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半可換に対する一考察

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1．発見。

数学では普通 $AB = BA$ のとき、A と B が可換であるという。しかし $AB = BA$ を分解すると、一般に $AB \leq BA$ 且つ $AB \geq BA$ となる。我々は前者を「A は B に劣可換である」といい、後者を「A は B に優可換である」ということにする。またどちらかが成り立つとき「A は B に半可換である」ということにする。ここで大事な考え方は、$AB \leq BA$ または $AB \geq BA$ を不等式に見えても不等式と見ないことである。このような考え方が一般的であるかどうかは不明であるが、最近 MathSciNet である論文を見てそのような考え方に至った。その論文というのは、Lech Maligranda による、

A simple proof of the Holder and the Minkowski inequality
Amer. Math. Monthly, 102-3(1995), 256-259

というもので、J. Rakosnik による Review には、

[The core of the paper is the following lemma: For $1 \leq p < \infty$ and any $a, b > 0$ we have
$\inf_{t>0}\left[\frac{1}{p}t^{1/p-1}a + (1-\frac{1}{p})t^{1/p}b\right] = a^{1/p}b^{1-1/p}$ and $\inf_{0<t<1}\left[t^{1-p}a^p + (1-t)^{1-p}b^p\right] = (a+b)^p$.
Two proofs are given. The first one is based on elementary caluculas. In the second one the Jensen inequality and the convexity of the functions $\exp(u)$ and u^p are used. The Holder and Minkowski inequalities are immediate consequences. It is known that the inequality $x^\alpha - \alpha x + \alpha - 1 \leq 0$ holds for $0 < \alpha < 1, x > 0$, and equality holds if and only if $x = 1$. This is a consequence of an easy application of the calculus. Putting $\alpha = 1/p$ and $x = a(bt)^{-1}$ we obtain the first formula above.]

と書いてあった。

実際、彼の key lemma をみて Holder の不等式も Minkowski の不等式も

「良く知られたある種の平均は、ある正線形汎関数と常に半可換である」

ことを主張していることに気付いた訳である。

以下の節でもう少し詳しく述べてみよう。

2．関数族 (m) と (M)。

先ず以下の記号を約束する：

D : a domain in $\boldsymbol{R}^n$, S : a real linear space with dual S^*, $S_0 \subseteq S$

Φ : a subset of S^* such that $(\varphi x_1 \cdots, \varphi x_n) \in D$ for all $x_1, \cdots, x_n \in S_0$ and $\varphi \in \Phi$

$\hat{s}(\varphi) = \varphi(s)$ $(\varphi \in \Phi, s \in S)$, $\hat{S} = \{\hat{s} : s \in S\}$

このとき $\hat{S}$ は半順序実線形空間をつくる。次に以下のような２つの関数 $m, M : D \to \boldsymbol{R}$ を考える：

(1) $m(\hat{x}_1, \cdots, \hat{x}_n), M(\hat{x}_1, \cdots, \hat{x}_n) \in \hat{S}$ for all $x_1, \cdots, x_n \in S_0$.

(2) $m(L\hat{x}_1, \cdots, L\hat{x}_n) \leq L((m(\hat{x}_1, \cdots, \hat{x}_n))$, $M(L\hat{x}_1, \cdots, L\hat{x}_1) \geq L((M(\hat{x}_1 \cdots, \hat{x}_n))$

for all $x_1, \cdots, x_n \in S_0$ and all order-preserving linear functionals L from $\hat{S}$ into $\boldsymbol{R}$ such that $(L\hat{x}_1, \cdots, L\hat{x}_n) \in D$ for all $x_1, \cdots, x_n \in S_0$.

Definition. We say that the above function m (M) belongs to a class (m) = (m ; D, S, S_0, Φ) (resp. (M) = (M ; D, S, S_0, Φ)).

Remark 1. Let $\alpha_1, \cdots, \alpha_n \in \boldsymbol{R}$. Then the following function on D belongs to both (m) and (M):

$$f(a_1, \cdots, a_n) = \alpha_1 a_1 + \cdots + \alpha_n a_n \quad ((a_1, \cdots, a_n) \in D)$$

This is a trivial case but it gives to us an impotant suggestion for a construction of functions which belong to the class (m) or (M).

数学は例に始まり例に終わるといわれるが、クラス (m) と (M) に属する非自明な簡単例を掲げよう：

$D = R^+ \times R^+, S = R^2, S_0 = R^+ \times R^+$

$\Phi = \{\varphi_1, \varphi_2\}$, where φ_i is the i-th coordinate function

Then $(\varphi x_1, \varphi x_2) \in D$ for all $x_1, x_2 \in S_0$ and $\varphi \in \Phi$. In this case,

(i) $M(a, b) = \left(\sqrt{a} + \sqrt{b}\right)^2$, $M(a, b) = \sqrt{ab} \in$ (M).

(ii) $m(a, b) = \sqrt{a^2 + b^2}$, $m(a, b) = a^2 / b \in$ (m).

Remark 2. Let L be an order preserving linear functional on $\hat{S}$ such that $(L\hat{x}_1, L\hat{x}_2) \in D$ for all $x_1, x_2 \in S_0$. Let $\alpha = L(1, 0)^\wedge$ and $\beta = L(0, 1)^\wedge$ and set $x_1 = (a, b)$ and $x_2 = (c, d) \in S_0$.

(i) Let $\Omega = \{1, 2\}$, $\mu(1) = \alpha, \mu(2) = \beta$, $|f(1)| = \sqrt{a}$, $|f(2)| = \sqrt{b}$, $|g(1)| = \sqrt{c}$ and $|g(2)| = \sqrt{d}$. Then

$$M(L\hat{x}_1, L\hat{x}_2) \geq L((M(\hat{x}_1, \hat{x}_2)) \Leftrightarrow \| f \|_2 + \| g \|_2 \geq \big\| |f| + |g| \big\|_2 \quad \text{when } M(a, b) = \left(\sqrt{a} + \sqrt{b}\right)^2$$

and

$$M(L\hat{x}_1, L\hat{x}_2) \geq L((M(\hat{x}_1, \hat{x}_2)) \Leftrightarrow \| f \|_2 \| g \|_2 \geq \| fg \|_1 \quad \text{when } M(a, b) = \sqrt{ab}.$$

(ii) Let $\Omega = \{1, 2\}$, $\mu(1) = \alpha, \mu(2) = \beta$, $|f(1)| = a^2$, $|f(2)| = b^2$, $|g(1)| = c^2$ and $|g(2)| = d^2$. Then

$$m(L\hat{x}_1, L\hat{x}_2) \leq L((m(\hat{x}_1, \hat{x}_2)) \Leftrightarrow \| f \|_{1/2} + \| g \|_{1/2} \leq \big\| |f| + |g| \big\|_{1/2} \quad \text{when } m(a, b) = \sqrt{a^2 + b^2}.$$

Also let $\Omega = \{1, 2\}$, $\mu(1) = \alpha, \mu(2) = \beta$, $|f(1)| = a^2$, $|f(2)| = b^2$, $|g(1)| = \frac{1}{c}$, $|g(2)| = \frac{1}{d}$. Then

$$m(L\hat{x}_1, L\hat{x}_2) \leq L((m(\hat{x}_1, \hat{x}_2)) \Leftrightarrow \| f \|_{1/2} \| g \|_{-1} \leq \| fg \|_1 \quad \text{when } m(a, b) = a^2 / b.$$

3. (m) または (M) に属する関数の１つの構成法。

Let T be a set and suppose that $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n : T \to R$ satisfy the following properties:

(1) For each $(a_1, \cdots, a_n) \in D$, $m(a_1, \cdots, a_n) \equiv \sup_{t \in T} \alpha_1(t)a_1 + \cdots + \alpha_n(t)a_n \in R$ and $M(a_1, \cdots, a_n) \equiv \inf_{t \in T} \beta_1(t)a_1 + \cdots + \beta_n(t)a_n \in R$.

(2) $m(\hat{x}_1, \cdots, \hat{x}_n) \in \hat{S}$ and $M(\hat{x}_1, \cdots, \hat{x}_n) \in \hat{S}$ for each $x_1, \cdots, x_n \in S_0$.

In this case, we have the following

Lemma 1. m belongs to the class (m) and M belongs to the class (M).

Proof. Let $x_1, \cdots, x_n \in S_0$ and L an order-preserving linear functional from $\hat{S}$ into R such that $(L\hat{x}_1, \cdots, L\hat{x}_n) \in D$ for all $x_1, \cdots, x_n \in S_0$. Note that

$$\alpha_1(t)\hat{x}_1 + \cdots + \alpha_n(t)\hat{x}_n \leq m(\hat{x}_1, \cdots, \hat{x}_n) \text{ and } \beta_1(t)\hat{x}_1 + \cdots + \beta_n(t)\hat{x}_n \geq M(\hat{x}_1, \cdots, \hat{x}_n)$$

for all $t \in T$. Then

$$\alpha_1(t)L\hat{x}_1 + \cdots + \alpha_n(t)L\hat{x}_n = L(\alpha_1(t)\hat{x}_1 + \cdots + \alpha_n(t)\hat{x}_n) \leq L(m(\hat{x}_1, \cdots, \hat{x}_n))$$

and

$$\beta_1(t)L\hat{x}_1 + \cdots + \beta_n(t)L\hat{x}_n = L(\beta_1(t)\hat{x}_1 + \cdots + \beta_n(t)\hat{x}_n) \geq L(M(\hat{x}_1, \cdots, \hat{x}_n))$$

for all $t \in T$. Therefore

$$m(L\hat{x}_1, \cdots, L\hat{x}_n) \leq L(m(\hat{x}_1, \cdots, \hat{x}_n)) \text{ and } M(L\hat{x}_1, \cdots, L\hat{x}_n) \geq L(M(\hat{x}_1, \cdots, \hat{x}_n)),$$

so that m belongs to the class (m) and M belongs to the calss (M). Q. E. D.

Remark 3. If T consists of a single point, then the above result becomes to the trivial case.

Remark 4. We can consider the case that L is a sub-affine or super-affine map. It seems that this case is more natural than the linear case.

上の構成法に対する具体的な例を掲げる。

3-1. Holder function.

$D = \boldsymbol{R}^+ \times \cdots \times \boldsymbol{R}^+$, S : a real linear space with dual S^*, $S_0 \subseteq S$,

Φ : a subset of S^* such that $(\varphi x_1 \cdots, \varphi x_n) \in D$ for all $x_1, \cdots, x_n \in S_0$ and $\varphi \in \Phi$,

$\hat{s}(\varphi) = \varphi(s)$ $(\varphi \in \Phi, s \in S)$, $\hat{S} = \{\hat{s} : s \in S\}$, $p_1 + \cdots + p_n = 1$,

$\text{Hör}(a_1, \cdots, a_n) = \prod_{i=1}^{n} a_i^{p_i}$ $\left((a_1, \cdots, a_n) \in D\right)$, $\text{Hör}(\hat{x}_1, \cdots, \hat{x}_n) \in \hat{S}$ for all $x_1, \cdots, x_n \in S_0$.

In this case, we have the following

Lemma 2. (i) If all p_i are positive, then the function Hör belongs to the class (M).

(ii) If the only one of $\{p_1, \cdots, p_n\}$ is positive, then the function Hör belongs to the class (m).

Proof. Let $T = \boldsymbol{R}^+ \times \cdots \times \boldsymbol{R}^+$.

(i) Suppose that all p_i are positive and let $(a_1, \cdots, a_n) \in D$. For each $t = (t_1, \cdots, t_n) \in T$, we have

$$\sum_{i=1}^{n} p_i t_i a_i \geq \prod_{i=1}^{n} (t_i a_i)^{p_i}$$

and hence

$$\sum_{i=1}^{n} \left(p_i t_i \prod_{j=1}^{n} t_j^{-p_j} \right) a_i \geq \prod_{i=1}^{n} a_i^{p_i} = \text{Hör}(a_1, \cdots, a_n).$$

Set

$\beta_1(t) = p_1 t_1 \prod_{j=1}^{n} t_j^{-p_j}, \cdots, \beta_n(t) = p_n t_n \prod_{j=1}^{n} t_j^{-p_j}$ and $h(t, a_1, \cdots, a_n) = \beta_1(t) a_1 + \cdots + \beta_n(t) a_n$ for each $t = (t_1, \cdots, t_n) \in T$. Then we have

$$\inf_{t \in T} h(t, a_1, \cdots, a_n) \geq \text{Hör}(a_1, \cdots, a_n).$$

Also since $h(t_*, a_1, \cdots, a_n) = \text{Hör}(a_1, \cdots, a_n)$ for $t_* = (a_1^{-1}, \cdots, a_n^{-1}) \in T$, it follows that $\inf_{t \in T} h(t, a_1, \cdots, a_n) = \text{Hör}(a_1, \cdots, a_n)$. Therefore the desired result follows from Lemma 1.

(ii) Suppose that the only one of $\{p_1, \cdots, p_n\}$ is positive and let $(a_1, \cdots, a_n) \in D$. For each $t = (t_1, \cdots, t_n) \in T$, we have $\sum_{i=1}^{n} p_i t_i a_i \geq \prod_{i=1}^{n} (t_i a_i)^{p_i}$. Then the desired result follows from the similar argument in (i). Q. E. D.

3-2. Minkowski type function.

$D = \boldsymbol{R}^+ \times \cdots \times \boldsymbol{R}^+$, $f : \boldsymbol{R}^+ \to \boldsymbol{R}$: a concave (convex) function with inverse, $\rho : \boldsymbol{R}^+ \to \boldsymbol{R}$,

$f_\rho(a_1, \cdots, a_n) = f\left(\sum_{i=1}^{n} f^{-1}(\rho(a_i)) \right)$ $\left((a_1, \cdots, a_n) \in D\right)$,

$f_{\rho^*}(\tau) \equiv \exists \inf_{s>0} \frac{\tau}{s} f\left(\frac{f^{-1}(\rho(s))}{\tau} \right)$ $(0 < \tau < 1)$ (resp. $f_\rho^*(\tau) \equiv \exists \sup_{s>0} \frac{\tau}{s} f\left(\frac{f^{-1}(\rho(s))}{\tau} \right)$ $(0 < \tau < 1)$)

$T = \{t = (t_1, \cdots, t_n) : t_1 + \cdots + t_n = 1,\ t_1, \cdots, t_n > 0\}$,

$\alpha_1(t) = f_{\varphi^*}(t_1), \cdots, \alpha_n(t) = f_{\varphi^*}(t_n)$ $(t \in T)$ (resp. $\beta_1(t) = f_\rho^*(t_1), \cdots, \beta_n(t_n) = f_\rho^*(t)$ $(t \in T)$,

$$h, H : T \times D \to R \ : \ h(t, a_1, \cdots, a_n) = \alpha_1(t)a_1 + \cdots + \alpha_n(t)a_n ,$$
$$H(t, a_1, \cdots, a_n) = \beta_1(t)a_1 + \cdots + \beta_n(t)a_n .$$

In this case, we have the following

Lemma 3. (i) If f is concave, then $\sup_{t\in T} h(t, a_1, \cdots, a_n) \le f_\rho(a_1, \cdots, a_n))$ for each $(a_1, \cdots, a_n) \in D$.

(ii) If f is convex, then $\inf_{t\in T} H(t, a_1, \cdots, a_n) \ge f_\rho(a_1, \cdots, a_n)$ for each $(a_1, \cdots, a_n) \in D$.

Proof. (i) Let $(a_1, \cdots, a_n) \in D$. For each $t = (t_1, \cdots, t_n) \in T$, we have

$$\sum_{i=1}^{n} t_i f(b_i) \le f\left(\sum_{i=1}^{n} t_i b_i\right) \quad ((b_1, \cdots, b_n) \in D),$$

and hence by putting $b_1 = f^{-1}(\rho(a_1)) / t_1, \cdots, b_n = f^{-1}(\rho(a_1)) / t_n$ in the above inequality,

$$\sum_{i=1}^{n} t_i f\left(\frac{f^{-1}(\rho(a_i))}{t_i}\right) \le f\left(\sum_{i=1}^{n} f^{-1}(\rho(a_i))\right).$$

Note also that $\sum_{i=1}^{n} f_{\rho^*}(t_i)a_i \le \sum_{i=1}^{n} t_i f\left(\frac{f^{-1}(\rho(a_i))}{t_i}\right)$ for each $t = (t_1, \cdots, t_n) \in T$. Then we have

$$\sum_{i=1}^{n} f_{\rho^*}(t_i)a_i \le f\left(\sum_{i=1}^{n} f^{-1}(\rho(a_i))\right)$$

for each $t = (t_1, \cdots, t_n) \in T$. Then we have desired result.

(ii) Similarly to the concave case. Q. E. D.

Definition. We say that f_φ is of Minkowski type if

$$f_\rho(a_1, \cdots, a_n) = \sup_{t\in T} h(t, a_1, \cdots, a_n) \ \text{(when } f \text{ is concave)}$$
$$= \inf_{t\in T} H(t, a_1, \cdots, a_n) \ \text{(when } f \text{ is convex)}$$

for each $(a_1, \cdots, a_n) \in D$.

In particular, let $p \ne 0$ and set $f(t) = t^p, \ \rho(t) = t \ (t > 0)$. Then

$$f_\rho(a_1, \cdots, a_n) = \left(a_1^{1/p} + \cdots + a_n^{1/p}\right)^p$$

is a Minkowski type function on D.

Let $D = R^+ \times \cdots \times R^+$, S : a real linear space with dual S^*, $S_0 \subseteq S$,

Φ : a subset of S^* such that $(\varphi x_1 \cdots, \varphi x_n) \in D$ for all $x_1, \cdots, x_n \in S_0$ and $\varphi \in \Phi$, $\hat{s}(\varphi) = \varphi(s)$ $(\varphi \in \Phi, s \in S)$, $\hat{S} = \{\hat{s} : s \in S\}$ and $f_\varphi(\hat{x}_1, \cdots, \hat{x}_n) \in \hat{S}$ for all $x_1, \cdots, x_n \in S_0$.

Then we have from Lemma 1 that

Lemma 4. (i) If f is concave, then f_ρ belongs to the class (m).

(ii) If f is convex, then f_ρ belongs to the class (M).

4．応用。

上の事柄から適当な空間を設定することにより、generalized Holder's inequality :

(i) $$\int |f_1|^{p_1} \cdots |f_n|^{p_n} d\mu \le \left(\int |f_1| d\mu\right)^{p_1} \cdots \left(\int |f_n| d\mu\right)^{p_n}$$

if all p_i are positive and $p_1 + \cdots + p_n = 1$,

(ii) $$\int |f_1|^{p_1} \cdots |f_n|^{p_n} d\mu \ge \left(\int |f_1| d\mu\right)^{p_1} \cdots \left(\int |f_n| d\mu\right)^{p_n}$$

if the only one of $\{p_1, \cdots, p_n\}$ is positive and $p_1 + \cdots + p_n = 1$,
及び generalized Minkowski's inequality :

(iii) $$\int \left(|f_1|^{1/p} + \cdots + |f_n|^{1/p} \right)^p d\mu \geq \left(\left(\int |f_1| d\mu \right)^{1/p} + \cdots + \left(\int |f_n| d\mu \right)^{1/p} \right)^p$$

if $0 < p < 1$,

(iv) $$\int \left(|f_1|^{1/p} + \cdots + |f_n|^{1/p} \right)^p d\mu \leq \left(\left(\int |f_1| d\mu \right)^{1/p} + \cdots + \left(\int |f_n| d\mu \right)^{1/p} \right)^p$$

if $p > 1$ or $p < 0$ を得ることができる。

5．問題。

5-1. Define a Holder type function.

5-2. Find a resonable new Minkowski type function.