satisfying (1). Consider the kernel function

$$K_\gamma(x,y)=|x-y|^{1-n}y_n^{-\gamma/p}.$$

To evaluate the size of exceptional sets, we use the capacity

$$C_{K_\gamma,p}(E;G)=\inf\int_D g(y)^p\,dy,$$

where E is a subset of an open set G in D and the infimum is taken over all nonnegative measurable functions g such that $g = 0$ outside G and

$$\int_D K_\gamma(x,y)g(y)dy \geqq 1 \qquad \text{for all } x \in E.$$

We say that $E \subset D$ is (K_γ, p)-thin at infinity if

$$\sum_{i=1}^{\infty} 2^{-i(n+\gamma-p)} C_{K_\gamma,p}(E_i; D_i) < \infty, \tag{2}$$

where $E_i = \{x \in E : 2^i \leqq |x| < 2^{i+1}\}$ and $D_i = \{x \in D : 2^{i-1} < |x| < 2^{i+2}\}$.

Our first aim in this paper is to establish the following theorem.

THEOREM 1 (cf. [4]). *Let $p > 1$, $-1 < \gamma < p-1$ and $n+\gamma-p \geqq 0$. If $u \in \mathbf{D}^{p,\gamma}$, then there exist a set $E \subset D$ and a number A such that E is (K_γ, p)-thin at infinity;*

$$\lim_{|x|\to\infty, x\in D-E} |x|^{(n+\gamma-p)/p}[u(x) - A] = 0$$

in case $n+\gamma-p > 0$ and

$$\lim_{|x|\to\infty, x\in D-E} (\log|x|)^{-1/p'}[u(x) - A] = 0$$

in case $n+\gamma-p = 0$, where $p' = p/(p-1)$.

In fact, if $1 \leqq q < p$ and $q < p/(1+\gamma)$, then Hölder's inequality gives

$$\int_G |\nabla u(x)|^q \, dx \;\leqq\; \left(\int_G x_n^{-\gamma q/(p-q)} \, dx\right)^{1-q/p} \left(\int_G |\nabla u(x)|^p x_n^\gamma \, dx\right)^{q/p} < \infty$$

for every bounded open set $G \subset D$. Hence we can find a locally q-precise extension $\overline{u}$ to $\mathbf{R}^n$ such that $\overline{u}(x', x_n) = u(x', x_n)$ for $x_n > 0$ and $\overline{u}(x', x_n) = u(x', -x_n)$ for $x_n < 0$. We denote by $B(x,r)$ the open ball centered at x with radius $r > 0$. In view of [13], we can find a number a such that

$$\overline{u}(x) = c_n \sum_{i=1}^{n} \int_{B(0,1)} \frac{x_i - y_i}{|x-y|^n} \frac{\partial \overline{u}}{\partial y_i}(y)dy + c_n \sum_{i=1}^{n} \int_{\mathbf{R}^n - B(0,1)} \left(\frac{x_i - y_i}{|x-y|^n} - \frac{-y_i}{|y|^n}\right) \frac{\partial \overline{u}}{\partial y_i}(y) \, dy + a$$

for almost every $x \in \mathbf{R}^n$. Here we see that the equality holds for every $x \in D$ except that in a set of $C_{K_\gamma,p}$-capacity zero. Now Theorem 1 is a consequence of [4].

3 Riesz decomposition

We denote by $\mathbf{D}_0^{p,\gamma}$ the space of all functions $u \in \mathbf{D}^{p,\gamma}$ having vertical limit zero at almost every boundary point of D, and by $\mathbf{HD}^{p,\gamma}$ the space of all harmonic functions on D in $\mathbf{D}^{p,\gamma}$. As in Deny-Lions [2], we have the following Riesz decomposition of $u \in \mathbf{D}^{p,\gamma}$.

THEOREM 2. *A function $u \in \mathbf{D}^{p,\gamma}$ is uniquely represented as*

$$u = u_0 + h, \tag{3}$$

where $u_0 \in \mathbf{D}_0^{p,\gamma}$ and $h \in \mathbf{HD}^{p,\gamma}$. More precisely, for fixed $\xi \in D$,

$$\begin{aligned} u_0(x) &= c_n \sum_{i=1}^{n} \int_D \left(\frac{x_i - y_i}{|x-y|^n} - \frac{\bar{x}_i - y_i}{|\bar{x}-y|^n} \right) \frac{\partial u}{\partial y_i}(y)\, dy, \\ h(x) &= 2c_n \sum_{i=1}^{n} \int_D \left(\frac{\bar{x}_i - y_i}{|\bar{x}-y|^n} - \frac{\bar{\xi}_i - y_i}{|\bar{\xi}-y|^n} \right) \frac{\partial u}{\partial y_i}(y)\, dy + A, \end{aligned}$$

where $\bar{x} = (x_1, ..., x_{n-1}, -x_n)$ for $x = (x_1, ..., x_{n-1}, x_n)$, $c_n = \Gamma(n/2)/(2\pi^{n/2})$ and A is a constant depending on u and ξ.

As applications we are concerned with the limits at infinity of functions in $\mathbf{D}_0^{p,\gamma}$ and $\mathbf{HD}^{p,\gamma}$.

Consider the kernel function

$$k_{\beta,\gamma}(x,y) = x_n^{1-\beta} y_n^{-\gamma/p} |x-y|^{1-n} |\bar{x}-y|^{-1}$$

for x and y in D. To evaluate the size of exceptional sets, we use the capacity

$$C_{k_{\beta,\gamma},p}(E;G) = \inf \int_D g(y)^p\, dy,$$

where E is a subset of an open set G in D and the infimum is taken over all nonnegative measurable functions g such that $g = 0$ outside G and

$$\int_D k_{\beta,\gamma}(x,y) g(y) dy \geqq 1 \qquad \text{for all } x \in E.$$

We say that $E \subset D$ is $(k_{\beta,\gamma}, p)$-thin at infinity if

$$\sum_{i=1}^{\infty} 2^{-i(n+\gamma-(1-\beta)p)} C_{k_{\beta,\gamma},p}(E_i; D_i) < \infty. \tag{4}$$

THEOREM 3. *Let $p > 1$, $-1 < \gamma < p-1$ and $0 \leqq \beta \leqq 1$. If $u \in \mathbf{D}_0^{p,\gamma}$, then there exists a set $E \subset D$ such that E is $(k_{\beta,\gamma}, p)$-thin at infinity and*

$$\lim_{|x| \to \infty, x \in D-E} x_n^{-\beta} |x|^{(n+\gamma-(1-\beta)p)/p} u(x) = 0.$$

THEOREM 4. *Let $p > 1$, $-1 < \gamma < p-1$ and $n + \gamma - p \geqq 0$. If $h \in \mathbf{HD}^{p,\gamma}$, then there exist a number A such that*

$$\lim_{|x| \to \infty, x \in D} x_n^{(n+\gamma-p)/p} [h(x) - A] = 0$$

in case $n+\gamma-p>0$ *and*

$$\lim_{|x|\to\infty, x\in D} (\max\{\log(1/x_n), \log|x|\})^{-1/p'} [h(x)-A] = 0$$

in case $n+\gamma-p=0$.

REMARK 1. *Let* $p>1$, $-1<\gamma<p-1$ *and* $n+\gamma-p>0$. *Then we can find a function* $h \in \mathbf{HD}^{p,\gamma}$ *such that*

$$\limsup_{|x|\to\infty, x\in D} |x|^{(n+\gamma-p)/p} h(x) = \infty$$

and

$$\lim_{|x|\to\infty, x\in D} x_n^{(n+\gamma-p)/p} h(x) = 0.$$

For proofs of these theorems, we refer to [14].

4 Examples of thin sets at infinity

We are concerned with the measure condition on sets which are thin at infinity.

For a measurable set $E \subset \mathbf{R}^n$, denote by $|E|$ the Lebesgue measure of E. Then we can prove that

$$|E|^{(1-(1-\beta)/n)p} \leqq M C_{k_{\beta,\gamma},p}(E; D_0) \tag{5}$$

and

$$C_{k_{\beta,\gamma},p}(rE; rD_0) = r^{n+\gamma-(1-\beta)p} C_{k_{\beta,\gamma},p}(E; D_0) \tag{6}$$

whenever $E \subset D \cap B(0,2) - B(0,1)$ and $r>0$. Hence we have the following result.

PROPOSITION 1. *Let* $0 \leqq \beta \leqq 1$ *and* $-1<\gamma<p-1$. *If* (4) *holds, then*

$$\sum_{i=1}^{\infty} \left(\frac{|E_i|}{|B_i|}\right)^{(1-(1-\beta)/n)p} < \infty,$$

where $E_i = E \cap B_{i+1} - B_i$ *with* $B_i = B(0,2^i) \cap D$.

If E is well situated, then we have stronger results as in the following.

PROPOSITION 2. *Let* $0 \leqq \beta \leqq 1$ *and* $-1<\gamma<p-1$. *Set* $F = \cup_{j=1}^{\infty} B_j$, *where* $B_j = B(x_j, s_j)$ *with* $2^j \leqq |x_j| < 2^{j+1}$ *and* $r_j = (x_j)_n > 2s_j$. *If* $p<n$ *and* F *is* $(k_{\beta,\gamma},p)$*-thin at infinity, then*

$$\sum_{j=1}^{\infty} \left(\frac{s_j}{2^j}\right)^{n-p} \left(\frac{r_j}{2^j}\right)^{\beta p+\gamma} < \infty; \tag{7}$$

conversely, if (7) *holds, then* F *is* $(k_{\beta,\gamma}, p)$*-thin at infinity.*

PROOF. First we show that if $p < n$, then

$$s^{n-p} r^{\beta p+\gamma} \leqq M C_{k_{\beta,\gamma},p}(B; D_0) \tag{8}$$

for $B = B(x_0, s)$ with $1 \leqq |x_0| < 2$ and $r = (x_0)_n > 2s$. Let g be a nonnegative measurable function such that $g = 0$ outside D_0 and

$$\int_D k_{\beta,\gamma}(x,y) g(y)\, dy \geqq 1$$

for every $x \in B$. Then we have by Fubini's theorem

$$\begin{aligned}
|B| &\leqq \int_B \left(\int_{D_0} k_{\beta,\gamma}(x,y) g(y) dy \right) dx \\
&= \int_{D_0} g(y) y_n^{-\gamma/p} \left(\int_B x_n^{1-\beta} |x-y|^{1-n} |\bar{x} - y|^{-1} dx \right) dy \\
&\leqq M r^{1-\beta} \int_{D_0} g(y) y_n^{-\gamma/p} \left(\int_B |x-y|^{1-n} |\bar{x} - y|^{-1} dx \right) dy.
\end{aligned}$$

We set

$$I(y) = \int_B |x-y|^{1-n} |\bar{x}-y|^{-1} dx$$

and

$$J = \int_{D_0} g(y) y_n^{-\gamma/p} \left(\int_B |x-y|^{1-n} |\bar{x}-y|^{-1} dx \right) dy.$$

If $|y - x_0| < 3s/2$, then

$$I(y) \leqq r^{-1} \int_B |x-y|^{1-n} dx \leqq M r^{-1} s,$$

so that we have by Hölder's inequality

$$\begin{aligned}
J_1 &= \int_{\{y \in D_0 : |y-x_0| < 3s/2\}} g(y) y_n^{-\gamma/p} I(y) dy \\
&\leqq M r^{-1} s \int_{\{y \in D_0 : |y-x_0| < 3s/2\}} g(y) y_n^{-\gamma/p} dy \\
&\leqq M r^{-1} s \left(\int_{\{y \in D_0 : |y-x_0| < 3s/2\}} y_n^{-\gamma p'/p} dy \right)^{1/p'} \left(\int_{D_0} g(y)^p dy \right)^{1/p} \\
&\leqq M s^n r^{-1-\gamma/p} s^{1-n/p} \left(\int_{D_0} g(y)^p dy \right)^{1/p}.
\end{aligned}$$

If $|y-x_0| \geqq 3s/2$ and $y_n \leqq x_n/2$, then $|x-y| \geqq M(|x'-y|+x_n) \geqq M(|x_0'-y|+r)$, so that

$$\begin{aligned} I_2(y) &= \int_{\{x\in B: y_n \leqq x_n/2\}} |x-y|^{1-n}|\bar{x}-y|^{-1}dx \\ &\leqq M(|x_0'-y|+r)^{-n}s^n. \end{aligned}$$

Hence we have by Hölder's inequality

$$\begin{aligned} J_2 &= \int_{\{y\in D_0: |x_0-y|\geqq 3s/2\}} g(y)y_n^{-\gamma/p}I_2(y)dy \\ &\leqq Ms^n \int_{D_0} g(y)y_n^{-\gamma/p}(|x_0'-y|+r)^{-n}dy \\ &\leqq Ms^n \left(\int_{D_0} y_n^{-\gamma p'/p}(|x_0'-y|+r)^{-p'n}dy\right)^{1/p'} \left(\int_{D_0} g(y)^p dy\right)^{1/p} \\ &\leqq Ms^n r^{-\gamma/p-n/p}\left(\int_{D_0} g(y)^p dy\right)^{1/p} \\ &\leqq Ms^n s^{1-n/p} r^{-1-\gamma/p}\left(\int_{D_0} g(y)^p dy\right)^{1/p}, \end{aligned}$$

since $p<n$. If $|y-x_0| \geqq 3s/2$ and $y_n > x_n/2$, then $|x-y| \geqq M(|x_0-y|+s)$ and $|\bar{x}-y| \geqq M(|x_0-y|+r)$, so that

$$\begin{aligned} I_3(y) &= \int_{\{x\in B: y_n > x_n/2\}} |x-y|^{1-n}|\bar{x}-y|^{-1}dx \\ &\leqq M(|x_0-y|+s)^{1-n}(|x_0-y|+r)^{-1}s^n. \end{aligned}$$

Consequently, it follows that

$$\begin{aligned} J_3 &= \int_{\{y\in D_0: |x_0-y|\geqq 3s/2, y_n > r/4\}} g(y)y_n^{-\gamma/p}I_3(y)dy \\ &\leqq Ms^n \int_{\{y\in D_0: |y-x_0|\geqq 3s/2, y_n>r/4\}} g(y)y_n^{-\gamma/p}(|x_0-y|+s)^{1-n}(|x_0-y|+r)^{-1}dy. \end{aligned}$$

Setting $t=|x_0-y|$ and $|(x_0)_n - y_n| = t\cos\theta$, we note that

$$(t+r)\cos\theta \leqq |(x_0)_n - y_n| + (x_0)_n \leqq 3y_n < 3(r+t)$$

when $y_n > r/4$. Using Hölder's inequality and applying the polar coordinates about x_0, we have

$$\begin{aligned} J_3 &\leqq Ms^n \left(\int_{3s/2}^{\infty} (t+s)^{p'(1-n)}(t+r)^{p'(-\gamma/p-1)}t^{n-1}dt\right)^{1/p'} \left(\int_{D_0} g(y)^p dy\right)^{1/p} \\ &\leqq Ms^n s^{1-n/p} r^{-1-\gamma/p}\left(\int_{D_0} g(y)^p dy\right)^{1/p} \end{aligned}$$

since $p < n$. Therefore we obtain

$$|B| \leqq M r^{-\beta-\gamma/p} s^{(p-n)/p} s^n \left(\int_{D_0} g(y)^p dy \right)^{1/p}.$$

Hence it follows from the definition of $C_{k_{\beta,\gamma},p}$ that

$$r^{\beta p+\gamma} s^{n-p} \leqq M C_{k_{\beta,\gamma},p}(B; D_0),$$

as required.

To obtain the converse inequality, note that for $x \in B$

$$\begin{aligned} \int_B x_n^{1-\beta} |x-y|^{1-n} |\bar{x}-y|^{-1} y_n^{-\gamma/p} dy &\geqq M r^{-\beta-\gamma/p} \int_B |x-y|^{1-n} dy \\ &\geqq M r^{-\beta-\gamma/p} s, \end{aligned}$$

so that

$$C_{k_{\beta,\gamma},p}(B; D_0) \leqq M r^{(\beta+\gamma/p)p} s^{-p} \int_B dy = M r^{\beta p+\gamma} s^{n-p}.$$

Thus the proof is completed.

PROPOSITION 3. *Let $0 \leqq \beta \leqq 1$ and $-1 < \gamma < p-1$. Set $V = \cup_{j=1}^{\infty} B(x_j, r_j) \cap D$ with $x_j \in \partial D$, $2^j \leqq |x_j| < 2^{j+1}$ and $0 < r_j \leqq 2^{j+1}$. If V is $(k_{\beta,\gamma}, p)$-thin at infinity, then*

$$\sum_{j=1}^{\infty} \left(\frac{r_j}{2^j} \right)^{n+\gamma-(1-\beta)p} < \infty; \tag{9}$$

conversely, if $\gamma > (1-\beta)p$ and (9) holds, then V is $(k_{\beta,\gamma}, p)$-thin at infinity.

PROOF. First we show that if $B_+ = B(x_0, r) \cap D$ with $x_0 \in \partial D$, $1 \leqq |x_0| < 2$ and $0 < r \leqq 2$, then

$$r^{n+\gamma-(1-\beta)p} \leqq M C_{k_{\beta,\gamma},p}(B_+; D_0). \tag{10}$$

Let g be a nonnegative measurable function such that $g = 0$ outside D_0 and

$$\int_D k_{\beta,\gamma}(x,y) g(y)\, dy \geqq 1$$

for every $x \in B_+$. Then we have by Fubini's theorem

$$\begin{aligned} |B_+| &\leqq \int_{B_+} \left(\int_{D_0} k_{\beta,\gamma}(x,y) g(y) dy \right) dx \\ &= \int_{D_0} g(y) y_n^{-\gamma/p} \left(\int_{B_+} x_n^{1-\beta} |x-y|^{1-n} |\bar{x}-y|^{-1} dx \right) dy. \end{aligned}$$

Here we see that if $|x_0 - y| > 2r$, then

$$\int_{B_+} x_n^{1-\beta}|x-y|^{1-n}|\bar{x}-y|^{-1}dx \leqq M|x_0-y|^{-n}r^{1-\beta+n}$$

and that if $|x_0 - y| \leqq 2r$, then

$$\begin{aligned}\int_{B_+} x_n^{1-\beta}|x-y|^{1-n}|\bar{x}-y|^{-1}dx &\leqq Mr^{1-\beta}\int_{B_+}|x-y|^{1-n}(|x-y|+y_n)^{-1}dx \\ &\leqq Mr^{1-\beta}\log(4r/y_n).\end{aligned}$$

Then we have by Hölder's inequality

$$\begin{aligned}J_1 &= r^{1-\beta}\int_{\{y\in D_0:|x_0-y|\leqq 2r\}} g(y)y_n^{-\gamma/p}\log(4r/y_n)dy \\ &\leqq r^{1-\beta}\left(\int_{\{y\in D_0:|x_0-y|\leqq 2r\}}\{\log(4r/y_n)\}^{p'}y_n^{-\gamma p'/p}dy\right)^{1/p'}\left(\int_{D_0} g(y)^p dy\right)^{1/p} \\ &\leqq Mr^{1-\beta-\gamma/p+n/p'}\left(\int_{D_0} g(y)^p dy\right)^{1/p}\end{aligned}$$

and

$$\begin{aligned}J_2 &= r^{1-\beta+n}\int_{\{y\in D_0:|x_0-y|>2r\}} g(y)y_n^{-\gamma/p}|x_0-y|^{-n}dy \\ &\leqq r^{1-\beta+n}\left(\int_{\{y\in D_0:|x_0-y|>2r\}} y_n^{-\gamma p'/p}|x_0-y|^{-p'n}dy\right)^{1/p'}\left(\int_{D_0} g(y)^p dy\right)^{1/p} \\ &\leqq Mr^{1-\beta-\gamma/p+n/p'}\left(\int_{D_0} g(y)^p dy\right)^{1/p}.\end{aligned}$$

Therefore we have

$$|B_+| \leqq Mr^{1-\beta-\gamma/p+n/p'}\left(\int_{D_0} g(y)^p dy\right)^{1/p},$$

so that it follows from the definition of $C_{k_{\beta,\gamma},p}$ that

$$r^{n+\gamma-(1-\beta)p} \leqq MC_{k_{\beta,\gamma},p}(B_+; D_0),$$

as required.

To obtain the converse inequality, note that for $x \in B_+$

$$\begin{aligned}&\int_{B_+} x_n^{1-\beta}|x-y|^{1-n}|\bar{x}-y|^{-1}y_n^{-\gamma/p}dy \\ \geqq& \int_{B_+\cap B(x,x_n/2)} x_n^{1-\beta}|x-y|^{1-n}|\bar{x}-y|^{-1}y_n^{-\gamma/p}dy \\ \geqq& Mx_n^{1-\beta-1-\gamma/p}\int_{B_+\cap B(x,x_n/2)}|x-y|^{1-n}dy \\ \geqq& Mx_n^{1-\beta-\gamma/p} \geqq Mr^{1-\beta-\gamma/p},\end{aligned}$$

since $1-\beta<\gamma/p$. Hence it follows from the definition of $C_{k_{\beta,\gamma},p}$ that

$$C_{k_{\beta,\gamma},p}(B_+;D_0) \leqq Mr^{-(1-\beta)p+\gamma}\int_{B_+} dy = Mr^{n+\gamma-(1-\beta)p}.$$

Thus the proof is completed.

For a nondecreasing function φ on $\mathbf{R}^1$ such that $0<\varphi(2t)\leqq M\varphi(t)$ for $t>0$ with a positive constant M, we set

$$T_\varphi=\{x=(x',x_n);\ 0<x_n<\varphi(|x'|)\}.$$

PROPOSITION 4 (cf. Aikawa [1, Proposition 5.1]). *Let $0<\beta\leqq 1$ and $p(1-\beta)-1<\gamma<p-1$. Assume further that*

$$\lim_{r\to\infty}\frac{\varphi(r)}{r}=0. \tag{11}$$

Then T_φ is $(k_{\beta,\gamma},p)$-thin at infinity if and only if

$$\int_1^\infty\left(\frac{\varphi(t)}{t}\right)^{p(-1+\beta)+\gamma+1}\frac{dt}{t}<\infty. \tag{12}$$

For example, $\varphi(r)=r[\log(1+r)]^{-\delta}$ satisfies (12), when $\delta\{p(-1+\beta)+\gamma+1\}>1$.

5 Limits of monotone functions

Finally we consider the limits at infinity for monotone BLD functions. A continuous function u is called monotone on $\mathbf{D}$ in the sense of Lebesgue (see [5]) if for every relatively compact open subset G of $\mathbf{D}$,

$$\max_{G\cup\partial G}u=\max_{\partial G}u \qquad \text{and} \qquad \min_{G\cup\partial G}u=\min_{\partial G}u.$$

For examples and fundamental properties of monotone functions, see [12] and [16]. Among them the following result is only needed for monotone functions.

LEMMA 1. *If u is a monotone BLD function on $B(x,2r)$ and $p>n-1$, then*

$$|u(z)-u(x)|^p\leqq Mr^{p-n}\int_{B(x,2r)}|\nabla u(y)|^p dy \tag{13}$$

for every $z\in B(x,r)$.

THEOREM 5. *Let $p > n-1$, $-1 < \gamma < p-1$ and $n+\gamma-p \geqq 0$. If u is a monotone function on D satisfying* (1), *then there exist a number A such that*

$$\lim_{|x|\to\infty, x\in D} x_n^{(n+\gamma-p)/p}[u(x)-A] = 0$$

in case $n+\gamma-p>0$ and

$$\lim_{|x|\to\infty, x\in D} (\max\{\log(1/x_n), \log|x|\})^{-1/p'}[u(x)-A] = 0$$

in case $n+\gamma-p=0$.

PROOF. For $x \in D$, let $r=|x|$, $C(x)=(0,...,0,r)$ and $\rho_{\mathbf{D}}(x)$ denote the distance of $x \in \mathbf{D}$ from the boundary ∂D, that is, $\rho_{\mathbf{D}}(x)=x_n$. We take a finite covering $\{B_j\}$, $B_j = B(X_j, 4^{-1}\rho_{\mathbf{D}}(X_j))$, such that

(i) $X_1 = x$ and $X_{N+1} = C(x)$;

(ii) $r/2 < |z| < 2r$ for $z \in A(r) = \cup_j 2B_j$, where $2B_j = B(X_j, 2^{-1}\rho_{\mathbf{D}}(X_j))$;

(iii) $B_j \cap B_{j+1} \neq \emptyset$ for each j;

(iv) $\sum_j \chi_{2B_j}$ is bounded, where χ_A denotes the characteristic function of A.

By the monotonicity of u, we see that

$$|u(y)-u(X_j)| \leqq M\rho_{\mathbf{D}}(X_j)^{(p-n)/p}\int_{2B_j}|\nabla u(z)|^p dz$$

for $y \in B_j$. First suppose $n+\gamma-p>0$. Using Theorem 1, we can find a number A and $C_1(x)$ such that $C_1(x) \in B_{N+1}$ and

$$\lim_{|x|\to\infty}|x|^{(n+\gamma-p)/p}[u(C_1(x))-A] = 0.$$

Then we have by Hölder's inequality

$$\begin{aligned}
|u(x)-A| &\leqq |u(X_1)-u(X_2)|+|u(X_2)-u(X_3)|+\cdots+|u(X_N)-u(X_{N+1})| \\
&\quad +|u(X_{N+1})-u(C_1(x))|+|u(C_1(x))-A| \\
&\leqq M\sum_j \rho_{\mathbf{D}}(X_j)^{(p-n-\gamma)/p}\left(\int_{2B_j}|\nabla u(z)|^p\rho_{\mathbf{D}}(z)^\gamma dz\right)^{1/p} + |u(C_1(x))-A| \\
&\leqq M\left(\sum_j \rho_{\mathbf{D}}(X_j)^{p'(p-n-\gamma)/p}\right)^{1/p'}\left(\int_{A(r)}|\nabla u(z)|^p\rho_{\mathbf{D}}(z)^\gamma dz\right)^{1/p} \\
&\quad + |u(C_1(x))-A| \\
&\leqq Mx_n^{(p-n-\gamma)/p}\left(\int_{\mathbf{D}-B(0,r/2)}|\nabla u(z)|^p\rho_{\mathbf{D}}(z)^\gamma dz\right)^{1/p} + |u(C_1(x))-A|,
\end{aligned}$$

which proves

$$\lim_{|x|\to\infty} x_n^{(n+\gamma-p)/p}[u(x) - A] = 0,$$

as required.

The case $n + \gamma - p = 0$ can be treated similarly.

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