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Mean value property for temperatures on an annulus domain

名古屋大学大学院・多元数理科学研究科　鈴木　紀明 (Noriaki SUZUKI)
Graduate School of Mathematics,
Nagoya University

§1. Introduction

Heat balls in $\mathbf{R}^{n+1}$ are characterized by some mean value identity for temperatures (solutions of the heat equation) in [3]. In this paper we give similar theorem for a heat annulus. The corresponding result for harmonic functions is given in [1] (see also [2]).

For a point in $(n+1)$-dimensional Euclidean space $\mathbf{R}^{n+1}$, we write

$$P = (x,t) = (x_1, \ldots, x_n, t,).$$

We use $W = W_n$ to denote the Gauss-Weierstrass kernel, defined by

$$W_n(x,t) \;:=\; \begin{cases} (4\pi t)^{-n/2}\exp(-|x|^2/4t) & \text{if } t > 0, \\ 0 & \text{if } t \le 0, \end{cases}$$

where $|x| := (x_1^2 + \cdots + x_n^2)^{1/2}$. The heat ball $\Omega(c)$ centered at the origin and radius $c > 0$ is defined by a level surface of W_n, that is,

$$\Omega(c) \;:=\; \{(x,t) \in \mathbf{R}^{n+1} : W_n(x,-t) > (4\pi c)^{-n/2}\}.$$

Clearly $\Omega(c) \subset \{|x|^2 < 2nc/e,\ -c < t < 0\}$. We consider the following mean values $M(u,c)$ over the heat sphere $\partial\Omega(c)$ and $V_\alpha(u,c)$ over the heat ball $\Omega(c)$:

$$M(u,c) \;:=\; \frac{1}{(4\pi c)^{n/2}} \int_{\partial\Omega(c)} Q(x,t)u(x,t)d\sigma(x,t)$$

where $Q(x,t) = |x|^2\{4|x|^2t^2 + (|x|^2 + 2nt)^2\}^{-1/2}(t < 0)$, $Q(0,0) = 1$, and

$$V_\alpha(u,c) := \alpha c^{-\alpha} \int_0^c r^{\alpha-1} M(u,r)dr, \tag{1.1}$$

for $\alpha > 0$. Then,

$$V_\alpha(u,c) = \frac{\alpha}{2^{n+1}n\pi^{n/2}c^\alpha} \int\int_{\Omega(c)} K_\alpha(x,t)u(x,t)dxdt,$$

where

$$K_\alpha(x,t) := \frac{|x|^2}{(-t)^{(n+4-2\alpha)/2}} \exp\left(\frac{(2\alpha - n)|x|^2}{4n(-t)}\right)$$

For $0 < c_1 < c_2$, we put

$$A(c_1, c_2) := \Omega(c_2) \setminus \overline{\Omega(c_1)}$$

and call $A(c_1, c_2)$ a heat annulus.

We have the following mean value property for temperatures.

Theorem 1. (I) Let $\alpha > 0$ and $c > 0$. If u is a temperature in $\Omega(c)$ and continuous on its closure $\overline{\Omega(c)}$, then $u(0,0) = V_\alpha(u, c)$, that is

$$u(0,0) = \frac{\alpha}{2^{n+1} n \pi^{n/2} c^\alpha} \int\int_{\Omega(c)} K_\alpha(x,t) u(x,t) dx dt. \tag{1.2}$$

(II) Let $\alpha > 0$ and $0 < c_1 < c_2$. If u is a temperatrure in $A(c_1, c_2)$ and continuous on its closure $\overline{A(c_1, c_2)}$, then

$$M(u, c) = \frac{\alpha}{n 2^{n+1} \pi^{n/2} (c_2^\alpha - c_1^\alpha)} \int\int_{A(c_1,c_2)} K_\alpha(x,t) u(x,t) dx dt, \tag{1.3}$$

where c is a constant defined by

$$c^{-n/2} := \begin{cases} \dfrac{\alpha(c_2^{\alpha-n/2} - c_1^{\alpha-n/2})}{(\alpha - n/2)(c_2^\alpha - c_1^\alpha)} & (\text{if } \alpha \neq n/2) \\ \dfrac{n \log(c_2/c_1)}{2(c_2^{n/2} - c_1^{n/2})} & (\text{if } \alpha = n/2). \end{cases} \tag{1.4}$$

The following converse assertions of Theorem 1 are our main results in this paper.

Theorem 2. (I) Let $\alpha > 0$, $c > 0$ and let D be a bounded open set in $\mathbf{R}^{n+1}$. If the following conditions are satisfies, then $D = \Omega(c)$:

(1) $(\chi_D - \chi_{\Omega(c)}) K_\alpha \in L^p(\mathbf{R}^{n+1})$ for some $p > n/2 + 1$.

(2) For all $(y, s) \in \mathbf{R}^{n+1} \setminus D$,

$$\frac{\alpha}{2^{n+1} n \pi^{n/2} c^\alpha} \int\int_D W(x - y, t - s) K_\alpha(x,t) dx dt = W(y, -s). \tag{1.5}$$

(II) Let $\alpha > 0$, $0 < c_1 < c_2$ and let D be a bounded open set in $\mathbf{R}^{n+1}$. Put c as in (1.4). If the following conditions are satisfies, then $D = A(c_1, c_2)$:

(1) D contains $\partial\Omega(c) \setminus \{(0,0)\}$.

(2) $(\chi_D - \chi_{A(c_1,c_2)}) K_\alpha \in L^p(\mathbf{R}^{n+1})$ for some $p > n/2 + 1$.

(3) For all $(y, s) \in \mathbf{R}^{n+1} \setminus D$,

$$\frac{\alpha}{n 2^{n+1} \pi^{n/2} (c_2^\alpha - c_1^\alpha)} \int\int_D W(x - y, t - s) K_\alpha(x,t) dx dt = M(W(\cdot - y, \cdot - s), c). \tag{1.6}$$

(4) $\inf\{s \,; (y, s) \in \Omega(c_1) \cap D^C\} = c_1$.

§2. Proof of Theorems

Proof of Theorem 1. (I) Since $u(0,0) = M(u,r)$ for $0 < r < c$ (see [4]), we have

$$V_\alpha(u,c) = \alpha c^{-\alpha}\int_0^c r^{\alpha-1}u(0,0)dr = u(0,0).$$

To show (II), we first remark that

$$M(u,r) = pr^{-n/2} + q \quad (c_1 < \forall r < c_2)$$

with some constants p, q (see [5]). Hence if $\alpha \neq n/2$, we have

$$\int_{c_1}^{c_2} r^{\alpha-1}M(u,r)dr = p\cdot\frac{\alpha(c_2^{\alpha-n/2} - c_1^{\alpha-n/2})}{(\alpha-n/2)(c_2^\alpha - c_1^\alpha)} + q = pc^{-n/2} + q = M(u,c).$$

On the other hand, by (1.1)

$$\int_{c_1}^{c_2} r^{\alpha-1}M(h,r)dr = \frac{1}{n2^{n+1}\pi^{n/2}}\int\int_{A(c_2,c_1)} K_\alpha(x,t)u(x,t)dxdt. \tag{2.1}$$

These equalities give (1.3). The case $\alpha = n/2$ is shown in a similar way.

Proof of Theorem 2. (I) In [3], we gave a proof for the case $\alpha = n/2$. Although its proof is valid for general $\alpha > 0$, we will repeat it for the sake of completeness.

Put $\beta := 2^{n+1}n\pi^{n/2}c^\alpha/\alpha$. By the volume mean value property of temperatures in [4], we have

$$\int\int_{\Omega(c)} K_\alpha(x,t)dxdt = \beta$$

and for every $(y,s) \in \mathbf{R}^{n+1}\setminus\Omega(c)$,

$$\int\int_{\Omega(c)} W(x-y,t-s)K_\alpha(x,t)dxdt = \beta W(y,-s). \tag{2.2}$$

Since D is bounded, there is $s < 0$ such that $(y,s) \notin \Omega(c)$ for all $y \in \mathbf{R}^n$, so that (1.5) gives

$$\begin{aligned}\beta &= \beta\int_{\mathbf{R}^n} W(y,-s)dy \\ &= \int_{\mathbf{R}^n}\left(\int\int_D W(x-y,t-s)K_\alpha(x,t)dxdt\right)dy \\ &= \int\int_D K_\alpha(x,t)dxdt\end{aligned}$$

and hence

$$\int\int_{\Omega(c)} K_\alpha(x,t)dxdt = \int\int_D K_\alpha(x,t)dxdt. \tag{2.3}$$

Now for every $(y,s) \in \mathbf{R}^{n+1}$, we put

$$\begin{aligned} v(y,s) &:= \int\int_D W(x-y,t-s)K_\alpha(x,t)dxdt \\ v_0(y,s) &:= \int\int_{\Omega(c)} W(x-y,t-s)K_\alpha(x,t)dxdt \\ u(y,s) &:= \beta W(y,-s) - v(y,s) \\ u_0(y,s) &:= \beta W(y,-s) - v_0(y,s). \end{aligned}$$

Then (1.5) implies that

$$u(y,s) = 0, \qquad \forall (y,s) \in \mathbf{R}^{n+1} \setminus D \tag{2.4}$$

and (2.2) implies

$$u_0(y,s) = 0, \qquad \forall (y,s) \in \mathbf{R}^{n+1} \setminus \Omega(c) \tag{2.5}$$

Further, for $r > 0$ we see

$$M(W(\cdot - y, \cdot - s), r) = W(y,-s) \wedge (4\pi r)^{-n/2} = \begin{cases} W(y,-s) & \text{if } (y,s) \notin \Omega(r) \\ (4\pi r)^{-n/2} & \text{if } (y,s) \in \Omega(r) \end{cases} \tag{2.6}$$

(see [5]), and hence

$$u_0(y,s) > 0, \qquad \forall (y,s) \in \Omega(c). \tag{2.7}$$

We assert that $v - v_0 \in C(\mathbf{R}^{n+1})$. Put $f := (\chi_D - \chi_{\Omega(c)})K_\alpha$. Take $s = 0$ in (1.5), we see $\mathrm{supp}(f) \subset \mathbf{R}^{n+1} \times (-\infty, 0]$. For each $a \leq 0$, let f_a denote the restriction of f to $\mathbf{R}^n \times (-\infty, a)$, and let $F_a := \mathrm{supp}(f) \cap (\mathbf{R}^n \times [a,0])$. If $a < 0$ then f_a is bounded, so that the function

$$(y,s) \mapsto \int\int_{\mathbf{R}^{n+1}} W(x-y,t-s)f_a(x,t)dxdt$$

is continuous. Since

$$H^*\left(\int\int_{F_a} W(x-y,t-s)f(x,t)dxdt\right) = 0 \qquad \forall (y,s) \in \mathbf{R}^{n+1} \setminus F_a$$

it follows that $v - v_0 \in C(\mathbf{R}^{n+1} \setminus F_a)$. Since a is arbitrary, $v - v_0 \in C(\mathbf{R}^{n+1} \setminus F_0)$. Finally, if $q := p/(p-1)$, the exponent conjugate to p, then $q < (n+2)/n$ and for some constant M we have

$$|v - v_0|(y,s) \leq M|s|^{(n+2-nq)/2q}\|f\|_p,$$

so that condition (1) in (I) implies that $(v - v_0)(y,s) \to 0$ as $s \to 0$.

To prove that $D = \Omega(c)$, it is sufficient to show that $\chi_D = \chi_{\Omega(c)}$ a.e. on $\mathbf{R}^{n+1}$. For then $u = u_0$, so that (2.4) and (2.7) imply that $\Omega(c) \subset D$. Therefore $\Omega(c) = D \setminus F$ for some relatively closed subset F of D with measure zero. Since $\overline{\Omega(c)}^{\,o} = \Omega(c)$, it follows that $D = \Omega(c)$.

Suppose that $\chi_D \neq \chi_{\Omega(c)}$ on a set of positive measure. Since $\chi_{\overline{\Omega(c)}} = \chi_{\Omega(c)}$ a.e., we can choose $P_0 \in D \setminus \overline{\Omega(c)}$, in view of (2.3). If L is any line through P_0, we can choose $Q_1, Q_2 \in L \cap \partial D$ such that P_0 belongs to the segment $\overline{Q_1Q_2}$. If Q_1 and Q_2 both belonged to $\overline{\Omega(c)}$, then by convexity P_0 would also belong to $\overline{\Omega(c)}$, which is false. Therefore $\partial D \setminus \overline{\Omega(c)} \neq \emptyset$. Moreover, $\partial D \setminus \overline{\Omega(c)}$ contains a point (y_0, s_0) with the property that every ball centred there meets $D_+ = D \cap (\mathbf{R}^n \times (s_0, \infty))$. For otherwise $\partial D \setminus \overline{\Omega(c)}$ would be contained in the union of a sequence of parallel hyperplanes, and so D would be unbounded. Choose a ball B, centred at (y_0, s_0), such that $B \cap \overline{\Omega(c)} = \emptyset$. The function u is an H^*-subtemperature on B, is not an H^*-temperature on $B \cap D$, and is zero at (y_0, s_0) by (2.4). Since $B \cap D_+ \neq \emptyset$, the maximum principle therefore implies that

$$\sup_B \, u > 0 .$$

Put

$$m \; = \; \max_{\mathbf{R}^{n+1}} (u - u_0) \qquad \text{and} \qquad E \; = \; (u - u_0)^{-1}(m) .$$

Since $B \cap \overline{\Omega(c)} = \emptyset$, we have $u_0 = 0$ on B. Therefore $\sup_B(u - u_0) > 0$, and hence $m > 0$. Because $u_0 \geq 0$ by (2.5) and (2.7), we have $u > 0$ on E, and hence $E \subseteq D$ by (2.4). On the other hand, for all $(x, t) \in D$ we have

$$H^*(u - u_0)(x, t) \; = \; (1 - \chi_{\Omega(c)}(x, t)) \, \frac{\|x\|^2}{t^2} \geq 0 ,$$

so that $u - u_0$ is an H^*-subtemperature on D. The maximum principle now implies that $E \cap \partial D \neq \emptyset$, a contradiction. Hence $D = \Omega(c)$.

To show (II), we first remark that

$$\text{(2.8)} \qquad \chi_{\Omega(r)} K_\alpha \notin L^{n/2+1}(\mathbf{R}^{n+1}), \quad (\forall r > 0).$$

Now applying $u \equiv 1$ to (1.3), we have

$$\frac{\alpha}{n 2^{n+1} \pi^{n/2} (c_2^\alpha - c_1^\alpha)} \int\int_{A(c_1, c_2)} K_\alpha(x, t) dx dt = 1.$$

Furthermore, by the usual limiting argument, (1.3) gives

$$\text{(2.8)} \quad M(W(\cdot - y, \cdot - s), c) = \frac{\alpha}{n 2^{n+1} \pi^{n/2} (c_2^\alpha - c_1^\alpha)} \int\int_{A(c_1, c_2)} W(x - y, t - s) K_\alpha(x, t) dx dt$$

for all $(y, s) \in \mathbf{R}^{n+1} \setminus A(c_1, c_2)$. As in (2.2), we have

$$\text{(2.9)} \qquad \int\int_D K_\alpha(x, t) dx dt = \int\int_{A(c_1, c_2)} K_\alpha(x, t) dx dt,$$

and as in the proof of (I), we put

$$\begin{aligned} v(y,s) &:= \frac{\alpha}{n2^{n+1}\pi^{n/2}(c_2^\alpha - c_1^\alpha)} \int\int_D W(x-y,t-s)K_\alpha(x,t)dxdt, \\ v_0(y,s) &:= \frac{\alpha}{n2^{n+1}\pi^{n/2}(c_2^\alpha - c_1^\alpha)} \int\int_{A(c_1,c_2)} W(x-y,t-s)K_\alpha(x,t)dxdt, \\ u(y,s) &:= M(W(\cdot - y, \cdot - s), c) - v(y,s), \\ u_0(y,s) &:= M(W(\cdot - y, \cdot - s), c) - v_0(y,s). \end{aligned}$$

for every $(y,s) \in \mathbf{R}^{n+1}$. Then $u - u_0 \in C(\mathbf{R}^{n+1})$ as in (I). Also, (1.6) implies

$$u(y,s) = 0, \quad \forall (y,s) \in \mathbf{R}^{n+1} \setminus D \tag{2.10}$$

and by (2.6) and (2.1) we have

$$\begin{cases} u_0(y,s) = 0, & \text{if } (y,s) \notin A(c_1,c_2) \\ u_0(y,s) > 0, & \text{if } (y,s) \in A(c_1,c_2). \end{cases} \tag{2.11}$$

If we assume $D \setminus \overline{\Omega(c_2)} \neq \emptyset$, then we have a contradiction as in the proof of (I). Hence $D \subset \Omega(c_2)$.

Next we pay attension to a set $\partial D \cap \Omega(c_1)$ and assume that this is empty. Then $D \subset A(c_1,c_2)$ or $D \supset \Omega(c_1)$. In the first case, we have $\chi_D = \chi_{A(c_1,c_2)}$ a.e. by (2.9) and hence $D = A(c_1,c_2)$ as in (I). The second case does not occur, because of (2.8). The rest of proof is to consider the case $\partial D \cap \Omega(c_1) \neq \emptyset$. Choose a point $(y_0,s_0) \in \partial D \cap \Omega(c_1)$ and take $r > 0$ such that a usual ball $B := B((y_0,s_0),r)$ is contained in $\Omega(c_1)$. If $B \cap D \cap \{(y,s); s > s_0\} \neq \emptyset$, we have a contradiction by the maximum principle as in (I). On the other hand, if $B \cap D \cap \{(y,s); s > s_0\} = \emptyset$ for all point $(y_0,s_0) \in \partial D \cap \Omega(c_1)$, we see that $D \cap \Omega(c_1) = \{(y,s); s < s_0\} \cap \Omega(c_1)$. This contradicts our assumption (4) of (II).

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