数理解析研究所講究録 1301 巻 2003 年 39-47

Some correspondences of reduction-procedures between natural deduction and sequent calculus

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Abstract

An interpretation from LK to NK with one conclusion and full logical symbols is defined. A procedure for transforming LK-proofs is shown such that one can recognize the existence of the redexes in the corresponding NK-proofs.

1 Introduction

Gentzen [3] introduced two types of logical systems. One is natural deduction and the other is sequent calculus. He proved his Hauptsatz as the cut-elimination theorem for the sequent calculus LK and LJ. Later, Prawitz [5, 6] proved Gentzen's Hauptsatz as the normalizatioin theorem for the natural deduction NK (in some restriction) and NJ. In general, the normalization procedure for natural deduction works 'faster' than the cut-elimination procedure for sequent calculus. Moreover, the former has stronger property than the latter such as the strong normalization theorem [5, 6, 4] and the Church-Rosser property [5, 6, 2]. In this paper, we introduce an interpretation from LK-proofs to NK-proofs. Here, NK means the classical natural deduction with one conclusion and containing full logical symbols primitively. Then we discuss a classification of cuts in LK according to their roles in NK and define procedures for eliminating the cuts in LK which cause no redex in NK. But the investigation of the representation in LK for the correspondings of the normalization procedure in NK is our further work. The construction of this paper is as follows. In the next section, we show our system of NK which is expressed in a system of typed terms. That is introduced in our previous papers [1, 2]. In section 3, we define our interpretation of LK to NK. In section 4, we define some kinds of cuts in LK which does not correspond with redexes in NK, and we show our lemmata which say that we can remove such inessential cuts.

2 λ^C-calculus for the representation of NK

In this section, we define the system of λ^C-calculus which is an extension of typed λ-calculus in Church-style. λ^C-calculus corresponds with classical natural deduction NK with one conclusion and full logical symbols. The system is introduced in [1, 2].

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2.1 Typed-terms

Types are formulas of a first order language $\mathcal{L}$ which contains $\supset$, $\wedge$, $\vee$, $\forall$, and $\exists$ as logical symbols, and $\bot$ as a propositional constant. We use $\neg A$ as an abbreviation of the formula $A \supset \bot$. Two formulas are identical if they vary only by bound variables. We use the following letters, possibly with sub- or superscripts, as metavariables.

- a, b, c : for individual variables of $\mathcal{L}$.
- p, q : for $\mathcal{L}$-terms.
- A, B, C, D : for $\mathcal{L}$-formulas.
- x, y, z : for λ^C-variables.
- r, s, t, u, v, w : for λ^C-terms.

There are assumed to be uncountably many λ^C-variables. *Typed λ^C-terms* and their *free λ^C-variables* are defined as follows, where the set of all free λ^C-variables of a λ^C-term t is denoted by $FV(t)$:

- If x is a λ^C-variable and A a $\mathcal{L}$-formula, then (x^A) is a λ^C-term of type A, and $FV((x^A)) = \{x^A\}$. Moreover, we call the term (x^A) an *axiom-term.*
- By the following schemata, we explain other constructions of terms. A schema of the form

$$\frac{\begin{array}{c}[x_1^{A_1}]\\ t_1 : B_1\end{array} \quad \cdots \quad \begin{array}{c}[x_n^{A_n}]\\ t_n : B_n\end{array}}{u : C}\ (*)$$

means that: if t_i is a λ^C-term of type B_i for each i, then u is a λ^C-term of type C, and $FV(u) = \bigcup_{1\leq i\leq n}(FV(t_i) \setminus \{x_i^{A_i}\})$. Moreover, we call u a *(*)-term.*

$$\frac{\begin{array}{c}[x^A]\\ t : B\end{array}}{(\lambda x^A.t) : A \supset B}\ (\supset I), \qquad \frac{t : A \supset B \quad u : A}{t(u) : B}\ (\supset E),$$

$$\frac{t : A \quad u : B}{\langle t, u\rangle : A \wedge B}\ (\wedge I), \quad \frac{t : A \wedge B}{t(\pi 0) : A}\ (\wedge E_0), \quad \frac{t : A \wedge B}{t(\pi 1) : B}\ (\wedge E_1),$$

$$\frac{t : A}{\{t, B\} : A \vee B}\ (\vee I_0), \quad \frac{t : B}{\{A, t\} : A \vee B}\ (\vee I_1), \quad \frac{t : A \vee B \quad \begin{array}{c}[x^A]\\ u : C\end{array} \quad \begin{array}{c}[y^B]\\ v : C\end{array}}{t(\lambda^{\vee} x^A.u, \lambda^{\vee} y^B.v) : C}\ (\vee E),$$

$$\frac{t : A}{(\lambda^{\forall} a.t) : \forall a A}\ (\forall I), \quad \frac{t : \forall a A}{t(p) : A[p/a]}\ (\forall E),$$

$$\frac{t : A[p/a]}{(\sigma_A^{p,a} t) : \exists a A}\ (\exists I), \quad \frac{t : \exists a A \quad \begin{array}{c}[x^A]\\ u : C\end{array}}{t(\lambda^{\exists} x^A.u) : C}\ (\exists E),$$

$$\frac{\begin{array}{c}[x^{\neg C}]\\ t:\bot\end{array}}{(\lambda^{\bot}x^{\neg C}.t):C}\ (\bot_c).$$

In the schemata above, $A[p/a]$ represents the type obtained from A by substituting p for all free occurrences of a in A. The constructions $(\forall I)$ and $(\exists E)$ are subject to the usual eigenvariable conditions for $\mathcal{L}$-variables. That is, in $(\forall I)$, the $\mathcal{L}$-variable a does not occur in the types of any free λ^C-variables in t, and in $(\exists E)$, the λ^C-variable a does not occur in the types of any free λ^C-variables in u except x^A, nor in C.

Note that any λ^C-term ends with a right bracket), $\rangle$, or } as a sequence of letters.

Eliminator. In each of the schemata $(\supset E)$, $(\wedge E)$, $(\vee E)$, $(\forall E)$, and $(\exists E)$, the newly constructed term is represented in the form $t\varepsilon$. We call this expression ε an *eliminator* from D_0 to D_1 where D_0 and D_1 are the type of t and $t\varepsilon$ respectively.

Note that an eliminator is a sequence of letters which starts with a left round bracket and ends with a right round bracket. Moreover, an eliminator itself is not a λ^C-term.

2.2 $\lambda^{\bot}$-regularity

An axiom term $(x^{\neg C})$ is $\lambda^{\bot}$-*nice* in a term t iff for every occurrence of $(x^{\neg C})$ in t there exists a term u of type C such that the occurrence of $(x^{\neg C})$ is in the form $(x^{\neg C})(u)$ as a subterm of t and also the occurrence of $(x^{\neg C})$ is not bound in t. A term w is $\lambda^{\bot}$-*regular* iff for every subterm $(\lambda^{\bot}x^{\neg C}.t)$ in w, $(x^{\neg C})$ is $\lambda^{\bot}$-nice in t.

To simplify the representation of the reduction, we assume that every λ^C-term is $\lambda^{\bot}$-regular. This causes no loss of generality, because a non-regular $\lambda^{\bot}$-abstraction can be transformed to a regular one as follows. If an occurrence $(x^{\neg C})$ violates the regularity of the $\lambda^{\bot}$-abstraction $(\lambda^{\bot}x^{\neg C}.t)$, then replace the occurrence $(x^{\neg C})$ by $(\lambda y^C.(x^{\neg C})(y^C))$ using a new variable y.

The reduction in the λ^C-calculus will be defined in the next subsection and will have the property that the contractum of any $\lambda^{\bot}$-regular λ^C-term is also $\lambda^{\bot}$-regular.

For a term $(\lambda^{\bot}x^{\neg C}.t)$, we use a notation for brackets as follows: for every occurrence of subterms of the form $(x^{\neg C})(u)$ in t, the right-most bracket of the subterm is indexed by x, that is, the subterm is denoted by $(x^{\neg C})(u)_x$.

2.3 Reductions

Now we define two kinds of contractions of λ^C-terms as follows. One is called *essential* and the other *structural*. They are denoted by $\rhd_e$ and $\rhd_s$ respectively.

- $(\lambda x^A.t)(u) \rhd_e t[u/x^A]$
- $\langle t,u\rangle(\pi 0) \rhd_e t,\quad \langle t,u\rangle(\pi 1) \rhd_e u$
- $\{t,B\}(\lambda^{\vee}x^A.u,\lambda^{\vee}y^B.v) \rhd_e u[t/x^A],\quad \{A,t\}(\lambda^{\vee}x^A.u,\lambda^{\vee}y^B.v) \rhd_e v[t/y^B]$
- $(\lambda^{\forall}a.t)(p) \rhd_e t[p/a]$
- $(\sigma^{p,a}_A t)(\lambda^{\exists}x^A.u) \rhd_e u[p/a;t/x^A]$

- $t(\lambda^{\vee} x^A.u, \lambda^{\vee} y^B.v)\varepsilon \rhd_s t(\lambda^{\vee} x^A.u\varepsilon, \lambda^{\vee} y^B.v\varepsilon)$
- $t(\lambda^{\exists} x^A.u)\varepsilon \rhd_s t(\lambda^{\exists} x^A.u\varepsilon)$
- $(\lambda^{\bot} x^{\neg C}.t)\varepsilon \rhd_s (\lambda^{\bot} x^{\neg D}.t[\ \varepsilon)_x/\ )_x, x^{\neg D}/x^{\neg C}])$

In the schemata above, ε represents an eliminator from C to D. The expression $t[u/x^A]$ represents the term obtained from t by substituting u for all occurrences (x^A) in t. The expression $t[\ \varepsilon)_x/\ )_x, x^{\neg D}/x^{\neg C}]$ represents the term obtained from t by substituting $\varepsilon)_x$ for all occurrences $)_x$ and $x^{\neg D}$ for all occurrences $x^{\neg C}$ in t simultaneously. The expression $t[p/a]$ represents the term obtained from t by substituting p for all occurrences a in t. Moreover, $u[p/a; t/x^A]$ is the abbreviation of $u[p/a][t/x^{A[p/a]}]$.

Note that our structural contractions consist of commutative contractions concerning elimination rules for disjunction and existential quantifier, and the same kind of contractions for the redex formed by a classical absurdity rule and an elimination rule.

3 Interpretation of LK to λ^C-calculus

In this section, we define a mapping from LK-proofs to λ^C-terms, and show some basic property of the mapping.

3.1 The mapping ϕ

ϕ maps each LK-proof $\mathcal{D}$ to a λ^C-term in the following conditions.

- If the conclusion of $\mathcal{D}$ is of the form $A_1, \cdots, A_n \to B_1, \cdots, B_m, C$, then the set of all free variables in $\phi\mathcal{D}$ is included in the set $\{x_1^{A_1}, \cdots, x_n^{A_n}, y_1^{\neg B_1}, \cdots, y_m^{\neg B_m}\}$, and the type of $\phi\mathcal{D}$ is C.
- If the conclusion of $\mathcal{D}$ is of the form $A_1, \cdots, A_n \to$, then the set of all free variables in $\phi\mathcal{D}$ is included in the set $\{x_1^{A_1}, \cdots, x_n^{A_n}\}$, and the type of $\phi\mathcal{D}$ is $\bot$.

We call each $x_i^{A_i}$ a *left-side corresponding variable* and each $y_j^{\neg B_j}$ a *right-side corresponding variable* of $\mathcal{D}$.

$\phi\mathcal{D}$ is defined by the induction on the construction of $\mathcal{D}$.

- If $\mathcal{D}$ is an axiom of the form $A \to A$, then $\phi\mathcal{D}$ is (x^A).
- By the following schemata, we explain other constructions of $\phi\mathcal{D}$. A schema of the form

  $$\frac{\Gamma_1 \to \Theta_1[t_1 : C_1], \cdots, \Gamma_n \to \Theta_n[t_n : C_n]}{\Gamma_0 \to \Theta_0[t_0 : C_0]}$$

  means that: if the LK-proofs whose last sequent are the upper sequents $\Gamma_1 \to \Theta_1, \cdots, \Gamma_n \to \Theta_n$ are mapped by ϕ to t_0 of type C_0, $\cdots$, t_n of type C_n respectively, then the LK-proof whose last sequent is the lower sequent $\Gamma_0 \to \Theta_0$ is mapped by ϕ to t_0 of type C_0,

$$\frac{\Gamma \to \Theta, A\ [u : A] \quad B, \Delta \to \Lambda\ [t : C]}{A \supset B, \Gamma, \Delta \to \Theta, \Lambda\ [t[(z^{A \supset B})(u)/x^B] : C]}\ (\supset L)$$ if Θ is empty or Λ is not empty,

$$\frac{\Gamma \to \Theta, A\ [u : A] \quad B, \Delta \to \Lambda\ [t : C]}{A \supset B, \Gamma, \Delta \to \Theta, \Lambda\ [(\lambda^{\perp} y_n^{\neg C_n}.t[(z^{A \supset B})(u)/x^B]) : C_n]}\ (\supset L)$$ otherwise, where $\Theta \equiv C_1, \cdots, C_n$,

$$\frac{A, \Gamma \to \Theta, B\ [t : B]}{\Gamma \to \Theta, A \supset B\ [(\lambda x^A.t) : A \supset B]}\ (\supset R),$$

$$\frac{A, \Gamma \to \Theta\ [t : C]}{A \wedge B, \Gamma \to \Theta\ [t[(z^{A \wedge B})(\pi 0)/x^A] : C]}\ (\wedge L_0), \quad \frac{B, \Gamma \to \Theta\ [t : C]}{A \wedge B, \Gamma \to \Theta\ [t[(z^{A \wedge B})(\pi 1)/x^B] : C]}\ (\wedge L_1),$$

$$\frac{\Gamma \to \Theta, A\ [t : A] \quad \Gamma \to \Theta, B\ [u : B]}{\Gamma \to \Theta, A \wedge B\ [\langle t, u \rangle : A \wedge B]}\ (\wedge R)$$

$$\frac{A, \Gamma \to \Theta\ [u : C] \quad B, \Gamma \to \Theta\ [v : C]}{A \vee B, \Gamma \to \Theta\ [(z^{A \vee B})(\lambda^{\vee} x^A.u, \lambda^{\vee} y^B.v) : C]}\ (\vee L)$$

$$\frac{\Gamma \to \Theta, A\ [t : A]}{\Gamma \to \Theta, A \vee B\ [\{t, B\} : A \vee B]}\ (\vee R_0), \quad \frac{\Gamma \to \Theta, B\ [t : B]}{\Gamma \to \Theta, A \vee B\ [\{A, t\} : A \vee B]}\ (\vee R_1),$$

$$\frac{\Gamma \to \Theta, A\ [u : A]}{\neg A, \Gamma \to \Theta\ [(z^{\neg A})(u) : \perp]}\ (\neg L)$$ if Θ is empty, $$\frac{\Gamma \to \Theta, A\ [u : A]}{\neg A, \Gamma \to \Theta\ [(\lambda^{\perp} y_n^{\neg C_n}.(z^{\neg A})(u)) : C_n]}\ (\neg L),$$ otherwise, where $\Theta \equiv C_1, \cdots, C_n$,

$$\frac{A, \Gamma \to \Theta\ [t : C]}{\Gamma \to \Theta, \neg A\ [(\lambda x^A.t) : \neg A]}\ (\neg R)$$ if Θ is empty, $$\frac{A, \Gamma \to \Theta\ [t : C]}{\Gamma \to \Theta, \neg A\ [(\lambda x^A.(z^{\neg C})(t)) : \neg A]}\ (\neg R),$$ otherwise,

$$\frac{A[p/a], \Gamma \to \Theta\ [t : C]}{\forall a A, \Gamma \to \Theta\ [t[(z^{\forall a A})(p)/x^{A[p/a]}] : C]}\ (\forall L), \quad \frac{\Gamma \to \Theta, A\ [t : A]}{\Gamma \to \Theta, \forall a A\ [(\lambda^{\forall} a.t) : \forall a A]}\ (\forall R),$$

$$\frac{A, \Gamma \to \Theta\ [u : C]}{\exists a A, \Gamma \to \Theta\ [(z^{\exists a A})(\lambda^{\exists} x^A.u) : C]}\ (\exists L), \quad \frac{\Gamma \to \Theta, A[p/a]\ [t : A[p/a]]}{\Gamma \to \Theta, \exists a A\ [(\sigma_A^{p,a} t) : \exists a A]}\ (\exists R),$$

$$\frac{\Gamma \to \Theta\ [t : C]}{A, \Gamma \to \Theta\ [t : C]}\ (WL),$$

$$\frac{\Gamma \to \Theta\ [t : C]}{\Gamma \to \Theta, A\ [(\lambda^{\perp} z^{\neg A}.t) : A]}\ (WR)$$ if Θ is empty, $$\frac{\Gamma \to \Theta\ [t : C]}{\Gamma \to \Theta, A\ [(\lambda^{\perp} z^{\neg A}.(y^{\neg C})(t)) : A]}\ (WR)$$ otherwise,

$$\frac{A, A, \Gamma \to \Theta\ [t : C]}{A, \Gamma \to \Theta\ [t[z^A/x_1^A; z^A/x_2^A] : C]}\ (CL),$$

$\dfrac{\Gamma \to \Theta, A, A\ [t : A]}{\Gamma \to \Theta, A\ [(\lambda^\perp y^{\neg A}.(y^{\neg A})(t)) : A]}\ (CR)$ where $y^{\neg A}$ is the right-side corresonding variable for the occurrence A in the upper sequent which is at the second place from right,

$$\frac{\Delta, A, B, \Gamma \to \Theta\ [t : C]}{\Delta, B, A, \Gamma \to \Theta\ [t : C]}\ (EL)$$

$\dfrac{\Gamma \to \Theta, A, B, \Lambda\ [t : C]}{\Gamma \to \Theta, B, A, \Lambda\ [t : C]}\ (ER)$ if Λ is not empty, $\dfrac{\Gamma \to \Theta, A, B, \Lambda\ [t : C]}{\Gamma \to \Theta, B, A, \Lambda\ [(\lambda^\perp x^{\neg A}.(z^{\neg B})(t)) : A]}\ (ER)$ otherwise,

$\dfrac{\Gamma \to \Theta, A\ [u : A] \quad A, \Delta \to \Lambda\ [t : C]}{\Gamma, \Delta \to \Theta, \Lambda\ [t[u/x^A] : C]}\ (Cut)$ if Θ is empty or Λ is not empty, where x^A is the left-side corresonding variable for the occurrence A in the right upper sequent, $\dfrac{\Gamma \to \Theta, A\ [u : A] \quad A, \Delta \to \Lambda\ [t : C]}{\Gamma, \Delta \to \Theta, \Lambda\ [(\lambda^\perp y_n^{\neg C_n}.t[u/x^A]) : C_n]}\ (Cut)$ otherwise, where $\Theta \equiv C_1, \cdots, C_n$, and x^A is as of the previous case.

Fact For any LK-proof $\mathcal{D}$, $\phi\mathcal{D}$ is $\lambda^\perp$-regular.

Proof If a λ^C-variable $x^{\neg C}$ is newly bounded by a $\lambda^\perp$in a step of the construction of $\phi\mathcal{D}$, then $x^{\neg C}$ is one of the right-side corresponding variables in the previous step. On the other hand, every right-side corresponding variables in $\mathcal{D}$ is λ^C-nice in $\phi\mathcal{D}$. □

We have also the next fact by definitions.

Fact If a LK-proof $\mathcal{D}$ is cut-free, then $\phi\mathcal{D}$ is normal.

Notice There exists a LK-proof which has cuts but whose image of ϕ is normal. For example, the following proof has such property:

$$\frac{A \to A \quad A \to A}{A \to A}$$

4 Correspondences between cuts and redexes

Definition (*assumption-predecessor*) Let F be an occurrence of a formula in the antecedent of a sequent in a LK-proof. An *assumption-predecessor* of F is defined as follows. (i) If F is $A \wedge B$, $A \wedge B$, $A \supset B$, $\forall aA$, A, A, B in the lower sequent of the schema $(\wedge L_0)$, $(\wedge L_1)$, $(\supset L)$, $(\forall L)$, (CL), (EL), (EL) respectively, then A, B, B, $A[p/a]$, two A's, A, B in the upper sequent of the schema is the assumption-predecessor of F. (ii) If F is one of the formula in Γ or Δ in the lower sequent of a schema, then the corresponding formulae in Γ's or Δ respectively in the upper sequents of the schema are the assumption-predecessors of F. (iii) Otherwise, F has no assumption-predecessor.

Definition (*weakening-assumption*) Let F be an occurrence of a formula in the antecedent of a sequent in a LK-proof. We call F a weakening-assumption if one of the following conditions holds. We define this notion inductively on the construction of LK-proofs. (i) F is the weakening formula A of the schema (WL). (ii) F is one of the formulae in Γ in the lower sequent of the schema $(\supset L)$, and B in the right upper sequent of the schema is a weakening-assumption. (iii) F is one of the formulae in Γ in the lower sequent of the schema (Cut), and A in the right upper sequent of the schema is a weakening-assumption.

We have the following two facts by induction on the construction of the LK-proof $\mathcal{D}$.

Fact Let $\Gamma, A, \Delta \to \Theta$ be the conclusion of a LK-proof $\mathcal{D}$, and x^A the left-side corresponding variable in $\phi\mathcal{D}$ for the occurrence A. Then the following conditions are equivalent: (i) x^A does not occur in $\phi\mathcal{D}$. (ii) The occurrence A is a weakening-assumption.

Fact Suppose $\Gamma, A, \Delta \to \Theta$ is the conclusion of a LK-proof $\mathcal{D}$, and the occurrence A is a weakening-assumption. Then we can transform $\mathcal{D}$ to a LK-proof $\mathcal{D}'$ whose conclusion is $\Gamma, \Delta \to \Theta$.

Definition (*weakening-assumtion-inference*) Let A (B) be the occurrence of the formula A (B resp.) in the right upper sequent $A, \Delta \to \Lambda$ ($B, \Delta \to \Lambda$ resp.) of an instance R of (Cut) rule ($(\supset L)$ rule resp.) in a LK-proof. We call R a *weakening-assumption-cut* (*weakening-assumption-*$(\supset L)$ resp.) if the occurrence A (B resp.) is a weakening-assumption. Moreover, we call an instance R of an inference rule in a LK-proof a (*weakening-assumption-inference*) if R is a weakening-assumption-cut or a weakening-assumption-$(\supset L)$.

By the previous fact, we have the next lemma.

Lemma 1 For any given LK-proof $\mathcal{D}$, we can transform $\mathcal{D}$ to an LK-proof $\mathcal{D}'$ of the same conclusion which has no weakening-assumption-inference.

Definition (*axiom-assumption*) Let F be an occurrence of a formula in the antecedent of a sequent in a LK-proof. We call F an *axiom-assumption* if F is not a weakening-assumption, and if one of the following conditions holds. We define this notion inductively on the construction of LK-proofs. (i) F is the formula-occurrence in the antecedent of an axiom sequent. (ii) F has one and only one assumptioin-predecessor which is identical as formula with F, and which is an axiom-assumption. (iii) F has two assumption-predecessor which are all axiom-assumptions, or one of which is an axiom-assumption and the other is a weakening-assumption.

We have the next fact by definitions.

Fact Let $\Gamma, A, \Delta \to \Theta$ be the conclusion of a LK-proof $\mathcal{D}$, and x^A the left-side corresponding variable in $\phi\mathcal{D}$ of the occurrence A. Then following two conditions are equivalent. (i) A is an axiom-assumption. (ii) x^A occurs in $\phi\mathcal{D}$, and any occurrence of x^A in $\phi\mathcal{D}$ is not of the form $(x^A)\varepsilon$ where ε is an eliminator.

Definition (*axiom-assumption-cut*) For an instance R of cut rule in a LK-proof, we call R an *axiom-assumption-cut* if the occurrence of the cut-formula A in the right upper sequent $A, \Delta \to \Lambda$ of R is an axiom-assumption.

Fact Suppose $\mathcal{D}_0$ is a LK-proof of $\Delta_0, A, \Delta_1 \to \Lambda$ which has no axiom-assumption-cut and the occurrence A in the last sequent of $\mathcal{D}_0$ is an axiom-assumption. Let $\mathcal{D}_1$ be a LK-proof of $\Gamma \to \Theta, A$. Then we can transform $\mathcal{D}_0$ to a LK-proof $\mathcal{D}'_0$ of $\Delta_0, \Gamma, \Delta_1 \to \Theta, \Lambda$ which has no axiom-assumption-cut. Moreover, if $\mathcal{D}_0$ and $\mathcal{D}_1$ have no weakening-assumption-rule, neither has $\mathcal{D}'_0$.

Proof By induction on the construction of $\mathcal{D}_0$. □

Using this fact and Lemma 1, we have the next lemma.

Lemma 2 For any given LK-proof $\mathcal{D}$, we can transform $\mathcal{D}$ to a LK-proof of the same conclusion which has neither weakening-assumption-rule nor axiom-assumption-cut.

Definition (*smooth $(\supset L)$, (Cut), (ER)*) We say that an instance R of $(\supset L)$ or (Cut) rule is *smooth* if the succedent of right upper sequent of R is not empty. We also say that an instance R of (ER) rule is *smooth* if the right-most formula in the succedent of upper sequent of R is not exchanged by R.

Definition (*smooth LK-proof*) Let $\mathcal{D}$ be a LK-proof. We say that $\mathcal{D}$ is *smooth* if $\mathcal{D}$ is an axiom or if the last inference of D is $(\wedge L_0)$, $(\wedge L_1)$, smooth $(\supset L)$, $(\forall L)$, (WL), (CL), (EL), smooth (ER), or smooth (Cut).

Definition (*slipping cut*) Let R be an instance of cut rule in a LK-proof $\mathcal{D}$. We call R a slipping cut if the subproof of $\mathcal{D}$ whose last seqeunt is the left upper sequent of R is smooth.

Lemma 3 Let $\mathcal{D}$ be a LK-proof whose last inference R is an instance of cut rule. If R is not a weakening-assumption-cut, an axiom-assumption-cut, nor a slipping cut, then $\phi\mathcal{D}$ is not normal.

Proof Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be the subproofs of $\mathcal{D}$ whose last sequent is left and right upper sequent of R respectively. Let $A, \Delta \to \Lambda$ be the conclusion of $\mathcal{D}_1$ and x^A the left-side corresponding variable in $\phi\mathcal{D}_1$ of the occurrence A. $\phi\mathcal{D}_0$ is $(\wedge I)$-, $(\vee I_0)$-, $(\vee I_1)$-, $(\supset I)$-, $(\forall I)$-, $(\exists I)$-, $(\vee E)$-, $(\exists E)$-, or $(\bot_c)$-term, because R is not a slipping cut. On the other hand, x^A occurs in $\phi\mathcal{D}_1$ because R is not a weakening-assumption-cut, and also there exists at least one occurrence of x^A in the form of $(x^A)\varepsilon$ where ε is an eliminator because R is not an axiom-assumption-cut. Therefore, $\phi\mathcal{D}$ has a

Fact Suppose $\mathcal{D}$ is a LK-proof whose last inference R is a slipping cut and there exists no slipping cut in $\mathcal{D}$ above R. Then we can transform $\mathcal{D}$ to a LK-proof $\mathcal{D}'$ of the same conclusion in which there exists no slipping cut. Moreover, if $\mathcal{D}$ has neither weakening-assumption-rule nor axiom assumption-cut, then neither has $\mathcal{D}'$.

Proof By induction on the constraction of the LK-proof whose last sequent is the left upper sequent of $\mathcal{D}$. □

By this fact and Lemma 2, we have the next lemma.

Lemma 4 For any given LK-proof $\mathcal{D}$, we can transform $\mathcal{D}$ to a LK-proof of the same conclusion which has neither weakening-assumption-rule, axiom-assumption-cut, nor slipping cut.

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