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SUPER-CRITICAL BUBBLING IN ELLIPTIC BOUNDARY VALUE PROBLEMS

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1. Introduction

The purpose of this note is to review some recent results concerning solvability of semilinear elliptic boundary value problems near the critical exponent. When the nonlinearity has a power growth, it is well known that the critical exponent $\frac{N+2}{N-2}$ sets a treshold where the solution set may change dramatically, and the effect of lower order terms in the nonlinearity and/or geometry-topology of the domain becomes crucial in the structure of this set. This has been a subject broadly studied over the last two decades, so that the results cited here constitute only partial account of progress made. Highly non-trivial understanding has been obtained on the effect of criticality in nonlinear elliptic problems, however this effect seems to hide many misterious aspects not yet unveiled, in particular rather little seems to be known on the structure of solution sets when the power is super-critical.

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$ with smooth boundary $\partial\Omega$. In what follows we will restrict ourselves to the two classical boundary value problems,

$$\begin{cases} -\Delta u = u^q + \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

and

$$\begin{cases} -d^2\Delta u + u = u^q & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where $q > 1$. While solvability of these problems is an elementary fact when $q < \frac{N+2}{N-2}$, this is no longer the case for $q \geq \frac{N+2}{N-2}$ due to the loss of compactness of Sobolev embeddings. Our aim is to analyze solutions exhibiting ***bubbling behavior*** to the above problems when one lets the exponent q approach $\frac{N+2}{N-2}$ from above.

2. Single-bubbling in (1.1)

Integrating the equation against a first eigenfunction of the Laplacian yields that a necessary condition for solvability of (1.1) is $\lambda < \lambda_1$. On the other hand, if $1 < q < \frac{N+2}{N-2}$ and $0 < \lambda < \lambda_1$ a solution may be found as follows. Let us consider the Rayleigh quotient

$$Q_\lambda(u) = \frac{\int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2}{\left(\int_\Omega |u|^{q+1}\right)^{\frac{2}{q+1}}}, \quad u \in H_0^1(\Omega) \setminus \{0\} \tag{2.1}$$

and set

$$S_\lambda = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} Q_\lambda(u). \tag{2.2}$$

S_λ is achieved thanks to compactness of Sobolev embedding if $q < \frac{N+2}{N-2}$, and a suitable scalar multiple of it turns out to be a solution of (1.1). The case $q \geq \frac{N+2}{N-2}$ is considerably more delicate: for $q = \frac{N+2}{N-2}$ compactness of the embedding is lost while for $q > \frac{N+2}{N-2}$ there is no such embedding. This obstruction is not just technical for the solvability question, but essential. Pohozaev [53] showed that if Ω is strictly star-shaped then no solution of (1.1) exists if $\lambda \leq 0$ and $q \geq \frac{N+2}{N-2}$.

Let $S(N)$ be the best constant in the critical Sobolev embedding,

$$S(N) = \inf_{u \in C_0^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}}. \tag{2.3}$$

Let us consider $q = \frac{N+2}{N-2}$ in (2.1) and the number

$$\lambda^* = \inf\{\lambda > 0 \,/ S_\lambda < S(N)\}. \tag{2.4}$$

In [12], Brezis and Nirenberg established that $\lambda^* = 0$ for $N \geq 4$ and $0 < \lambda^* < \lambda_1$ for $N = 3$. As a consequence S_λ is achieved for $\lambda^* < \lambda < \lambda_1$ and hence (1.1) is solvable in this range. In case that Ω is a ball and $N = 3$ it is shown in [12] that $\lambda^* = \frac{\lambda_1}{4}$ and that no solution exists for $\lambda \leq \lambda^*$.

Thus $\lambda > 0$ taken at the appropriate range makes compactness restored and therefore solvability holds. Pohozaev's result shows that solvability at the critical exponent for, say, $\lambda = 0$ is strongly linked to the effect of topology and/or geometry. In fact, in sharp contrast with that non-existence result is the observation due to Kazdan and Warner [39] that compactness of Sobolev's embedding is regained within the class of radially symmetric functions at any exponent if Ω is a radially symmetric annulus, $\Omega = \{a < |x| < b\}$, thus yielding existence of a radial solution to Problem (1.1) for any exponent $q > 1$. Without symmetry the question is harder. This issue was first considered by Coron [17] who found that (1.1) is solvable when $q = \frac{N+2}{N-2}$ and $\lambda = 0$ in any domain exhibiting a sufficiently small hole. Bahri and Coron [9] extended notably this result proving that if $q = \frac{N+2}{N-2}$, $\lambda = 0$ and some homology group of Ω with coefficients in $\mathbf{Z}_2$ is not trivial, then (1.1) has at least one solution, in particular in any three-dimensional domain which is not contractible to a point. Examples showing that this condition is actually not necessary for solvability were found by Dancer [18], Ding [29] and Passaseo [51], showing that geometry and not only topology influences existence. In [11] it is raised the question whether the presence of non-trivial topology in the domain suffices for existence in the super-critical case, as it is the case in the symmetric annulus. The answer is actually negative in general. Passaseo in [52] found examples of domains with non-trivial topology for which (1.1) is not solvable for $\lambda = 0$ in case that the power q is sufficiently large. The question of existence for super-critical powers close to critical remained however open. This note will survey some results, which in particular establish the presence of solutions to (1.1) for slightly super-critical powers, which become *unbounded* as the exponent $q = \frac{N+2}{N-2}$ is approached.

2.1. **Blowing-up solutions.** By a *blowing-up solution* for (1.1) near the critical exponent we mean an unbounded sequence of solutions u_n of (1.1) for $\lambda = \lambda_n$ bounded, and $q = q_n \to \frac{N+2}{N-2}$. Setting

$$M_n = \alpha^{-1} \max_{\Omega} u_n = \alpha^{-1} u_n(x_n) \to +\infty$$

we see then that the scaled function

$$v_n(y) = M_n\, u_n(x_n + M_n^{(q_n-1)/2}\, y),$$

satisfies

$$\Delta v_n + v_n^{q_n} + M_n^{-(q_n-1)} \lambda_n v_n = 0$$

in the expanding domain $\Omega_n = M_n^{(q_n-1)/2}(\Omega - x_n)$. Assuming for instance that x_n stays away from the boundary of Ω, elliptic regularity implies that locally over compacts around the origin, v_n converges up to subsequences to a positive solution of

$$\Delta w + w^p = 0$$

in entire space, with $w(0) = \max w = \alpha$. It is known, see [15], that for the convenient choice $\alpha_N = (N(N-2))^{\frac{N-2}{4}}$, this solution is explicitly given by

$$w(z) = \alpha_N \left(\frac{1}{1 + |z|^2} \right)^{\frac{N-2}{2}}. \tag{2.5}$$

which corresponds precisely to an extremal of $S(N)$, see [8, 57]. Coming back to the original variable, we expect then that "near x_n " the behavior of $u_n(y)$ can be approximated as

$$u_n(y) = \alpha_N \left(\frac{1}{1 + M_n^{\frac{4}{N-2}}\, |x - x_n|^2} \right)^{\frac{N-2}{2}} M_n\, (1 + o(1))\,. \tag{2.6}$$

A point to be made is that since the convergence in expanded variables is only local over compacts, it is not at all clear how far from x_n the approximation (2.6) holds true, even if only one maximum point x_n exists. Roughly speaking, we say that the solution solution $u_n(x)$ exhibits single-bubbling around x_n if (2.6) holds with $o(1) \to 0$ uniformly in some fixed open subset

2.2. **Super-critical bubbling for $\lambda = 0$.** As we mentioned above, the question of existence remained open concerning powers close to critical from above. In [24, 25] this issue has been adressed for a class of domains which includes that considered by Coron in [17], for $\lambda = 0$. It is established that a solution to (1.1) exists for $\lambda = 0$, $q = \frac{N+2}{N-2} + \varepsilon$ with any small $\varepsilon > 0$ if for instance Ω is a smooth domain exhibiting a sufficiently small hole. Unlike the proofs by Coron or by Bahri-Coron, which are indirect, the solutions are found constructively: considering ε as a small parameter, the solution exhibits single-bubbling around exactly two points and ceases to exist when $\varepsilon = 0$. More precisely, let $\mathcal{D}$ be a bounded, smooth domain in $\mathbb{R}^N$, $N \geq 3$, and P a point of $\mathcal{D}$. Let us consider the domain

$$\Omega = \mathcal{D} \setminus \overline{B}(P, \mu) \tag{2.7}$$

where $\mu > 0$ is a small number. Then there exists a $\mu_0 > 0$, which depends on $\mathcal{D}$ and the point P such that if $0 < \mu < \mu_0$ is fixed and Ω is the domain given by (2.7), then the following holds: There exists $\varepsilon_0 > 0$ and a solution u_ε, $0 < \varepsilon < \varepsilon_0$ of (1.1) with $\lambda = 0$ of the form

$$u_\varepsilon(x) = \sum_{j=1}^{2} \alpha_N \left(\frac{1}{1 + \varepsilon^{-\frac{2}{N-2}} \Lambda_{j\varepsilon}^{-2} |x - \xi_j^\varepsilon|^2} \right)^{\frac{N-2}{2}} \Lambda_{j\varepsilon}^{\frac{N-2}{2}} \varepsilon^{\frac{1}{2}} \left(1 + o(1) \right), \tag{2.8}$$

where $o(1) \to 0$ uniformly as $\varepsilon \to 0$. The numbers $\Lambda_{j\varepsilon}$ and the points ξ_j^ε converge (up to subsequences) to a critical point of certain function built upon the Green's function of Ω. The role of Green's function in concentration phenomena associated to almost-critical problems on the subcritical side, namely $q = \frac{N+2}{N-2} - \varepsilon$, has already been considered in several works, see for instance [13, 54, 10]. The above result is extended in [25] to the case of a domain exhibiting multiple small holes, showing that these two-spike solutions can actually be "glued" yielding existence of multiple solutions.

The assumption of "small hole" is used in an essential way in the proof. The case of a symmetric annulus with larger inner radius for instance is not covered by the result in [24]. It is however proven in [26] that the concentration phenomena involved is in fact much richer than may be apriori expected, at least in the case of domains exhibiting symmetries. In particular, we find the presence of a large number of geometrically distinct solutions to problem (1.1) when Ω is an annulus,

$$A_a^b = \{ x \ / \ a < |x| < b \}, \tag{2.9}$$

for given $0 < a < b$, provided that $\varepsilon > 0$ is sufficiently small. More precisely, we find that a k-spike solution of (1.1) exists for any k sufficiently large. This is also the case for any solid of revolution around the x_3-axis in $\mathbb{R}^3$, symmetric on the variable x_3, which does not contain the origin. The k-spike solution found has its maxima on the vertices of a regular polygon contained in the plane $x_3 = 0$.

These facts lead naturally to conjecture that in a domain with nontrivial topology, k-bubble solutions exist whenever k is sufficiently large. Recently in [46] it was shown that for $q = \frac{N+2}{N-2}$ a solution exists for any *negative*, sufficiently small value of λ, in the small-hole situation. The solution found is again a double spike blowing-up as $\lambda \uparrow 0$.

3. Multiple-bubbling in (1.1). The radial case

The solutions in the previous section exhibit single bubbling around a finite number of points. In this section we consider the case of $\Omega = B$, the unit ball in $\mathbb{R}^N$, and search for radial solutions to Problem (1.1). As we will see, for $q = \frac{N+2}{N-2} + \varepsilon$ and certain range $\lambda = o(1)$, depending on ε, one can see bubbling solutions. Somewhat surprisingly, much more than single-bubble solutions is going on in this problem: we find the presence of *towers* constituted by superposition of bubbles of different blow-up orders. In fact, given any number $k \geq 1$, there is an ε-dependent range for λ for which there exist solutions of the form

$$u_\varepsilon(y) = \alpha_N \sum_{j=1}^{k} \left(\frac{1}{1 + M_j^{\frac{4}{N-2}} |y|^2} \right)^{\frac{N-2}{2}} M_j \,(1 + o(1)) \quad \text{as } y \to 0 , \tag{3.1}$$

where $M_j \to +\infty$ and $M_j = o(M_{j+1})$ for all j. This is in strong contrast with the case in which $\varepsilon = 0$ and one lets $\lambda \downarrow 0$ or $\lambda = 0$ and $\varepsilon \uparrow 0$ where only a single bubble is present, as established by Brezis and Peletier [13], also see [54, 38]. For simplicity in the exposition, we restrict ourselves in this section to the case $N \geq 5$. We have the validity of the following result, established in [20]

Theorem 1. [20] *Assume $N \geq 5$ and $q = \frac{N+2}{N-2} + \varepsilon$. Then, given an integer $k \geq 1$, there exists a number $\mu_k > 0$ such that if $\mu > \mu_k$ and*

$$\lambda = \mu \, \varepsilon^{\frac{N-4}{N-2}} ,$$

then there are constants $0 < \alpha_j^- < \alpha_j^+$, $j = 1, \dots, k$ which depend on k, N and μ and two solutions $u_\varepsilon^\pm$ of Problem (1.1) of the form

$$u_\varepsilon^\pm(y) = \alpha_N \sum_{j=1}^{k} \left(\frac{1}{1 + \left[\alpha_j^\pm \, \varepsilon^{\frac{1}{2}-j}\right]^{\frac{4}{N-2}} |y|^2} \right)^{\frac{N-2}{2}} \alpha_j^\pm \, \varepsilon^{\frac{1}{2}-j} \,(1 + o(1)) , \tag{3.2}$$

where $o(1) \to 0$ uniformly on B as $\varepsilon \to 0$.

We shall next sketch the proof of Theorem 1. The problem of finding radial solutions u to Problem (1.1) corresponds to that of solving the boundary value problem

$$u'' + \tfrac{N-1}{r} u' + u^{p+\varepsilon} + \lambda u = 0 , \quad u'(0) = 0 , \quad u(1) = 0 . \tag{3.3}$$

Here and in what follows $p = \frac{N+2}{N-2}$ and we write simply $u = u(r)$ with $r = |y|$. We transform the problem by means of the following change of variable

$$v(x) = \left(\tfrac{2}{p-1}\right)^{-\frac{2}{p-1+\varepsilon}} r^{\frac{2}{p-1}}\, u(r) \quad \text{with } r = e^{-\frac{p-1}{2}x}\,, \quad x \in (0, +\infty)\,, \tag{3.4}$$

a variation of the so-called Emden-Fowler transformation, first introduced in [31]. Problem (3.3) then becomes

$$\begin{cases} v'' - v + e^{\varepsilon x}\, v^{p+\varepsilon} + \left(\frac{p-1}{2}\right)^2 \lambda\, e^{-(p-1)x}\, v = 0 & \text{on } (0, \infty)\,, \\ v(0) = 0\,, \quad v > 0\,, \quad v(x) \to 0 \text{ as } x \to +\infty\,. \end{cases} \tag{3.5}$$

The energy functional associated to Problem (3.5) is given by

$$E_\varepsilon(w) = I_\varepsilon(w) - \frac{1}{2}\left(\frac{p-1}{2}\right)^2 \lambda \int_0^\infty e^{-(p-1)x} |w|^2\, dx \tag{3.6}$$

with

$$I_\varepsilon(w) = \frac{1}{2}\int_0^\infty |w'|^2\, dx + \frac{1}{2}\int_0^\infty |w|^2\, dx - \frac{1}{p+\varepsilon+1}\int_0^\infty e^{\varepsilon x} |w|^{p+\varepsilon+1}\, dx\,. \tag{3.7}$$

Let us consider the unique solution $U(x)$ to the problem

$$\begin{cases} U'' - U + U^p = 0 & \text{on } (-\infty, \infty) \\ U'(0) = 0 \\ U > 0\,, \quad U(x) \to 0 \text{ as } x \to \pm\infty \end{cases} \tag{3.8}$$

This solution is nothing but the one given by the Emden-Fowler tranformation (with $\varepsilon = 0$) of the radial solution of $\Delta w + w^p = 0$ given by (2.5), namely

$$U(x) = \left(\frac{4N}{N-2}\right)^{\frac{N-2}{4}} e^{-x}\left(1 + e^{-\frac{4}{N-2}x}\right)^{-\frac{N-2}{2}}\,. \tag{3.9}$$

Let us consider points $0 < \xi_1 < \xi_2 < \cdots < \xi_k$. We look for a solution of (3.5) of the form

$$v(x) = \sum_{i=1}^{k} (U(x - \xi_i) + \pi_i) + \phi \tag{3.10}$$

where ϕ is small and $\pi_i(x) = -U(\xi_i)\, e^{-x}$. The correction π_i is meant to make the ansatz satisfy the Dirichlet boundary conditions. A main observation is that $v(x) \sim \sum_{i=1}^{k} U(x - \xi_i)$ solves (3.5) if and only if (going back in the change of variables)

$$u(r) \sim \alpha_N \sum_{i=1}^{k} \left(\frac{1}{1 + e^{\frac{4\xi_i}{N-2}} r^2}\right)^{\frac{N-2}{2}} e^{\xi_i}$$

solves (3.3). Therefore the ansatz given for v provides (for large values of the ξ_i's), a bubble-tower solution for (1.1) of the form (3.1) with $M_i = e^{\xi_i}$.

Let us write

$$U_i(x) = U(x-\xi_i)\ , \quad V_i = U_i + \pi_i\ , \quad \pi_i(x) = -U(\xi_i)\, e^{-x}\ , \quad V = \sum_{i=1}^{k} V_i\ . \tag{3.11}$$

It is easily checked that V_i is nonnegative on $\mathbb{R}^+$. We shall work out asymptotics for the associated energy functional at the function V, assuming that the numbers ξ_i are large and also very far apart but at comparable distances from each other.

We make the following choices for the points ξ_i:

$$\begin{aligned} \xi_1 &= -\tfrac{1}{2}\log\varepsilon + \log\Lambda_1\ , \\ \xi_{i+1} - \xi_i &= -\log\varepsilon - \log\Lambda_{i+1}\ , \quad i = 1,\dots,k-1\ , \end{aligned} \tag{3.12}$$

where the Λ_i's are positive parameters. For notational convenience, we also set $\Lambda = (\Lambda_1, \Lambda_2, \dots, \Lambda_k)$. The advantage of the above choice is the validity of the expansion of the energy E_ε defined by (3.6) given as follows.

Lemma 3.1. *Let $N \geq 5$. Fix a small number $\delta > 0$ and assume that*

$$\delta < \Lambda_i < \delta^{-1} \quad \textit{for all } i = 1,\dots,k\ . \tag{3.13}$$

Assume also that $\lambda = \mu\,\varepsilon^{\frac{N-4}{N-2}}$ for some $\mu > 0$. Let V be given by (3.11). Then, with the choice (3.12) of the points ξ_i, there are positive numbers a_i, $i = 0,\dots,5$, depending only on N such that the following expansion holds:

$$E_\varepsilon(V) = k\,a_0 + \varepsilon\,\Psi_k(\Lambda) + \frac{k^2}{2}\,a_3\,\varepsilon\,\log\varepsilon + a_5\,\varepsilon + \varepsilon\,\theta_\varepsilon(\Lambda)\ , \quad \textit{where} \tag{3.14}$$

$$\Psi_k(\Lambda) = a_1\,\Lambda_1^{-2} - k\,a_3\log\Lambda_1 - a_4\,\mu\,\Lambda_1^{-(p-1)} + \sum_{i=2}^{k}\left[(k-i+1)\,a_3\log\Lambda_i - a_2\,\Lambda_i\right]\ , \tag{3.15}$$

and as $\varepsilon \to 0$, the term $\theta_\varepsilon(\Lambda)$ converges to 0 uniformly and in the C^1-sense on the set of Λ_i's satisfying contraints (3.13).

If there is indeed a solution of (3.5) of the form $v = V + \phi$, with V as in the statement of the lemma, and ϕ small, it is natural to expect that this occurs if the vector $\Lambda = (\Lambda_1, \dots, \Lambda_k)$ corresponds to a critical point of the function Ψ_k. This is in fact true, as it follows from,a Lyapunov-Schmidt reduction procedure. Before, let us analyze the critical points of Ψ_k:

$$\Psi_k(\Lambda) = \varphi_k^\mu(\Lambda_1) + \sum_{i=2}^{k}\varphi_i(\Lambda_i)\ ,$$

$\varphi_k^\mu(s) = a_1\,s^{-2} - k\,a_3\log s - a_4\,\mu\,s^{-(p-1)}$ and $\varphi_i(s) = (k-i+1)\,a_3\log s - a_2\,s$.

Let us observe that there is a number $\mu_k > 0$ such that φ_k^{μ} has exactly two critical points: a nondegenerate maximum, $s_k^+(\mu)$, and a nondegenerate minimum, $s_k^-(\mu)$. On the other hand, each of the functions φ_j has exactly one nondegenerate critical point, a maximum,

$$s = (k - j + 1)\, b_3, \quad \text{for each } j = 2, \dots , k\, ,$$

with b_3 certain positive constant depending on N.

Then we have:

Lemma 3.2. *Assume that $\mu > \mu_k$. Then, the function $\Psi_k(\Lambda)$ has exactly two critical points, given by*

$$\Lambda^{\pm} = (\, s_k^{\pm}(\mu), (k-1)\, b_3, (k-2)\, b_3, \dots , b_3\,)\, .$$

These critical points are nondegenerate.

Let us consider again points $0 < \xi_1 < \xi_2 < \ldots < \xi_k$, which are for now arbitrary. We keep the notations U_i, V_i and V defined by (3.11). Additionally we define

$$Z_i(x) = U_i'(x) - U_i'(0)e^{-x}\, , \quad i = 1, \dots , k$$

and consider the problem of finding a function ϕ for which there are constants c_i, $i = 1, \dots , k$, such that, in $(0, \infty)$

$$\begin{cases} -(V+\phi)'' + (V+\phi) - e^{\varepsilon x}(V+\phi)_+^{p+\varepsilon} - \lambda\, (\frac{p-1}{2})^2 e^{-(p-1)x}(V+\phi) = \sum_{i=1}^{k} c_i\, Z_i\, , \\ \phi(0) = 0\, , \quad \lim_{x\to+\infty} \phi(x) = 0\, , \\ \int_0^{\infty} Z_i\, \phi\, dx = 0 \quad \text{for all } i = 1, \dots , k\, . \end{cases} \tag{3.16}$$

This problem turns out to be solvable for points ξ_i chosen in a convenient range. After this, the original problem becomes reduced to adjusting the points ξ_i so that $c_i = 0$ for all i.

In order to solve Problem (3.16), let us consider the linearized operator around V defined as

$$\mathcal{L}_\varepsilon \phi \;=\; -\phi'' \,+\, \phi \,-\, (p+\varepsilon)e^{\varepsilon x}V^{p+\varepsilon-1}\phi \,-\, \lambda\, (\tfrac{p-1}{2})^2\, e^{-(p-1)x}\phi\, .$$

Then problem (3.16) can be rewritten as

$$\begin{cases} \mathcal{L}_\varepsilon \phi \;=\; N_\varepsilon(\phi) \,+\, R_\varepsilon \,+\, \sum_{i=1}^{k} c_i\, Z_i \quad \text{in } (0, \infty)\, , \\ \phi(0) = 0, \quad \lim_{x\to+\infty} \phi(x) = 0\, , \\ \int_0^{\infty} Z_i\, \phi\, dx = 0 \quad \text{for all } i = 1, \dots , k\, . \end{cases} \tag{3.17}$$

where

$$N_\varepsilon(\phi) = e^{\varepsilon x}\left[(V+\phi)_+^{p+\varepsilon} - V^{p+\varepsilon} - (p+\varepsilon)V^{p+\varepsilon-1}\phi\right] \quad \text{and} \tag{3.18}$$

$$R_\varepsilon = e^{\varepsilon x}[V^{p+\varepsilon} - V^p] + V^p[e^{\varepsilon x} - 1] + [V^p - \sum_{i=1}^k V_i^p] + \lambda\left(\tfrac{p-1}{2}\right)^2 e^{-(p-1)x}\, V\,.$$

The operator $\mathcal{L}_\varepsilon$ turns out to be boundedly invertible under the orthogonality conditions for an appropriate norm. We introduce the following norm which depends on the points ξ_i. For a small, fixed positive number σ and a function $\psi(x)$ defined on $(0,\infty)$, let us set

$$\|\psi\|_* = \sup_{x>0}\left(\sum_{i=1}^k e^{-\sigma|x-\xi_i|}\right)^{-1}|\psi(x)|\,. \tag{3.19}$$

Consider the linear problem of, given a function h, finding ϕ such that

$$\begin{cases} \mathcal{L}_\varepsilon\phi = h(x) + \sum_{i=1}^k c_i Z_i \quad \text{in } (0,\infty)\,, \\[2mm] \phi(0) = 0\,, \quad \lim_{x\to+\infty}\phi(x) = 0\,, \\[2mm] \int_0^\infty Z_i\,\phi\,dx = 0 \quad \text{for all } i = 1,\dots,k\,, \end{cases} \tag{3.20}$$

for certain constants c_i. Then we have the validity of the following result.

Lemma 3.3. *There exist positive numbers ε_0, δ_0, δ_1, R_0, and a constant $C>0$ such that if the scalar λ and the points $0<\xi_1<\xi_2<\cdots<\xi_k$ satisfy*

$$R_0 < \xi_1, \quad R_0 < \min_{1\le i<k}(\xi_{i+1}-\xi_i)\,, \quad \xi_k < \frac{\delta_0}{\varepsilon}\,, \quad \lambda < \delta_1\,, \tag{3.21}$$

then for all $0<\varepsilon<\varepsilon_0$ and all $h\in C[0,\infty)$ with $\|h\|_* < +\infty$, Problem (3.20) admits a unique solution $\phi =: T_\varepsilon(h)$. Besides,

$$\|T_\varepsilon(h)\|_* \le C\,\|h\|_* \quad \textit{and} \quad |c_i| \le C\,\|h\|_*\,.$$

Now we are ready to solve Problem (3.16). We shall do this after restricting conveniently the range of the parameters ξ_i and λ. Let us consider for a number M large but fixed, the following conditions:

$$\begin{cases} \xi_1 > \frac{1}{2}\log(M\varepsilon)^{-1}\,, \quad \log(M\varepsilon)^{-1} < \min_{1\le i<k}(\xi_{i+1}-\xi_i)\,, \\[2mm] \qquad \xi_k < k\log(M\varepsilon)^{-1}\,, \quad \lambda < M\,\varepsilon^{\frac{N-4}{N-2}}\,. \end{cases} \tag{3.22}$$

Useful facts that we easily check is that under relations (3.22), N_ε and R_ε defined by (3.18) satisfy for all small $\varepsilon>0$ and $\|\phi\|_* \le \frac{1}{4}$ the estimates:

$$\|N_\varepsilon(\phi)\|_* \le C\,\|\phi\|_*^p \quad \text{and} \quad \|R^\varepsilon\|_* \le C\,\varepsilon^{1-\sigma}\,, \tag{3.23}$$

provided that σ is chosen small enough.

Lemma 3.4. *Assume that relations (3.22) hold. Then there is a constant $C > 0$ such that, for all $\varepsilon > 0$ small enough, there exists a unique solution $\phi = \phi(\xi)$ to problem (3.16) which besides satisfies*

$$\|\phi\|_* \leq C\varepsilon^{1-\sigma} .$$

Moreover, the map $\xi \mapsto \phi(\xi)$ is of class C^1 for the $\|\cdot\|_*$-norm and

$$\|D_\xi\phi\|_* \leq C\varepsilon^{1-\sigma} .$$

Proof. We will only prove the existence statement Problem (3.16) is equivalent to solving a fixed point problem. Indeed ϕ is a solution of (3.16) if and only if

$$\phi = T_\varepsilon(N_\varepsilon(\phi) + R_\varepsilon) =: A_\varepsilon(\phi) .$$

Thus we need to prove that the operator A_ε defined above is a contraction in a proper region. Let us consider the set

$$\mathcal{F}_r = \{\phi \in C[0,\infty) \, : \, \|\phi\|_* \leq r\,\varepsilon^{1-\sigma}\}$$

with r a positive number to be fixed later. From Proposition 3.3 and (3.23), we get

$$\|A_\varepsilon(\phi)\|_* \leq C\|N_\varepsilon(\phi) + R_\varepsilon\|_* \leq C[(r\,\varepsilon)^p + \varepsilon^{1-\sigma}] < r\,\varepsilon^{1-\sigma}$$

for all small ε, provided that r is chosen large enough, but independent of ε. Thus A_ε maps $\mathcal{F}_r$ into itself for this choice of r. Moreover, A_ε turns out to be a contraction mapping in this region. This follows from the fact that N_ε defines a contraction in the $\|\cdot\|_*$-norm, which can be proved in a straightforward way. This concludes the proof. □

Now let us fix a large number M and assume that conditions (3.22) hold true for $\xi = (\xi_1, \dots, \xi_k)$ and λ. According to the previous results, our problem has been reduced to that of finding points ξ_i so that the constants c_i which appear in (3.17), for the solution ϕ given by Lemma 3.4, are all zero. Thus we need to solve the system of equations

$$c_i(\xi) = 0 \quad \text{for all } i = 1, \dots, k . \tag{3.24}$$

If (3.24) holds, then $v = V + \phi$ will be a solution to (3.16) with the desired form. This system turns out to be equivalent to a variational problem, which we introduce next.

Let us consider the functional

$$\mathcal{I}_\varepsilon(\xi) = E_\varepsilon(V + \phi) ,$$

where $\phi = \phi(\xi)$ is given by Lemma 3.4 and E_ε is defined by (3.6). We claim that solving system (3.24) is equivalent to finding a critical point of this functional. In fact, integrating (3.16) against Z_i and using the definition of E_ε and ϕ, we obtain

$$DE_\varepsilon(V + \phi)[Z_i] = 0 \quad \text{for all } i = 1, \dots, k . \tag{3.25}$$

Now, it is easily checked that

$$\frac{\partial}{\partial \xi_i}(V + \phi) = Z_i + o(1) ,$$

with $o(1) \to 0$ in the *-norm as $\varepsilon \to 0$. We can decompose each of the $o(1)$ terms above as the sum of a small term which lies in the vector space spanned by the Z_i's, and a function η with $\int_0^{+\infty} Z_i\, \eta\, dx = 0$ for all i. Again, from equation (3.16), we get $DJ_\varepsilon(V + \phi)[\eta] = 0$. What we have shown is that system (3.25) is equivalent to

$$\nabla \mathcal{I}_\varepsilon(\xi) = 0 .$$

The following fact is crucial to find critical points of $\mathcal{I}_\varepsilon$.

Lemma 3.5. *Assume that $\sigma < \frac{1}{2}$ in the definition of the *-norm. Then the following expansion holds*

$$\mathcal{I}_\varepsilon(\xi) = E_\varepsilon(V) + o(\varepsilon) ,$$

where the term $o(\varepsilon)$ is uniform in the C^1-sense over all points satisfying constraint (3.22), for given $M > 0$.

Proof of Theorem 1. Let us assume $\mu > \mu_k$. We need to find a critical point of $\mathcal{I}_\varepsilon(\xi)$. We consider the change of variable $\xi = \xi(\Lambda)$

$$\xi_1 = -\frac{1}{2}\log\varepsilon - \log\Lambda_1 , \quad \xi_{i+1} - \xi_i = -\log\varepsilon - \log\Lambda_i , \quad i \geq 2 ,$$

where the Λ_i's are positive parameters, and we denote $\Lambda = (\Lambda_1, \dots , \Lambda_k)$. Thus it suffices to find a critical point of

$$\Phi_\varepsilon(\Lambda) \equiv \varepsilon^{-1}\nabla\mathcal{I}_\varepsilon(\xi(\Lambda)) .$$

From the above lemma and the decomposition (3.14) given in Lemma 3.1, which actually holds with the $o(\varepsilon)$ term in the C^1 sense uniformly on points satisfying constraints (3.22), we obtain

$$\nabla\Phi_\varepsilon(\Lambda) = \nabla\Psi_k(\Lambda) + o(1) ,$$

where $o(1) \to 0$ uniformly on points Λ satisfying (3.13). We assume that for our fixed $\mu > \mu_k$, the critical points $\Lambda^\pm$ of Ψ_k in Lemma 3.5 satisfy this constraint. Since the critical points $\Lambda^\pm$ are nondegenerate, it follows that the local degrees $\deg(\nabla\Psi_k, \mathcal{V}_\pm, 0)$ are well defined and they are non-zero. Here $\mathcal{V}_\pm$ are arbitrarily small neighborhoods of the points $\Lambda^\pm$ in $\mathbb{R}^k$. We also conclude that $\deg(\nabla\mathcal{I}_\varepsilon, \mathcal{V}_\pm, 0) \neq 0$ for all sufficiently small ε. Hence we may find critical points $\Lambda_\varepsilon^\pm$ of Φ_ε with

$$\Lambda_\varepsilon^\pm = \Lambda^\pm + o(1), \quad \lim_{\varepsilon\to 0} o(1) = 0 .$$

For $\xi_\varepsilon^\pm = \xi(\Lambda_\varepsilon^\pm)$, the functions $v^\pm = V + \phi(\xi_\varepsilon^\pm)$ are solutions of Problem (3.5). From the equation satisfied by ϕ, (3.16), and its smallness in the *-norm, we derive that $v = V(1 + o(1))$, where $o(1) \to 0$ uniformly on $(0, \infty)$.

Further, if we set simply $\xi^{\pm} \equiv \xi(\Lambda^{\pm})$, then it is also true that

$$v^{\pm}(x) = \sum_{1=1}^{k} U(x - \xi_i^{\pm})\,(1 + o(1))\,,$$

again with $o(1) \to 0$ uniformly on $(0, \infty)$. Finally, if we go back in the change of variables (3.4) to a solution of (1.1), the explicit form of the parameters $\Lambda^{\pm}$ found in Lemma 3.2 provides the expression (3.2) for the solutions. This concludes the proof of Theorem 1. □

4. Super-critical bubbling in a Neumann problem

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$ with smooth boundary $\partial\Omega$. The boundary value problem

$$\begin{cases} -d^2\Delta u + u = u^q & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

where $q > 1$ and $d > 0$, has deserved a lot of attention in recent years. It arises for instance as the *shadow system* associated to activator-inhibitor systems in mathematical theory of biological pattern formation such as the Gierer-Meinhardt model and in certain models of chemotaxis, see references in [45]. In such models, and related ones, it is particularly meaningful the presence of solutions exhibiting peaks of concentration, namely one or several local maxima around which the solution remains strictly positive, while being very small away from them.

The works [45, 48, 49] have dealt with precise analysis of least energy solutions to this problem in the subcritical case, $1 < q < \frac{N+2}{N-2}$ namely solutions which minimize the Rayleigh quotient

$$Q(u) = \frac{d^2 \int_\Omega |\nabla u|^2 + \int_\Omega |u|^2}{\left(\int_\Omega |u|^{q+1}\right)^{\frac{2}{q+1}}}, \quad u \in H^1(\Omega) \setminus \{0\}, \tag{4.2}$$

for small d. From those works, it became known that for d sufficiently small, a minimizer u_d of Q has a unique local maximum point x_d which is located on the boundary. Besides, $H(x_d) \to \max_{x \in \partial\Omega} H(x)$ where H denotes mean curvature of $\partial\Omega$ and

$$u_d(x) \sim W\left(\frac{x - x_d}{d}\right), \tag{4.3}$$

where W denotes the (unique) radially symmetric solution of

$$\Delta W - W + W^p = 0 \quad \text{in } \mathbb{R}^N \tag{4.4}$$

$$W > 0, \quad \lim_{|x| \to +\infty} W(x) = 0.$$

This solution decays exponentially which implies indeed the presence of a very sharp, bounded spike for the solution around x_d. See also [23] for a short proof of these facts.

Solutions other than least energy with similar qualitative behavior around one or several points of the boundary or inside the domain have been found by several authors, see [19, 27, 33, 37, 34, 40, 42, 60] and their references. In particular, it is known from [60] that such a spike solution exists around any non-degenerate critical point of $H(x)$.

Phenomena of this type occur as well in the critical case $q = \frac{N+2}{N-2}$, however several important differences are present. For instance, since compactness of the embedding of $H^1(\Omega)$ into $L^{q+1}(\Omega)$ is lost, existence of minimizers of $Q(u)$ becomes non-obvious (and in general not true for large d as recently established in [44]). It is the case however, as shown in [1, 58], that such a minimizer does exist if d is sufficiently small. However the asymptotic profile (4.3) is lost. In fact, as a consequence of Pohozaev's identity, no solution to (4.4) for $q \geq \frac{N+2}{N-2}$ exists. The profile and asymptotic behavior of this least energy solution has been analyzed in [4, 50, 56]. Again only one local maximum point x_d located around a point of maximum mean curvature of $\partial\Omega$ exists. However, unlike the subcritical case now its maximum value $M_d = u_d(x_d) \to +\infty$. The asymptotic profile of u_d is now, at leading order

$$u_d(x) \sim (M_d/\alpha_N)\, w(\,(M_d/\alpha_N)^{\frac{p-1}{2}}(x - x_d)\,)$$

where $p = \frac{N+2}{N-2}$ and w is given by (2.5). The energy level of u_d is now well approximated by

$$d^{-2}Q(u_d) \sim \frac{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2}{(\,\frac{1}{2}\int_{\mathbb{R}^N} |w|^{p+1})^{\frac{2}{p+1}}}. \tag{4.5}$$

Construction of solutions with this type of *bubbling behavior* around one or more critical points of mean curvature has been achieved for instance in [2, 3, 32, 35, 55, 59]. An important difference with the subcritical case is that now mean curvature is required to be positive at these critical points. In fact, non-negativity of curvature is actually necessary for existence [5, 56, 36]. Recently in [36], behavior of solutions with energy values (4.5) have been thoroughly characterized, improving previous results in [5]. In particular blow-up points for such solutions are shown to be *simple*, in the sense that an appropriate constant multiple of $w(x)$ bounds globally from above the scaled solution around its maximum point. This type of estimates for bubbling for other elliptic problems at the critical exponent are found in [41, 43].

Little is known for Problem (4.1) when the power q is supercritical, namely $q > \frac{N+2}{N-2}$. Sobolev embedding no longer holds, so that variational construction of solutions becomes difficult. Here we consider this case for powers close to critical, where now we let the parameter d be fixed, with no loss of generality $d = 1$. Our first result establishes existence of boundary bubbling solutions when q approaches critical from the super-critical side, namely $q = \frac{N+2}{N-2} + \varepsilon$ with small $\varepsilon > 0$. Given a non-degenerate critical point of mean curvature (or, more generally, a situation of topologically non-trivial critical point) with *positive critical value*, a solution exhibiting boundary bubbling around such a point as $\varepsilon \to 0$ exists. Thus we deal with the semilinear elliptic problem

$$\begin{cases} -\Delta u + u = u^{\frac{N+2}{N-2}+\varepsilon} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{4.6}$$

where $\varepsilon > 0$. Let $H(x)$ denote mean curvature of $\partial\Omega$. We explain next what we mean by *topologically non-trivial critical point situation* for $H(x)$, which includes as special cases, local minima, maxima or non-degenerate critical points.

Let Λ be a (relative) open subset of $\partial\Omega$ with smooth boundary. We say that *H links non-trivially in Λ at critical level $\mathcal{H}_\Lambda$ relative to B and B_0* if B and B_0 are closed subsets of $\bar{\Lambda}$ such that B is conected and $B_0 \subset B$ such that the following conditions hold: if we set

$$\Gamma = \{\Phi \in C(B,\Lambda) \ / \ \Phi|_{B_0} = Id\}$$

then

$$\sup_{y\in B_0} H(y) < \mathcal{H}_\Lambda \equiv \inf_{\Phi\in\Gamma} \sup_{y\in B} H(\Phi(y)),$$

and for all $y \in \partial\Lambda$ such that $H(y) = \mathcal{H}_\Lambda$, there exists a vector τ_y tangent to $\partial\Lambda$ at y such that

$$\nabla H(y) \cdot \tau_y \neq 0.$$

Standard deformation arguments show that under these conditions a critical point $\bar{y} \in \Lambda$ of H with $H(\bar{y}) = \mathcal{H}_\Lambda$ in fact exists. It is easy to check that the above conditions hold if

$$\inf_{x\in\Lambda} H(x) < \inf_{x\in\partial\Lambda} H(x), \qquad \text{or} \qquad \sup_{x\in\Lambda} H(x) > \inf_{x\in\partial\Lambda} H(x),$$

namely the case of (possibly degenerate) local minimum or maximum points of H. They also hold if Λ is any small neighborhood of a non-degenerate critical point of H. This notion of local linking was used in [27] to build up boundary spikes in the subcritical case of (4.1), and was previously used in [22]. An alternative notion of non-trivial critical point of H was used in this context in [42].

Our first result is the following.

Theorem 2. [28] *Assume that $N \geq 4$ and that there is an open, smooth subset Λ of $\partial\Omega$ where mean curvature $H(x)$ notrivially links at critical level $\mathcal{H}_\Lambda$. If additionally $\mathcal{H}_\Lambda > 0$, for all sufficiently small $\varepsilon > 0$ there is a solution $u_\varepsilon(x)$ of (4.6) of the following form,*

$$u_\varepsilon(y) = \alpha_N \left(\frac{1}{1+\lambda^2\varepsilon^{-2}|y-\zeta_\varepsilon|^2}\right)^{\frac{N-2}{2}} \lambda^{\frac{N-2}{2}}\varepsilon^{-\frac{N-2}{2}}(1+o(1))$$

where $o(1) \to 0$ uniformly in $\overline{\Omega}$,

$$\lambda = \gamma_N \mathcal{H}_\Lambda,$$

$\gamma_N > 0$ is a explicit constant, and ζ_ε is a point in Λ such that

$$H(\zeta_\varepsilon) \to \mathcal{H}_\Lambda\,, \qquad \nabla H(\zeta_\varepsilon) \to 0,$$

as $\varepsilon \to 0$. The same statement holds true for dimension $N = 3$, where now

$$u_\varepsilon(y) \;=\; \alpha_3 \left(\frac{1}{1 \,+\, \lambda^2 \varepsilon^{-2}\, |\log\varepsilon|^2\, |y - \zeta_\varepsilon|^2} \right)^{\frac{1}{2}} \lambda^{\frac{1}{2}} \varepsilon^{-\frac{1}{2}}\, |\log\varepsilon|^{\frac{1}{2}}\, (\,1 \,+\, o(1)\,).$$

Recently in [16] it has been found that if $N \geq 4$, d is left fixed and one considers the exponent q as a parameter approaching the critical exponent *from below*, then single-bubbling solutions exist in certain cases. In particular, they find existence of single-bubble solutions with maximum points located on the boundary, near critical points of mean curvature with *negative value.*

The situation we deal with is more delicate because of breaking of Sobolev's embedding. This makes the approach of construction of solutions employed in [16] or in the above quoted references for the case $q = \frac{N+2}{N-2}$, $d \to 0$, technically not-applicable. We also observe that an inverse phenomenon compared with bubbling in the latter situation arises: the blow-up rate actually decreases as the value of curvature $\mathcal{H}_\Lambda$ does. Blow-up is instead enhanced for $q = \frac{N+2}{N-2}$, $d \to 0$ as the critical value of curvature decreases to zero.

Our second result shows that in analogy to Theorem 1, *super-critical bubbling* does not need to be *simple.* In fact we are able to construct solutions with just one maximum point for which multiple bubbling is present. For instance if Ω is a ball, there exists a solution whose shape is that of a tower, constituted by superposition of an arbitrary number of single-bubbles of different blow-up orders. This phenomenon actually takes place just provided that Ω is symmetric with respect to the first $(N-1)$ variables, and $0 \in \partial\Omega$ is a point with positive mean curvature.

Theorem 3. [28] *Assume that $0 \in \partial\Omega$, $H(0) > 0$ and $N \geq 4$. Moreover, assume that for any $i = 1, \dots, N-1$,*

$$\text{if} \quad (y_1, \dots, y_i, \dots, y_N) \in \Omega \quad \text{then} \quad (y_1, \dots, -y_i, \dots, y_N) \in \Omega.$$

Then, given $k \geq 1$, there exists for all sufficiently small $\varepsilon > 0$ a solution u_ε of (4.6) of the form

$$u_\varepsilon(y) = \alpha_N \sum_{i=1}^{k} \left(\frac{1}{1 + \lambda_i^2 \varepsilon^{-2+(1-i)\frac{4}{N-2}} |y|^2} \right)^{\frac{N-2}{2}} \lambda_i^{\frac{N-2}{2}} \varepsilon^{-\frac{N-2}{2} - i + 1} \,(1 + o(1))$$

where $o(1) \to 0$ uniformly in $\overline{\Omega}$. Here

$$\lambda_i = \frac{H(0)}{k} \left[\gamma_N \beta_N^{i-1} \frac{(k-i)!}{(k-1)!} \right]^{\frac{2}{N-2}},$$

for $i=1,\dots,k$, where the positive constants γ_N, β_N are explicit. The same statement holds true for $N=3$ except that now

$$u_\varepsilon(y) = \alpha_3 \sum_{i=1}^{k} \left(\frac{1}{1+\lambda_i^2 \varepsilon^{2-4i} \,|\log \varepsilon|^2 \,|y|^2} \right)^{\frac12} \lambda_i^{\frac12} \varepsilon^{\frac12 - i} \,|\log\varepsilon|^{\frac12} \,(1 + o(1))$$

The solution predicted by this theorem is a superposition of k bubbles with respective blow-up orders $\varepsilon^{-\frac{N-2}{2}-i+1}$ for $N \geq 4$ and $\varepsilon^{\frac12 - i}|\log\varepsilon|^{\frac12}$ for $N=3$, $i=1,\dots,k$.

The proofs of Theorems 2 and 3 rely on a form of Lyapunov-Schmidt procedure similar to that used in Theorem 1 which reduces the construction of the seeked solutions to a finite-dimensional variational problem. In order to overcome the supercritical nature of the problem, we work out this reduction in some ad-hoc weighted L^∞ spaces. Very useful for this purpose, especially in the description of the multi-bubbling effect, is the introduction of polar coordinates around a reference point $\zeta \in \partial\Omega$, and then a transformation of the radial coordinate similar to (3.4), after which dilations are converted into translations in a one-dimensional variable. More precisely, we set

$$\rho = |y-\zeta| \quad \text{and} \quad \theta = \frac{y-\zeta}{|y-\zeta|}. \tag{4.7}$$

Here $(\rho,\theta) \in \tilde\Omega_\zeta$, which is a subset of $\tilde S = (0,+\infty) \times S^{N-1}$, and then

$$v(x,\theta) = \left(\frac{2}{p-1}\right)^{\frac{2}{p-1+\varepsilon}} \rho^{\frac{2}{p-1}} \tilde u(\rho,\theta), \quad \rho = e^{-\frac{p-1}{2}x}. \tag{4.8}$$

We denote by D the ζ-dependent subset of $S = \mathbb{R}\times S^{N-1}$ where the variables (x,θ) vary. After these changes of variables, problem (4.6) becomes

$$\begin{cases} \left(\frac{2}{p-1}\right)^2 \Delta_{S^{N-1}} v + v'' - v + e^{\varepsilon x} v^{p+\varepsilon} - \left(\frac{2}{p-1}\right)^2 e^{-(p-1)x} v = 0 & \text{in } D \\ v > 0 & \text{in } D_\zeta \\ (\frac{2}{p-1}) \nabla_\theta v \cdot \nu^\theta + \frac{\partial v}{\partial x}\nu^x + v\nu^x = 0 & \text{on } \partial D. \end{cases} \tag{4.9}$$

Here $' = \frac{\partial}{\partial x}$. This language is especially useful in the analysis of the linearized operator around a proper ansatz similar to that in (3.10). Estimates for solutions of the associated linearized operator in weighted norms, which would appear quite involved in original variables, take here natural forms. After this analysis, the finite dimensional variational problem can be studied in a fairly direct way. To be remarked is that the symmetry assumption in the multi-bubble case avoids that the reduced problem analogous to that in the proof of Theorem 1 be overdetermined.

5. Duality sub-supercritical bubbling in Problem (1.1)

Precise asymptotics for radial blowing-up solutions of (1.1) in a ball respectively when $\lambda \leq \lambda^*$, $q = \frac{N+2}{N-2} - \varepsilon$ and when $q = \frac{N+2}{N-2}$, $\lambda = \lambda^* + \varepsilon$ were found by Atkinson and Peletier [6, 7] and by Brezis and Peletier [13]. The results in [13] strongly suggested the role of Green's function in the location of blow-up for single-bubble solutions u of (1.1) in a general domain, a fact later confirmed from results by Rey [54] and Han [38]. Let us consider Green's function $G_0(x,y)$ of Ω, which for given $x \in \Omega$ solves

$$-\Delta_y G_0 = \delta_x \quad y \in \Omega\,,$$
$$G_0(x,y) = 0 \quad y \in \partial\Omega\,,$$

where δ_x is the Dirac mass centered at x. We consider Robin's function $g_0(x)$ defined as

$$g_0(x) = H_0(x,x)$$

where

$$H_0(x,y) = \frac{c_N}{|y-x|^{N-2}} - G_0(x,y).$$

g_0 is a smooth, strictly positive function which goes to $+\infty$ as x approaches $\partial\Omega$. Rey [54] found that for $N \geq 4$ solutions u_λ of (1.1) for $q = \frac{N+2}{N-2}$, $\lambda > 0$ with energy $Q_\lambda(u_\lambda) = S(N) + o(1)$ as $\lambda \to 0$ constitute single-bubbles with blow-up points around a critical point of g_0. Reciprocally, he finds existence of single-bubble solutions with blowing-up points near any non-degenerate critical point of $g_0(x)$. For $N = 3$, rather than g_0, the results of [13] suggest that the object responsible for the presence of blowing-up solutions is the Robin's function g_λ defined as follows. Let $\lambda < \lambda_1$ and consider Green's function $G_\lambda(x,y)$, solution for given $x \in \Omega$ of

$$-\Delta_y G_\lambda - \lambda G_\lambda = \delta_x \quad y \in \Omega\,,$$
$$G_\lambda(x,y) = 0 \quad y \in \partial\Omega\,.$$

Then we define

$$g_\lambda(x) = H_\lambda(x,x)$$

where

$$H_\lambda(x,y) = \frac{1}{4\pi|y-x|} - G_\lambda(x,y).$$

$g_\lambda(x)$ is again a smooth function which goes to $+\infty$ as x approaches $\partial\Omega$. Unlike g_0, its minimum value is not necessarity positive. In fact this number is decreasing in λ. It is strictly positive when λ is close to 0 and approaches $-\infty$ as $\lambda \uparrow \lambda_1$. The number λ_* given by

$$\lambda_* = \sup\{\lambda > 0 \;/\; \min_\Omega g_\lambda > 0\}, \tag{5.1}$$

which equals $\frac{\lambda_1}{4}$ in the case of a ball, is suggested in [13] to be precisely the least value of λ for which a least energy solution of (1.1) exists in dimension $N = 3$. This has been recently established by Druet in [30]. Besides, it is shown that least energy solutions u_λ for $\lambda \downarrow \lambda_*$ constitute a single-bubble with blowing-up near the set where g_{λ_*} attains its minimum value zero.

We consider here the role of *non-trivial critical values* of g_λ in existence of solutions of (1.1) in dimension $N = 3$. In fact their role is intimate, not only in the critical case $q = 5$ and in the sub-critical $q = 5 - \varepsilon$. More interesting, their connection with solvability of (1.1) for powers above critical is found. In fact phenomena apparently unknown even in the case of the ball is established, which put in evidence an amusing duality between the sub and super-critical cases. We also find parallel results in dimensions $N \geq 4$, where the relevant object is g_0 rather than g_λ. For the sake of focusing, we only state below our results for dimension 3. The meaning of a non-trivial critical value of g_λ is the same introduced in Theorem 2: Let $\mathcal{D}$ be an open subset of Ω with smooth boundary. We recall that g_λ *links non-trivially in* $\mathcal{D}$ *at critical level* $\mathcal{G}_\lambda$ *relative to* B *and* B_0 if B and B_0 are closed subsets of $\bar{\mathcal{D}}$ with B conected and $B_0 \subset B$ such that the following conditions hold: if we set

$$\Gamma = \{\Phi \in C(B, \mathcal{D}) \ / \ \Phi|_{B_0} = Id\}$$

then

$$\sup_{y \in B_0} g_\lambda(y) < \mathcal{G}_\lambda \equiv \inf_{\Phi \in \Gamma} \sup_{y \in B} g_\lambda(\Phi(y)),$$

and for all $y \in \partial\mathcal{D}$ such that $g_\lambda(y) = \mathcal{G}_\lambda$, there exists a vector τ_y tangent to $\partial\mathcal{D}$ at y such that

$$\nabla g_\lambda(y) \cdot \tau_y \neq 0.$$

Under these conditions a critical point $\bar{y} \in \mathcal{D}$ of g_λ with $g_\lambda(\bar{y}) = \mathcal{G}_\lambda$ in fact exists.

Theorem 4. [21] *Let us assume that $N = 3$ and that there is a set $\mathcal{D}$ where g_λ has a non-trivial critical level $\mathcal{G}_\lambda$.*
(a) *Assume that $\mathcal{G}_\lambda < 0$ $\quad q = 5 + \varepsilon$. Then Problem* (1.1) *is solvable for all sufficiently small $\varepsilon > 0$. More precisely, there exists a solution u_ε of* (1.1) *of the form*

$$u_\varepsilon(y) = \alpha_3 \left(\frac{1}{1 + M_\varepsilon^4 \, |y - \zeta_\varepsilon|^2} \right)^{\frac{1}{2}} M_\varepsilon \, (1 + o(1))$$

where $o(1) \to 0$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$,

$$M_\varepsilon = \frac{2^{\frac{3}{2}}}{3^{\frac{1}{8}} \pi} (-\mathcal{G}_\lambda)^{1/2} \varepsilon^{-\frac{1}{2}} .$$

and ζ_ε is a point in $\mathcal{D}$ such that $g_\lambda(\zeta_\varepsilon) \to \mathcal{G}_\lambda$, $\quad \nabla g_\lambda(\zeta_\varepsilon) \to 0, \quad$ as $\varepsilon \to 0$.

(b) *Assume that $\mathcal{G}_\lambda > 0$, $\quad q = 5 - \varepsilon$. Then Problem* (1.1) *has a solution u_ε of* (1.1) *exactly as in part (a) but with*

$$M_\varepsilon = \frac{2^{\frac{3}{2}}}{3^{\frac{1}{8}} \pi} (\mathcal{G}_\lambda)^{1/2} \varepsilon^{-\frac{1}{2}} .$$

The result of part (b) recovers the asymptotics found for the radial solution of (1.1) when Ω is a ball and $0 < \lambda < \frac{\lambda_1}{4}$ in Theorem 1 of [13]. As a consequence of Part (a), we find the following solvability result for the super-critical case of (1.1).

Corollary 1. *If $N = 3$ and $\lambda_* < \lambda < \lambda_1$ where λ_* is given by* (5.1), *then Problem* (1.1) *is solvable for $q = 5 + \varepsilon$ and all sufficiently small $\varepsilon > 0$. More presicely, a single-bubble solution exists with blow-up point near the minimum set of g_λ.*

Our next result exhibits a rather striking phenomenon taking place in the super-critical case $q = 5 + \varepsilon$. Not only the single-bubble solution above predicted or that of part (a) exists. In fact, under the presence of symmetries, unbounded solutions with just one maximum point, but for which the approximation (2.6) does not hold globally, appear. This solution has the shape of a *tower* constituted by a superposition of an arbitrary number of single bubbles. We say that $\Omega \subset \mathbb{R}^N$ is symmetric with respect to the coordinate axes if for any $i = 1, \dots, N$,

$$(y_1, \dots, y_i, \dots, y_N) \in \Omega \quad \text{implies} \quad (y_1, \dots, -y_i, \dots, y_N) \in \Omega.$$

Theorem 5. [21] *Assume that $N = 3$, $0 \in \Omega$, and that Ω is symmetric with respect to the coordinate axes. Assume also that $g_\lambda(0) < 0$ and $q = 5 + \varepsilon$. Then, given $k \geq 1$, there exists for all sufficiently small $\varepsilon > 0$ a solution u_ε of Problem* (1.1) *of the form*

$$u_\varepsilon(x) = \alpha_3 \sum_{j=1}^{k} \left(\frac{1}{1 + M_{j\varepsilon}^4 |x|^2} \right)^{\frac{1}{2}} M_{j\varepsilon} \left(1 + o(1)\right)$$

where $o(1) \to 0$ uniformly in $\overline{\Omega}$ and

$$M_{j\varepsilon} = (-g_\lambda(0))^{1/2} \left[c\beta^{i-1} \frac{(k-i)!}{(k-1)!} \right]^2 \varepsilon^{\frac{1}{2}-j},$$

for $j = 1, \dots, k$, where c and β are explicit constants.

The solution predicted by this theorem is a superposition of k bubbles with respective blow-up orders $\varepsilon^{\frac{1}{2}-j}$ $j = 1, \dots, k$.

Our next result refers to phenomena associated to a non-trivial critical value zero of g_λ for a number $\lambda = \lambda_{**}$, which apply in particular to the number λ_* in (5.1). For the statement we make the following observation. Since g_λ and its derivative depend continuously on λ, it turns out that if $g_{\lambda_{**}}$ non-trivially links in $\mathcal{D}$ relative to B and B_0 at level $\mathcal{G}_{\lambda_{**}}$, then so does g_λ at a well defined critical level $\mathcal{G}_\lambda$ for all λ sufficiently close to λ_{**} . Besides, since g_λ is strictly decreasing in λ

$$\mathcal{G}_{\lambda^1} < \mathcal{G}_{\lambda_{**}} < \mathcal{G}_{\lambda^2}$$

whenever $\lambda^1 > \lambda_{**} > \lambda^2$. To fix ideas, let us think of the local minimum situation in $\mathcal{D}$,

$$\mathcal{G}_{\lambda_{**}} = \inf_{x \in \mathcal{D}} g_{\lambda_{**}}(x) < \inf_{x \in \partial\mathcal{D}} g_{\lambda_{**}}(x)$$

then for λ close to λ_{**} we take

$$\mathcal{G}_\lambda = \inf_{x \in \mathcal{D}} g_\lambda(x).$$

Theorem 6. [21] *Let us assume that $N = 3$ and that for a number $\lambda = \lambda_{**}$ and an open, smooth subset $\mathcal{D}$ of Ω, $g_{\lambda_{**}}$ has a nontrivial critical value $\mathcal{G}_{\lambda_{**}} = 0$. Consider as well for λ close to λ_{**} the associated non-trivial critical value $\mathcal{G}_\lambda$.*

(a) *Assume that $q = 5 + \varepsilon$. Let*

$$\gamma > \frac{\sqrt{16 - 3\pi}}{4} 3^{\frac{1}{8}} \sqrt{\pi}$$

be fixed and assume additionally that $\lambda > \lambda_{**}$ is the unique number for which

$$\mathcal{G}_\lambda = -\gamma \sqrt{\lambda} \varepsilon^{\frac{1}{2}}.$$

Then for all ε sufficiently small there exist two solutions $u_\varepsilon^\pm$ to Problem (1.1) *of the form*

$$u_\varepsilon^\pm(x) = \alpha_3 \left(\frac{1}{1 + (M_\varepsilon^\pm)^4 |x - \zeta_\varepsilon|^2} \right)^{\frac{1}{2}} M_\varepsilon^\pm (1 + o(1)) \qquad (5.2)$$

where $o(1) \to 0$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$,

$$M_\varepsilon^\pm = m_\pm(\gamma) \varepsilon^{-\frac{1}{4}} .$$

where $m_\pm(\gamma)$ are the two positive roots of

$$am^2 - \gamma m + b = 0$$

with

$$a = 2\lambda(1 - \frac{3}{16}\pi), \quad b = \frac{3^{\frac{1}{4}}}{8}\pi \qquad (5.3)$$

and ζ_ε is a point in $\mathcal{D}$ such that

$$g_\lambda(\zeta_\varepsilon) \to 0, \quad \nabla g_\lambda(\zeta_\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0. \qquad (5.4)$$

(b) *Assume that $q = 5 - \varepsilon$. Let $\gamma \in (-\infty, +\infty)$ be fixed and assume additionally that λ (close to λ_{**}) is the unique number for which*

$$\mathcal{G}_\lambda = \gamma \sqrt{\lambda} \varepsilon^{\frac{1}{2}}.$$

Then for all ε sufficiently small there exist a solutions u_ε to Problem (1.1) *of the form (5.2) with $M_\varepsilon^\pm$ replaced by M_ε where*

$$M_\varepsilon = m(\gamma) \varepsilon^{-\frac{1}{4}} .$$

where $m(\gamma)$ is the unique positive root of

$$am^2 + \gamma m - b = 0$$

with a, b as in (5.3) and ζ_ε satisfies (5.4).

(c) *Assume that $q = 5$. Then for all $\lambda > \lambda_{**}$ sufficiently close to λ_{**} there* *exists a solution u_λ of Problem* (1.1) *of the form (5.2) with ζ_ε replaced by a point ζ_λ in $\mathcal{D}$ as in (5.4), with $M_\varepsilon^\pm$ now replaced by M_λ where*

$$M_\lambda = \left[\frac{5\pi}{2\,3^{\frac{1}{4}}}|\mathcal{G}_\lambda|^2 - 2\lambda(1 - \frac{3}{16}\pi)\right]^{\frac{1}{2}} (-\mathcal{G}_\lambda)^{-\frac{1}{2}}$$

Part (c) shows that a general domain may in principle have several *Brezis-Nirenberg numbers* λ_{**}, other than λ_*, where a "branch" of solutions u_λ comes down to the right of it. The result of part (b) recovers the asymptotics found in Theorem 2 of [13] for the radial solution in a ball when $\gamma = 0$.

It is illustrative to describe the results of Theorems 4-6 in terms of the bifurcation branch for the positive solutions of (1.1) in a ball which stems from $\lambda = \lambda_1$, $u = 0$, for any value of q. This branch does not have turning points for $q = 5$ (uniqueness of the positive radial solution is known from [61]) and blows-up at $\lambda = \frac{\lambda_1}{4}$. On the other hand, as soon as $\varepsilon > 0$, $q = 5+\varepsilon$ the branch turns right near the asymptote and then lives until getting close to λ_1. This "upper part" of the branch is the one described in Theorem 4, part (a). It is of course reasonable to ask how the turning point looks like, in particular showing the presence of two solutions for λ slightly to the right of it. This is the interpretation Theorem 6, part (a). Formal asymptotics of this first turning point, which are fully recovered by this result, were found by Budd and Norbury [14].

It is of course natural to ask what is the behavior of this branch "later". The result of Theorem 5 partly answers this question: for $\varepsilon > 0$ the branch oscillates wildly between $\frac{\lambda_1}{4}$ and λ_1, giving rise for fixed λ between these numbers to an arbitrarily large number of solutions. The towers of Theorem 5 may be interpreted as the solution found on the branch between the k-th and $k+1$ turning points.

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