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Dynamic Asset Allocation under Uncertainty

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Abstract

In this paper, we analyze temporal learning effects of the asset allocation decision of an investor, who has a long investment horizon. The investor has an uncertainty about the mean return of the risky stock (the state variable). Based on the work of Brennan (1998), it is shown theoretically that in the case with only the uncertainty of the mean return of the stock, the investor tends to increase an investment ratio on the risky stock by learning about the real state variable as time passes. The learning effect works in two ways, the reduction of the state variable uncertainty and the improvement of the state variable assessment. That is, the investor tends to decrease the hedge demand from the uncertainty of the state variable. The investor also improves the assessment of the state variable using the observed stock price at the same time.

1 Introduction

In this paper, we consider an asset allocation problem with an uncertainty of a state variable. Especially, we analyze the simple case when the investment opportunity set is constant in time, and the variance of the stock return is known in advance but the expected stock return (the state variable) is uncertain. Brennan (1998) uses the same setting and shows the relationship between the investment fraction of the stock and the remaining period by using numerical examples based on the Ibbotson and Sinquefield (1995) data. But we will analytically examine the temporal change of the fraction solving the stochastic differential equation of the (estimated) expected state variables. In this way, we can see the forward looking change of the investment fraction on the stock.

Asset allocation theory started with the setting that an investor maximizes his utility in a single period horizon, and an investment opportunity set is known. Merton (1971) extended the setting to the multi–period horizon with a known investment opportunity set. With this setting the investor knows an expected stock return, i.e., the investment opportunity set. Under the assumption of an iso–elastic utility function, the investor's investment fraction of the stock is constant with this setting.

Detemple (1986), Dothan and Feldman (1986) and Gennotte (1986) extended the study further with an uncertainty of an expected stock return in a continuous–time setting. The investor cannot observe the state variables but knows the stochastic law of the state variable's process. In this setting, the investor can learn about the unobservable state variables observing the stock returns, and the real values of the state variables are gradually getting to be revealed to the investor. These authors show that the investor with non–logarithmic utility hedges against the uncertainty of the state variables. But the clearer the state variables are as time passes, the closer the investor's investment fraction of the stock is to the case without the uncertainty.

Brennan, Schwartz and Lagnado (1997) apply the Merton's continuous–time model to the data analysis of the dynamic asset allocation in the case without the uncertainty of the state variables. They analyze the dynamic asset allocation using the data of a few state variables, but they ignore the fact that the stochastic processes of the state variables are estimated. So the results include the problem of the estimation risk. To find out the effect of the estimation risk on the investment fraction, Kandel and Stambaugh (1996) consider the estimation risk in a discrete–time model. They examine stock return predictability and the effect of estimation risk when asset returns are partially predictable and the coefficients of the predictive relation are estimated rather than known. With this setting, they show that uncertainty about the parameters of the conditional return distribution (estimation risk) affects the investor's optimal portfolio decision. Xia (2001) treats the estimation risk with the uncertainty of the state variables in a continuous model. She examines the effects of the uncertainty about stock return predictability on optimal dynamic portfolio choice. She shows that the investor hedges not only against the uncertainty of the state variables but also against the uncertainty of the estimated parameter. In this paper, we consider the simple model that the investment opportunity set is constant, and the expected return of the stock is unknown in the absence of the estimated risk. By considering the simple case, the temporal change of the investment fraction can be obtained theoretically.

The paper is organized as follows. In Section 2, the continuous model of the asset process is defined. In Section 3, we solve the investor's optimization problem. In Section 4, we consider the complete information case that there is no uncertainty about the state variable. In Section 5, we discuss the temporal change of the investment fraction on the stock. Section 6 is devoted to a discrete–time approximate model. Section 7 concludes the paper.

2 Continuous–Time Model

Let us consider an investor with a long horizon who maximizes the expected bequest at the end of the horizon, T (> 0). The investor can trade continuously in a riskless asset or a single risky stock. The real return on the riskfree asset is assumed to be constant, r. The stock price process $(S(t); t \in [0,T])$ is assumed to follow a stochastic differential equation with a drift affected by an unobservable state variable process $(\mu(t); t \in [0,T])$:

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu(t)\mathrm{d}t + \sigma \mathrm{d}B_1(t), \quad t \in [0,T], \tag{1}$$

where $S(0)$ is a constant and σ (> 0)is a constant diffusion parameter, and $(\mu(t))$ follows a stochastic differential equation:

$$\mathrm{d}\mu(t) = a\mu(t)\mathrm{d}t + b\mathrm{d}B_2(t), \quad t \in [0,T], \tag{2}$$

where $\mu(0)$ is a random variable, a and b are constant parameters, and $((B_1(t), B_2(t)); t \in [0,T])$ is a two dimensional standard Brownian motion. All of uncertainties in the economy are assumed to be generated by $\mu(0)$ and $((B_1(t), B_2(t))$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $W(t)$ denote the investor's wealth at time $t \in [0,T]$. The stochastic process $(W(t); t \in [0,T])$ is given by:

$$\mathrm{d}W(t) = \alpha(t)W(t)\frac{\mathrm{d}S(t)}{S(t)} + r(1-\alpha(t))W(t)\mathrm{d}t, \quad t \in [0,T],$$

or equivalently,

$$\begin{aligned} \frac{\mathrm{d}W(t)}{W(t)} &= \alpha(t)\{\mu(t)\mathrm{d}t + \sigma\mathrm{d}B_1(t)\} + r(1-\alpha(t))\mathrm{d}t \\ &= \{r + \alpha(t)(\mu(t) - r)\}\,\mathrm{d}t + \alpha(t)\sigma\mathrm{d}B_1(t), \quad t \in [0,T], \end{aligned} \tag{3}$$

where $\alpha(t)$ is the fraction of the wealth that is invested in the risky stock at time t.

Following Merton (1971), we consider that the investor maximizes the expected utility from his bequest at the end of the time horizon, T:

$$\max_{(\alpha(u); t \le u \le T)} \mathbb{E}\left[U(W(T),T)\,|\mathcal{F}^S(t)\right], \quad t \in [0,T], \tag{4}$$

where $U(W(T),T)$ is the bequest function, which is assumed to be a concave and twice–differentiable function of the bequest (final wealth) $W(T)$, and $\mathcal{F}^S(t)$ is the σ–algebra generated by $(S(u); 0 \le u \le t)$.

First, we will consider the case treated by Brennan (1998): the investor knows that the investment opportunity set is constant but the mean return of the risky stock is uncertain. In this case, $a = b = 0$ and $\mathrm{d}\mu(t) = 0$, $t \in [0,T]$. Further, the investor is assumed to know the diffusion parameter σ of the stock return. We assume that the investor's initial prior information of $\mu := \mu(0)$ is represented by the normal distribution with mean $m(0)$ and variance $\gamma(0)$ (> 0).

Theorem 1. *Let $m(t) := \mathbb{E}\left[\mu \,|\mathcal{F}^S(t)\right]$ and $\gamma(t) := \mathbb{E}\left[(\mu - m(t))^2 \,|\mathcal{F}^S(t)\right]$ denote the mean and variance conditional on the σ-algebra $\mathcal{F}^S(t)$ at time $t \in [0,T]$. The stochastic differential equation of $(m(t); t \in [0,T])$ and the ordinary differential equation of $(\gamma(t); t \in [0,T])$ are given below, respectively:*

$$\begin{aligned} \mathrm{d}m(t) &= \frac{\gamma(t)}{\sigma^2}\left(\frac{\mathrm{d}S(t)}{S(t)} - m(t)\mathrm{d}t\right) \\ &= \frac{\gamma(t)}{\sigma^2}(\mu - m(t))\mathrm{d}t + \frac{\gamma(t)}{\sigma}\mathrm{d}B_1(t), \quad t \in [0,T]; \qquad (5) \\ \mathrm{d}\gamma(t) &= -\left(\frac{\gamma(t)}{\sigma}\right)^2 \mathrm{d}t, \quad t \in [0,T]. \qquad (6) \end{aligned}$$

Proof. See Liptser and Shiryayev (2000). □

Theorem 2. *We can solve the above stochastic differential equations as follows:*

$$m(t) = \frac{m(0) + \gamma(0)\displaystyle\int_0^t \frac{1}{\sigma^2}\left(\frac{\mathrm{d}S(u)}{S(u)}\right)}{1 + \gamma(0)\displaystyle\int_0^t \left(\frac{1}{\sigma}\right)^2 \mathrm{d}u}, \quad t \in [0,T]; \qquad (7)$$

$$\gamma(t) = \frac{\gamma(0)}{1 + \gamma(0)\displaystyle\int_0^t \left(\frac{1}{\sigma}\right)^2 \mathrm{d}u}, \quad t \in [0,T]. \qquad (8)$$

Proof. See Liptser and Shiryayev (2000). □

From Theorem 2, $m(t)$ and $\gamma(t)$ can be calculated as follows:

$$\begin{aligned} m(t) &= \frac{m(0) + \gamma(0)\displaystyle\int_0^t \frac{1}{\sigma^2}\left(\frac{\mathrm{d}S(u)}{S(u)}\right)}{1 + \gamma(0)\displaystyle\int_0^t \left(\frac{1}{\sigma}\right)^2 \mathrm{d}u} \\ &= \frac{1}{1 + \frac{\gamma(0)}{\sigma^2}t}\left\{m(0) + \frac{\gamma(0)}{\sigma^2}\int_0^t (\mu\mathrm{d}t + \sigma\mathrm{d}B_1(t))\right\} \\ &= \frac{1}{1 + \frac{\gamma(0)}{\sigma^2}t}\left\{m(0) + \frac{\gamma(0)}{\sigma^2}(\mu t + \sigma B_1(t))\right\}, \quad t \in [0,T]; \qquad (9) \\ \gamma(t) &= \frac{\gamma(0)}{1 + \gamma(0)\displaystyle\int_0^t \left(\frac{1}{\sigma}\right)^2 \mathrm{d}u} \\ &= \frac{\gamma(0)}{1 + \frac{\gamma(0)}{\sigma^2}t}, \quad t \in [0,T]. \qquad (10) \end{aligned}$$

In order to examine the temporal behavior of $(m(t))$, we consider its expectation:

$$\mathbb{E}[m(t)] = \frac{m(0) + \frac{\gamma(0)}{\sigma^2}\mu t}{1 + \frac{\gamma(0)}{\sigma^2}t}, \quad t \in [0,T]. \qquad (11)$$

By differentiating the above formula with respect to σ, we have

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{m(0) + \frac{\gamma(0)}{\sigma^2}\mu t}{1 + \frac{\gamma(0)}{\sigma^2}t}\right) = \frac{2\gamma(0)\sigma t}{(\sigma^2 + \gamma(0)t)^2}(m(0) - \mu), \quad t \in [0,T]. \qquad (12)$$

So if $m(0) < \mu$, then $\mathbb{E}[m(t)]$ decreases as σ increases.

If we differentiate it with respect to t, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[m(t)] &= \frac{\frac{\gamma(0)}{\sigma^2}\mu\left(1+\frac{\gamma(0)}{\sigma^2}t\right) - \left(m(0)+\frac{\gamma(0)}{\sigma^2}\mu t\right)\frac{\gamma(0)}{\sigma^2}}{\left(1+\frac{\gamma(0)}{\sigma^2}t\right)^2} \\ &= \frac{\frac{\gamma(0)}{\sigma^2}(\mu - m(0))}{\left(1+\frac{\gamma(0)}{\sigma^2}t\right)^2}, \quad t \in [0,T]. \end{aligned} \tag{13}$$

As a result, $\mathbb{E}[m(t)]$ increases as t increases if $\mu > m(0)$, while it decreases as t passes if $\mu < m(0)$. Accordingly, the investor tends to improves monotonically his assessment of μ even if his initial assessment is different from the real value of μ. Furthermore, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = \frac{-\frac{\gamma(0)}{\sigma^2}}{\left(1+\frac{\gamma(0)}{\sigma^2}t\right)^2} < 0, \quad t \in [0,T], \tag{14}$$

the mean square error of investor's estimator of μ is decreasing as time passes. Observing the stock price, $S(t)$, the investor learns what real value μ is, and his expectation is gradually getting close to μ.

3 The Investor's Optimization Problem

The investor's indirect utility function is characterized on his wealth level $W(t)$, his current assessment of the coefficient $m(t)$, and time t. Therefore, the investor's expected utility of the bequest under an optimal policy is:

$$J(W(t),m(t),t) = \max_{(\alpha(s);t\le s\le T)} \mathbb{E}\left[U(W(T),T)\,|\mathcal{F}^S(t)\right], \quad t \in [0,T] \tag{15}$$

with a terminal condition $J(W(T),m(T),T) = U(W(T),T)$. Hence, using Itô's Lemma:

$$\mathrm{d}J = J_W\mathrm{d}W + J_m\mathrm{d}m + J_t\mathrm{d}t + \frac{1}{2}J_{WW}(\mathrm{d}W)^2 + \frac{1}{2}J_{mm}(\mathrm{d}m)^2 + J_{Wm}\mathrm{d}W\cdot\mathrm{d}m,$$

we have

$$\begin{aligned} \mathbb{E}[\mathrm{d}J] &= \left[J_W\left\{r+\alpha(\mathbb{E}[\mu]-r)\}\right]W + J_m\frac{\gamma}{\sigma^2}(\mathbb{E}[\mu]-m) + J_t + \frac{1}{2}J_{WW}\alpha^2\sigma^2W^2 + \frac{1}{2}J_{mm}\frac{\gamma^2}{\sigma^2} + J_{Wm}W\alpha\gamma\right]\mathrm{d}t \\ &= \left[J_W\left\{r+\alpha(m-r)\right\}W + J_m\frac{\gamma}{\sigma^2}(m-m) + J_t + \frac{1}{2}J_{WW}\alpha^2\sigma^2W^2 + \frac{1}{2}J_{mm}\frac{\gamma^2}{\sigma^2} + J_{Wm}W\alpha\gamma\right]\mathrm{d}t. \end{aligned} \tag{16}$$

Hereafter, we assume that the bequest function $U(W(T),T)$ displays Constant Relative Risk Aversion (CRRA):

$$U(W(T),T) = \frac{W(T)^{1-\delta}}{1-\delta}, \tag{17}$$

where $\delta > 0$ is the degree of relative risk aversion. Under this assumption on the bequest function, $J(W(t),m(t),t)$ may be separable in wealth $W(t)$, and be written as

$$J(W(t),m(t),t) = \frac{W(t)^{1-\delta}}{1-\delta}\Phi(m(t),t), \quad t \in [0,T] \tag{18}$$

for some function $\Phi(m(t),t)$. Substituting (18) to (16), we have

$$\mathbb{E}[\mathrm{d}J] = \frac{W^{1-\delta}}{1-\delta}\left[(1-\delta)\left\{r+\alpha(m-r)\right\}\Phi + \Phi_t - \frac{1}{2}(1-\delta)\delta\alpha^2\sigma^2\Phi + \frac{1}{2}\Phi_{mm}\frac{\gamma^2}{\sigma^2} + (1-\delta)\alpha\gamma\Phi_m\right]\mathrm{d}t. \tag{19}$$

The Bellman principle implies that under an optimal policy $(\alpha(t); 0 \le t \le T)$,

$$\max_{\alpha} \psi(W(t), m(t), t, \alpha) = \psi(W(t), m(t), t, \alpha(t)) = 0, \quad t \in [0, T], \tag{20}$$

where ψ is the function defined by the RHS of (19). The HJB equation for $\Phi(m(t), t)$ becomes:

$$\max_{\alpha} \left\{ \frac{W^{1-\delta}}{1-\delta} \left[(1-\delta)\left\{r + \alpha(m-r)\right\}\Phi + \Phi_t - \frac{1}{2}(1-\delta)\delta\alpha^2\sigma^2\Phi + \frac{1}{2}\Phi_{mm}\frac{\gamma^2}{\sigma^2} + (1-\delta)\alpha\gamma\Phi_m \right] \right\} = 0 \tag{21}$$

with a boundary condition $\Phi(m(T), T) = 1$.

Dependent on δ, $\Phi(m(t), t)$, $t \in [0, T]$ is the solution the following control problem. When $0 < \delta < 1$,

$$\max_{\alpha} \left[(1-\delta)\left\{r + \alpha(m-r)\right\}\Phi + \Phi_t - \frac{1}{2}(1-\delta)\delta\alpha^2\sigma^2\Phi + \frac{1}{2}\Phi_{mm}\frac{\gamma^2}{\sigma^2} + (1-\delta)\alpha\gamma\Phi_m \right] = 0. \tag{22}$$

When $1 < \delta$,

$$\min_{\alpha} \left[(1-\delta)\left\{r + \alpha(m-r)\right\}\Phi + \Phi_t - \frac{1}{2}(1-\delta)\delta\alpha^2\sigma^2\Phi + \frac{1}{2}\Phi_{mm}\frac{\gamma^2}{\sigma^2} + (1-\delta)\alpha\gamma\Phi_m \right] = 0. \tag{23}$$

The optimal portfolio policy $(\alpha(t); 0 \le t \le T)$ is given by the FOC of the above HJB equation:

$$(1-\delta)(m-r)\Phi - (1-\delta)\delta\alpha\sigma^2\Phi + (1-\delta)\gamma\Phi_m = 0.$$

Proposition 1. *The optimal investment fraction on the stock, $\alpha(t)$ at time $t \in [0, T]$, can be represented as:*

$$\alpha(t) = \frac{m(t) - r + \gamma(t)\dfrac{\Phi_m(m(t), t)}{\Phi(m(t), t)}}{\delta\sigma^2}, \quad t \in [0, T], \tag{24}$$

where $m(t)$ and $\gamma(t)$ are defined by (9) and (10), respectively, and $\Phi(m(t), t)$ is the solution to the control problem (22) and (23). □

Under the conjecture that

$$J_m(W(t), m(t), t) > 0, \quad t \in [0, T] \tag{25}$$

and the case of $\delta > 1$, we have

$$\Phi_m(m(t), t) < 0, \quad t \in [0, T],$$

because

$$J_m(W(t), m(t), t) = \frac{W(t)^{1-\delta}}{1-\delta}\Phi_m(m(t), t) > 0, \quad t \in [0, T].$$

Further, we have

$$\Phi(m(t), t) > 0, \quad t \in [0, T]$$

because

$$J(W(t), m(t), t) = \frac{W(t)^{1-\delta}}{1-\delta}\Phi(m(t), t) < 0, \quad t \in [0, T]. \tag{26}$$

Although the fact that $J_m(W(t), m(t), t) > 0$, $t \in [0, T]$ is a plausible conjecture, we can theoretically prove this partially for a discrete–time model in Section 6. Hence, under this conjecture, we have

$$\frac{\Phi_m(m(t), t)}{\Phi(m(t), t)} < 0, \quad t \in [0, T].$$

We can consider that $\gamma(t)\Phi_m(m(t), t)/\Phi(m(t), t)$ is the hedge for the uncertainty of μ.

Next, we consider an asymptotic case when the terminal point of time horizon, T becomes sufficiently large.

Proposition 2. *If the terminal point of time horizon, T tends to infinity, then $\mathbb{E}[m(t)]$, $\gamma(t)$, and $\mathbb{E}[\alpha(t)]$ converge to the following values, respectively:*

$$\mathbb{E}[m(t)] = \frac{\frac{m(0)}{t} + \frac{\gamma(0)}{\sigma^2}\mu}{\frac{1}{t} + \frac{\gamma(0)}{\sigma^2}} \to \mu \quad as \quad t \to \infty; \tag{27}$$

$$\gamma(t) \to 0 \quad as \quad t \to \infty; \tag{28}$$

$$\mathbb{E}[\alpha(t)] = \frac{\mathbb{E}[m(t)] - r + \gamma(t)\mathbb{E}\left[\frac{\Phi_m(m(t),t)}{\Phi(m(t),t)}\right]}{\delta\sigma^2} \to \frac{\mu - r}{\delta\sigma^2} =: \alpha^+ \quad as \quad t \to \infty. \tag{29}$$

□

As we have investigated the temporal variation of $m(t)$ and $\gamma(t)$, $\mathbb{E}[m(t)]$ increases and $\gamma(t)$ decreases as time t passes. But we do not know the temporal movement of $\mathbb{E}[\Phi_m(m(t),t)/\Phi(m(t),t)]$. Although we need to know the trend of $\Phi_m(m(t),t)/\Phi(m(t),t)$ in order to clarify that of the optimal investment portion $\alpha(t)$. We assume that $\mathbb{E}[\Phi_m(m(t),t)/\Phi(m(t),t)]$ (< 0) does not decrease so dramatically that the speed of decrease of $\mathbb{E}[\Phi_m(m(t),t)/\Phi(m(t),t)]$ does not overcome the speed of convergence of $\gamma(t)$ to zero. Under this assumption, $\gamma(t)\mathbb{E}[\Phi_m(m(t),t)/\Phi(m(t),t)]$ converges to zero as $t \to \infty$.

4 Complete Information Case: No Uncertainty and No Learning

Let us consider the complete information case that there is no learning effect. In general, the investor does not know the process $(\mu(t))$. However, we analyze the special case when the process $(\mu(t))$ is observable. Since the investor observes a realization of the process $(\mu(t))$, this is a special case without the learning effect.

The stochastic differential equations are assumed to be:

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu(t)\mathrm{d}t + \sigma \mathrm{d}B_1(t), \quad t \in [0,T], \tag{30}$$

where $\mu(t)$ follows a stochastic differential equation:

$$\mathrm{d}\mu(t) = a\mu(t)\mathrm{d}t + b\mathrm{d}B_2(t), \quad t \in [0,T] \qquad \text{(observable)}. \tag{31}$$

The followings are obvious:

$$\begin{aligned} m(t) &= \mathbb{E}[\mu(t)|\mathcal{F}^{S,\mu}(t)] \\ &= \mu(t); \qquad (32) \\ \gamma(t) &= \mathbb{E}\left[(\mu(t) - m(t))^2 \,|\mathcal{F}^{S,\mu}(t)\right] \\ &= 0. \qquad (33) \end{aligned}$$

Hence, in this case, we can get an optimal policy $(\alpha^*(t); 0 \le t \le T)$ as follows:

$$\alpha^*(t) = \frac{\mu(t) - r}{\delta\sigma^2}, \quad t \in [0,T]. \tag{34}$$

When $a > 0$, the investor increases the expected fraction $\mathbb{E}[\alpha(t)]$ of his investment on stock as time t increases. This is because he knows that $\mu(t)$ tends to increase, and the increase of $\mu(t)$ causes the high stock return. Especially, in the case of $a = b = 0$, we can get:

$$\alpha^*(t) = \frac{\mu(0) - r}{\delta\sigma^2} = \frac{\mu - r}{\delta\sigma^2} = \alpha^+, \quad t \in [0,T].$$

Even when $a = b = 0$, μ is the random variable. However, once μ is realized to be a constant value, the drift of the stock return is fixed as the constant value. In other words, the investment opportunity set, μ, becomes to be constant in time. Accordingly, the investor's investment fraction of stock becomes a constant.

5 The Learning Effect

We find that $\alpha^* = \alpha^+$ from the result of Sections 3 and 4. Observing the stock price, the investor learns about the real value of the state variable μ as time passes. The investor will finally get to know the real value of μ as $t \to \infty$. If the investor has a finite terminal horizon T, his assessment $m(T)$ of $\mu(T)$ is not correctly the same as the real value of μ.

If $\mu > m(0)$ the investor tends to increase the fraction of investment on stock as he improves the assessment of μ. Let τ be the remaining time to the terminal horizon T. The longer τ is, the larger the difference of investment fraction, $\alpha^* - \alpha(t)$. At time t, since the investor has much uncertainty about μ, he hesitates to invest a large fraction of his money on the stock. As he gradually knows about the real value of μ, he tends to increase the fraction.

6 A Discrete–Time Approximate Model

We verify whether the conjecture,

$$J_m(W(t), m(t), t) > 0, \quad t \in [0, T]$$

is appropriate. In order to do this, we consider an approximate model in a discrete–time setting. When we can observe the stock price process, $(S(t); 0 \le t \le T)$ with a time interval $\Delta > 0$. Since $(S(t); 0 \le t \le T)$ follows the geometric Brownian motion,

$$X_n^\Delta := \ln \frac{S(n\Delta)}{S((n-1)\Delta)} = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta + \sigma\left\{B_1(n\Delta) - B_1((n-1)\Delta)\right\}, \quad n = 1, 2, \cdots, \frac{T}{\Delta}, \tag{35}$$

where T/Δ is assumed to be an integer, for convenience. Hence,

$$X_n^\Delta \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta, \sigma^2\Delta\right), \quad n = 1, 2, \cdots, \frac{T}{\Delta}. \tag{36}$$

Let's consider a simple case first. Assume that a random variable X has a normal distribution $N(C, v)$ and that the mean C is also a random variable which has a normal prior distribution $N(M_n, \Gamma_n)$.

Hence, the probability density functions of X and C are given by

$$\begin{aligned} \mathbb{P}(X \in dx | C = c) &= \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{(x-c)^2}{2v}\right) dx, \quad x \in \mathbb{R}; \quad (37)\\ \mathbb{P}(C \in dc) &= \frac{1}{\sqrt{2\pi\Gamma_n}} \exp\left(-\frac{(c-M_n)^2}{2\Gamma_n}\right) dc, \quad c \in \mathbb{R}, \quad (38) \end{aligned}$$

respectively. So, by the Bayes formula, the posterior probability density function of C given an observation $\{X = x\}$ $(x \in \mathbb{R})$ becomes:

$$\begin{aligned} \mathbb{P}(C \in dc | X = x) dx &\propto \mathbb{P}(C \in dc)\mathbb{P}(X \in dx | C = c) \\ &\propto \exp\left(-\frac{1}{2}\left\{\frac{(c-M_n)^2}{\Gamma_n} + \frac{(x-c)^2}{v}\right\}\right) dc \cdot dx, \quad c \in \mathbb{R}. \end{aligned} \tag{39}$$

Since

$$\begin{aligned} \frac{(c-M_n)^2}{\Gamma_n} + \frac{(x-c)^2}{v} &= \frac{1}{\Gamma_n v}\left\{v(c-M_n)^2 + \Gamma_n(c-x)^2\right\} \\ &= \frac{1}{\Gamma_n v}\left\{(v+\Gamma_n)c^2 - 2(vM_n + \Gamma_n x)c + vM_n^2 + \Gamma_n x^2\right\} \\ &= \frac{1}{\Gamma_n v}\left\{(v+\Gamma_n)\left(c - \frac{vM_n + \Gamma_n x}{v+\Gamma_n}\right)^2 - \frac{(vM_n + \Gamma_n x)^2}{v+\Gamma_n} + vM_n^2 + \Gamma_n x^2\right\}, \end{aligned} \tag{40}$$

we have

$$\mathbb{P}(C \in dc | X = x) \propto \exp\left(-\frac{1}{2\frac{\Gamma_n v}{v+\Gamma_n}}\left(c - \frac{vM_n + \Gamma_n x}{v+\Gamma_n}\right)^2\right) dc, \quad c \in \mathbb{R}. \tag{41}$$

In order to apply the above result to our model in which the stock price follows a (discrete–time) geometric Brownian motion, we set

$$\begin{aligned} C &= \left(\mu - \frac{1}{2}\sigma^2\right)\Delta; &(42)\\ v &= \sigma^2\Delta. &(43)\end{aligned}$$

Let us define the conditional expectation and variance by

$$\begin{aligned} M_n &:= \mathbb{E}\left[C\,|\mathcal{F}_n^S\right] \\ &= \mathbb{E}\left[\left(\mu - \frac{1}{2}\sigma^2\right)\Delta\middle|\mathcal{F}_n^S\right] \\ &= \left(m_n - \frac{1}{2}\sigma^2\right)\Delta, \quad n = 0, 1, \cdots, \frac{T}{\Delta}; &(44)\\ \Gamma_n &:= \mathbb{E}\left[(C - M_n)^2|\mathcal{F}_n^S\right] \\ &= \mathbb{E}\left[\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta - \left(m_n - \frac{1}{2}\sigma^2\right)\Delta\right\}^2\middle|\mathcal{F}_n^S\right] \\ &= \mathbb{E}\left[(\mu - m_n)^2|\mathcal{F}_n^S\right]\Delta^2 \\ &= \gamma_n\Delta^2, \quad n = 0, 1, \cdots, \frac{T}{\Delta} &(45)\end{aligned}$$

where, for $n = 0, 1, \cdots, T/\Delta$,

$$\begin{aligned} \mathcal{F}_n^S &:= \sigma\left(S(0), S(\Delta), \cdots, S(n\Delta)\right); \\ m_n &:= \mathbb{E}[\mu|\mathcal{F}_n^S]; \\ \gamma_n &:= \mathbb{E}[(\mu - m_n)^2|\mathcal{F}_n^S]. \end{aligned}$$

Then, according to the previous results, they can be updated by the following formulas:

$$\begin{aligned} M_{n+1} &= \frac{vM_n + \Gamma_n x}{v + \Gamma_n}, \quad n = 0, 1, \cdots, \frac{T}{\Delta} - 1 &(46)\\ \Gamma_{n+1} &= \frac{\Gamma_n v}{v + \Gamma_n}, \quad n = 0, 1, \cdots, \frac{T}{\Delta} - 1. &(47)\end{aligned}$$

From Eqs. (44) and (45), Eq. (46) becomes

$$\left(m_{n+1} - \frac{1}{2}\sigma^2\right)\Delta = \frac{\sigma^2\Delta\left(m_n - \frac{1}{2}\sigma^2\right)\Delta + \gamma_n\Delta^2 X_{n+1}^{\Delta}}{\sigma^2\Delta + \gamma_n\Delta^2}. \tag{48}$$

Substituting (35) to (48), we have

$$\left(m_{n+1} - \frac{1}{2}\sigma^2\right)\Delta = \frac{\sigma^2\Delta\left(m_n - \frac{1}{2}\sigma^2\right)\Delta + \gamma_n\Delta^2\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta + \sigma(B_1((n+1)\Delta) - B_1(n\Delta))\right\}}{\sigma^2\Delta + \gamma_n\Delta^2}. \tag{49}$$

As a result, we can get the following stochastic difference equation:

$$m_{n+1} - m_n = \frac{\gamma_n}{\sigma^2 + \gamma_n\Delta}\left\{(\mu - m_n)\Delta + \sigma\left\{B_1((n+1)\Delta) - B_1(n\Delta)\right\}\right\}. \tag{50}$$

Substituting Eqs. (43) and (45) to Eq. (47), we have

$$\gamma_{n+1} = \frac{\gamma_n\sigma^2}{\sigma^2 + \gamma_n\Delta}. \tag{51}$$

Hence we get the following (deterministic) difference:

$$\gamma_{n+1} - \gamma_n = -\frac{\gamma_n^2 \Delta}{\sigma^2 + \gamma_n \Delta}. \tag{52}$$

Comparing the equations (5), (6) with (51), (52), we can conclude that this discrete–time model closely approximates the continuous–time model when Δ is sufficiently small.

The investor optimizes his expected bequest at the end of the time horizon, T. Let $V(W_n, m_n, n)$ be the optimal value function for this discrete–time model when the process starts from the state (W_n, m_n, n). Then, the optimality equation is:

$$V(W_n, m_n, n) = \max_{\alpha} \mathbb{E}\left[V(W_{n+1}, m_{n+1}, n+1)\,|\mathcal{F}_n^S\right], \quad n = 0, 1, \cdots, \frac{T}{\Delta} - 1, \tag{53}$$

with a terminal condition $V(W_N, m_N, N) = U(W_N, T)$ for $N := T/\Delta$.

Proposition 3. *m_{n+1} increases in m_n and X_n^Δ.*

Proof. From (52), this is clearly true. □

Proposition 4. *If $\alpha_n > 0$, W_{n+1} increases in X_n^Δ.*

Proof. Note that

$$\begin{aligned} W_{n+1} &= f(W_n, \alpha_n, X_n^\Delta) \\ &:= \alpha_n W_n \exp\left(X_n^\Delta\right) + (1 - \alpha_n) W_n (1 + r) \\ &= \alpha_n W_n \frac{S((n+1)\Delta)}{S(n\Delta)} + (1 - \alpha_n) W_n (1 + r) \end{aligned} \tag{54}$$

From the assumption, $\alpha_n > 0$, we can conclude that W_{n+1} increases in X_n^Δ. □

Proposition 5. *$V(W_n, m_n, n)$ increases in W_n.*

Proof. Suppose that $W_n < \widehat{W}_n$. The investor who has $\widehat{W}_n$ can spend W_n to attain $V(W_n, m_n, n)$ and invest the remaining $\widehat{W}_n - W_n$ for the riskless bond by which the investor can obtain the additional utility for sure. Hence $V(W_n, m_n, n) < V(\widehat{W}_n, m_n, n)$. □

Proposition 6. *If $\alpha_n > 0$ $\mathbb{P}$–a.s., then $V(W_n, m_n, n)$ increases in m_n.*

Proof. The optimality equation can be rewritten as

$$\begin{aligned} V(W_n, m_n, n) &= \max_{\alpha} \mathbb{E}\left[V(W_{n+1}, m_{n+1}, n+1)\,|\mathcal{F}_n^S\right] \\ &= \max_{\alpha} \int V\left(f(W_n, \alpha, x), h(m_n, \gamma_n, x, n+1), n+1\right) p(x|m_n, \gamma_n, n) \mathrm{d}x, \end{aligned} \tag{55}$$

where

$$\begin{aligned} f(W_n, \alpha, x) &:= \alpha W_n \mathrm{e}^x + (1 - \alpha) W_n (1 + r); \\ h(m_n, \gamma_n, x, n+1) &:= \frac{1}{\sigma^2 + \gamma_n \Delta}\left[\sigma^2 m_n + \gamma_n \mathrm{e}^x + \frac{1}{2}\gamma_n \sigma^2 \Delta\right]; \\ p(x|m_n, \gamma_n, n) &:= \frac{1}{\sqrt{2\pi\gamma_n}} \exp\left(-\frac{(x - m_n)^2}{2\gamma_n}\right). \end{aligned} \tag{56}$$

It is noted that, when γ_n is fixed and $m_n < \widehat{m}_n$, p.d.f. $p(\cdot|\widehat{m}_n, \gamma_n, n)$ is greater than p.d.f. $p(\cdot|m_n, \gamma_n, n)$ in the sense of the first order stochastic dominance (usual stochastic order). From Proposition 4, $W_{n+1} = f(W_n, \alpha, x)$ increases in x. From the Propositions 3, 4, and 5, it suffices to show that $V(W_{n+1}, m_{n+1}, n+1)$ increases in m_{n+1} by induction in n.

In the case of $n' = N := T/\Delta$,

$$V(W_N, m_N, N) = U(W_N, T) \tag{57}$$

is increasing in W_N.

When $n' = n+1$, suppose that $V(W_{n+1}, m_{n+1}, n+1)$ increases in m_{n+1}.

Then, when $n' = n$,

$$V(W_n, m_n, n) = \max_{\alpha} \int V\left(f(W_n, \alpha, x), h(m_n, \gamma_n, x, n+1), n+1\right) p(x|m_n, \gamma_n, n)\mathrm{d}x. \tag{58}$$

Because $m_{n+1} = h(m_n, \gamma_n, x, n+1)$ increases in m_n, $V(W_n, m_n, n)$ increases in m_n. □

7 Conclusion

In the dynamic asset allocation problem, we usually treat the case that an investment opportunity set is known. We analyze the case that the investment opportunity set is unknown but constant. Especially, we consider the case that the volatility of the stock return is known, but the expected return of the stock is unknown. We theoretically analyze the temporal change of the fraction of investment on the risky asset. It is shown that the fraction converges to the one of the no uncertainty case as the investor learns about the real state variable and the terminal point of the time horizon goes infinity. The learning process gives two effects on the investment fraction. One is improving the assessment of the state variable. The other is the reduction of the hedge demand against the uncertainty of the state variable learning about the state variable.

An extension of the paper would be to analyze theoretically the learning effects when the investment opportunity set follows the stochastic process and unobservable.

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