数理解析研究所講究録 1296 巻 2002 年 124-127

On the Number of Poles of the First Painlevé Transcendents and Higher Order Anlogues

Shun Shimomura

Department of Mathematics, Keio University

下村 俊 (慶応大・理工)

Let $w(z)$ be an arbitrary solution of the first Painlevé equation

$$\text{(PI)} \qquad w'' = 6w^2 + z.$$

Then, $w(z)$ is a transcendental meromorphic function, and every pole is double. Denote by $n(r, w)$ the number of poles inside the circle $|z| < r$. In this note, we prove the following:

Theorem A. *The growth order of* $w(z)$ *is not less than* $5/2$*, namely*

$$\limsup_{r\to\infty} \frac{\log n(r,w)}{\log r} \geq \frac{5}{2}. \tag{1}$$

For another proof of this result, see [2].

It is known that the equations

$$\text{(PI}_4\text{)} \qquad w^{(4)} = 20ww'' + 10(w')^2 - 40w^3 + 16z,$$

$$\text{(PI}_6\text{)} \qquad w^{(6)} = 28ww^{(4)} + 56w'w^{(3)} + 42(w'')^2 - 280(w^2w'' + w(w')^2 - w^4) + 64z$$

are higher order analogues for (PI). Denote by $w_4(z)$ (resp. $w_6(z)$) an arbitrary meromorphic solution of (PI_4) (resp. (PI_6)). It is easy to see that $w_4(z)$ (resp. $w_6(z)$) is transcendental and every pole is double. The following result is proved by the same argument as in the proof of Theorem A.

Theorem B. *We have*

$$\limsup_{r\to\infty} \frac{\log n(r,w_4)}{\log r} \geq \frac{7}{3}, \tag{2}$$

$$\limsup_{t\to\infty} \frac{\log n(r,w_6)}{\log r} \geq \frac{9}{4}. \tag{3}$$

Remark. For solutions of (PI), a more precise result is known (see [3], [4]):

$$\frac{r^{5/2}}{\log r} \ll n(r,w) \ll r^{5/2}. \tag{4}$$

(We write $f(r) \ll g(r)$ if $f(r) = O(g(r))$ as $r \to \infty$.)

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. Proof of Theorem A

In what follows, for simplicity, we use the abbreviation $n(r) := n(r, w)$. To prove (1), we suppose the contrary:

$$\limsup_{r\to\infty} \frac{\log n(r)}{\log r} < \frac{5}{2}, \tag{5}$$

namely, for some $\varepsilon > 0$,

$$n(r) \ll r^{5/2-\varepsilon}. \tag{6}$$

Starting from this supposition, we would like to derive a contradiction. By $\{a_j\}_{j=1}^{\infty}$ we denote the distinct poles of $w(z)$ arranged as $|a_1| \le \cdots \le |a_j| \le \cdots$ (by a Clunie resoning ([1, §9.2]), $w(z)$ has infinitely many poles). By virtue of (6), $w(z)$ is written in the form

$$w(z) = \Phi(z) + \phi(z), \tag{7}$$

$$\Phi(z) = \sum_{a_j} ((z - a_j)^{-2} - a_j^{-2}), \tag{8}$$

where $\phi(z)$ is an entire function; in the right-hand side of (8), if $a_1 = 0$, the term $(z - a_1)^{-2} - a_1^{-2}$ should be replaced by z^{-2}. Under supposition (6), we have the following lemmas whose proofs will be given afterward:

Lemma 1.1. *For arbitrary $r > 1$, there exists z_0 such that*

$$0.7r \le |z_0| \le r, \qquad \sum_{|a_j|<2r} |z_0 - a_j|^{-2} \ll r^{1/2-\varepsilon/2}.$$

Lemma 1.2. *We have, for $|z| \le r$,*

$$\sum_{|a_j|\ge 2r} |(z - a_j)^{-2} - a_j^{-2}| \ll r^{1/2-\varepsilon}, \qquad \sum_{|a_j|\ge 2r} |z - a_j|^{-4} \ll 1,$$

and

$$\sum_{|a_j|<2r} |a_j^{-2}| \ll r^{1/2-\varepsilon}.$$

Lemma 1.3. *There exists a set $E^* \subset (0, \infty)$ with finite linear measure such that*

$$\sum_{a_j} |(z - a_j)^{-2} - a_j^{-2}| \ll |z|^9 \quad \text{for } |z| \in (0, \infty) \setminus E^*.$$

Observing that $6w(z) = w''(z)/w(z) - z/w(z)$, we have

$$m(r, w) \ll m(r, w''/w) + \log r \ll \log r,$$

where

$$m(r, w) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |w(re^{i\theta})| d\theta, \quad \log^+ x = \max\{0, \log x\}$$

(for the notation and basic results in the Nevanlinna theory, see [1]). By Lemma 1.3, for $r \in (0, \infty) \setminus E^*$,

$$T(r, \phi) = m(r, \phi) = m(r, w - \Phi) \leq m(r, w) + m(r, \Phi) \ll \log r.$$

This implies that $\phi(z) \in \mathbf{C}[z]$. Note that $|\Phi(z)| \leq \left|\sum_{|a_j|<2r}\right| + \left|\sum_{|a_j|\geq 2r}\right|$. By Lemmas 1.1 and 1.2, for every $r > 1$, there exists z_0, $0.7r \leq |z_0| \leq r$ such that

$$|\Phi(z_0)| \ll r^{1/2-\varepsilon/2}, \qquad |\Phi''(z_0)| \ll r^{1-\varepsilon}.$$

Combining $w(z_0) = (w''(z_0) - z_0)^{1/2}/\sqrt{6}$ with these estimates, we have

$$|\phi(z_0)| \ll |\Phi(z_0)| + (|w''(z_0)| + |z_0|)^{1/2} \ll r^{1/2} + |\phi(z_0)|^{1/2},$$

which implies that $\phi(z) \equiv C \in \mathbf{C}$. Hence, from $z_0 = w''(z_0) - 6w(z_0)^2$, it follows that

$$0.7r \leq |z_0| \ll |w''(z_0)| + 6|w(z_0)|^2 \ll r^{1-\varepsilon},$$

which is a contradiction. We have thus proved Theorem A.

2. Proofs of the lemmas

2.1. Proof of Lemma 1.1. Put $D_r = \{z \,|\, |z| < r\}$ and $\Delta_0^\delta = \mathbf{C} \setminus \left(\bigcup_{j\geq 0} U_j^\delta\right)$; where $U_j^\delta = \left\{z \mid |z - a_j| < \delta |a_j|^{-1/4}\right\}$ if $a_j \neq 0$, and $U_0^\delta = \{z \,|\, |z| < \delta\}$ if $a_0 = 0$. Since, by (6),

$$\sum_{0<|a_j|<r} |a_j|^{-1/2} = \int_0^r \rho^{-1/2} dn(\rho) = \left[\rho^{-1/2} n(\rho)\right]_0^r + \frac{1}{2}\int_0^r \rho^{-3/2} n(\rho) d\rho \ll r^2,$$

we can take δ so small that $3\pi r^2/4 \leq \mu(\Delta_0^\delta \cap D_r) < \pi r^2$ for every $r > 1$, where $\mu(X)$ denotes the area of a domain X. It is easy to see that

$$\iint_{D_r \setminus U_j^\delta} \frac{dx\,dy}{|z - a_j|^2} \leq \iint_{\substack{\delta |a_j|^{-1/4} \leq \rho \leq 3r \\ 0 \leq \theta \leq 2\pi}} \rho^{-1} d\rho d\theta \ll \log r,$$

if $|a_j| < 2r$, and if $r > 1$; and hence

$$\iint_{\Delta_0^\delta \cap D_r} \sum_{|a_j|<2r} |z - a_j|^{-2} dx dy \ll n(2r) \log r \leq K_0 r^{5/2-\varepsilon/2}, \tag{9}$$

where K_0 is some positive number. Now consider the set

$$E_r = \left\{z \in \Delta_0^\delta \cap D_r \mid \sum\nolimits_{|a_j|<2r} |z - a_j|^{-2} \leq 4\pi^{-1} K_0 r^{1/2-\varepsilon/2}\right\}.$$

Suppose that $\mu(E_r) < \pi r^2/2$. Then

$$\iint_{\Delta_0^\delta \cap D_r \setminus E_r} \sum_{|a_j|<2r} |z - a_j|^{-2} dx dy > 4\pi^{-1} K_0 r^{1/2-\varepsilon/2}\left(\frac{3\pi r^2}{4} - \frac{\pi r^2}{2}\right) = K_0 r^{5/2-\varepsilon/2},$$

which contradicts (9). Hence $\mu(E_r) \geq \pi r^2/2$. Since $\mu(\{z \,|\, |z| < 0.7r\}) = 0.49\pi r^2$, we have $\{z \,|\, 0.7r \leq |z| \leq r\} \cap E_r \neq \emptyset$, which implies the conclusion.

2.2. Proof of Lemma 1.2. For $|a_j| \geq 2r$, and for $z \in D_r$, observe that $|z/a_j| \leq 1/2$. Since

$$|(z-a_j)^{-2} - a_j^{-2}| = 2|z||a_j|^{-3}|1-(z/a_j)/2||1-z/a_j|^{-2} \leq 10r|a_j|^{-3},$$

we have, by (6), that

$$\sum_{|a_j|\geq 2r} \left|(z-a_j)^{-2} - a_j^{-2}\right| \ll r \sum_{|a_j|\geq 2r} |a_j|^{-3} \ll r\int_{2r}^{\infty} t^{-3}dn(t)$$
$$\ll r\int_{2r}^{\infty} t^{-4}n(t)dt \ll r^{1/2-\varepsilon},$$

and that

$$\sum_{|a_j|<2r} |a_j^{-2}| = \int_0^{2r} t^{-2}dn(t) \ll r^{1/2-\varepsilon} + \int_0^{2r} t^{-3}n(t)dt \ll r^{1/2-\varepsilon}.$$

2.3. Proof of Lemma 1.3. We put

$$E^* = (0, |a_1|+1) \cup \Big(\bigcup_{j=2}^{\infty}(|a_j| - |a_j|^{-3}, |a_j| + |a_j|^{-3})\Big).$$

By (6), the total length of E^* is finite. If $|z| \notin E^*$, then

$$\Big(\sum_{0<|a_j|<2|z|} + \sum_{|a_j|\geq 2|z|}\Big)|(z-a_j)^{-2} - a_j^{-2}| \ll (|z|^6+1)n(2|z|) + |z|^{1/2} \ll |z|^9.$$

REFERENCES

1. Laine, I., *Nevanlinna theory and complex differential equations*, de Gruyter, Berlin, New York, 1993.
2. Mues, E. and Redheffer, R., *On the growth of the logarithmic derivatives*, J. London Math. Soc. **8** (1974), 412–425.
3. Shimomura, S., *Growth of the first, the second and the fourth Painlevé transcendents*, Math. Proc. Camb. Phil. Soc., to appear.
4. Shimomura, S., *Lower estimates for the growth of Painlevé transsccendents*, Funkcial. Ekvac., to appear.