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The polar decomposition of the product of operators and its applications to binormal and centered operators

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1 Introduction

This report is based on the following preprint:

- Masatoshi Ito, Takeaki Yamazaki and Masahiro Yanagida, *On the polar decomposition of the Aluthge transformation and related results*, to appear in J. Operator Theory.

- Masatoshi Ito, Takeaki Yamazaki and Masahiro Yanagida, *On the polar decomposition of the product of two operators and its applications*, to appear in Integral Equations Operator Theory.

In what follows, an operator means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T\geq 0$) if $(Tx,x)\geq 0$ for all $x\in H$.

It is well known that every operator T can be decomposed into $T=U|T|$ with a partial isometry U, where $|T|=(T^{*}T)^{\frac{1}{2}}$. U is determined uniquely by the kernel condition $N(U)=N(T)$, then this decomposition is called the polar decomposition. On the polar decomposition of the product of two operators, Furuta [4] showed a result in case they are doubly commutative. But it has not been obtained in the general case.

Let $T = U|T|$ be the polar decomposition. Then

$$\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$$

is called the Aluthge transformation of T [1], which is very useful for the study of non-normal operators. The polar decomposition of the Aluthge transformation was discussed in [1], but the complete solution of this problem has not been obtained.

Throughout this report, $\tilde{T}_n$ denotes the n-th iterated Aluthge transformation of T [8], that is,

$$\tilde{T}_n = \widetilde{(\tilde{T}_{n-1})} \quad \text{and} \quad \tilde{T}_0 = T.$$

One of the authors showed some properties of the n-th Aluthge transformation which are parallel to those of powers of operators in [11], [12] and [13].

An operator T is said to be binormal if

$$[|T|, |T^*|] = 0,$$

where $[A, B] = AB - BA$ for operators A and B, and T is said to be centered if the following sequence

$$\ldots, T^3(T^3)^*, T^2(T^2)^*, TT^*, T^*T, (T^2)^*T^2, (T^3)^*T^3, \ldots$$

is commutative. Binormal and centered operators were defined by Campbell [2] and Morrel-Muhly [9], respectively. Binormal operators are called weakly centered in [10]. Relations among these classes and that of quasinormal operators ($\Longleftrightarrow T^*TT = TT^*T$) are easily obtained as follows:

$$\text{quasinormal} \subsetneqq \text{centered} \subsetneqq \text{binormal}.$$

In fact, $T = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & 0 \end{pmatrix}$ is centered and not quasinormal, and $T = \begin{pmatrix} 0 & & & \\ A & 0 & & \\ & I & 0 & \\ & & A & 0 \end{pmatrix}$ is binormal and not centered, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. We remark that every weighted shift is centered, and every binormal and paranormal ($\Longleftrightarrow \|T^2x\| \geq \|Tx\|^2$ for every unit vector x) operator is hyponormal ($\Longleftrightarrow T^*T \geq TT^*$) [3].

In this report, firstly, we shall show results on the polar decomposition of the product and the Aluthge transformation of operators. Secondly, as applications of these results, we shall show several properties and characterizations of binormal and centered operators from the viewpoint of the polar decomposition and the Aluthge transformation.

2 The polar decomposition of the product and the Aluthge transformation of operators

The following result shows the polar decomposition of the product of two operators.

Theorem 2.1. *Let* $T = U|T|$, $S = V|S|$ *and*

$$|T||S^*| = W\big||T||S^*|\big| \tag{2.1}$$

be the polar decompositions. Then $TS = UWV|TS|$ *is also the polar decomposition.*

Theorem 2.1 is a generalization of the following result since U (resp. V) and S (resp. T) are doubly commutative and $W = U^*UVV^*$ in case T and S are doubly commutative.

Theorem 2.A ([4]). *Let* $T = U|T|$ *and* $S = V|S|$ *be the polar decompositions. If* T *and* S *are doubly commutative (i.e.,* $[T, S] = [T, S^*] = 0$*), then*

$$TS = UV|TS|$$

is the polar decomposition.

Theorem 2.1 also implies the following result on the polar decomposition of the Aluthge transformation.

Theorem 2.2. *Let* $T = U|T|$ *and*

$$|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = W\big||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\big|$$

be the polar decompositions. Then $\tilde{T} = WU|\tilde{T}|$ *is also the polar decomposition.*

Proof of Theorem 2.1.
(i) Firstly, we shall show that $TS = UWV|TS|$ holds. We remark that $(V|S|A|S|V^*)^\alpha = V(|S|A|S|)^\alpha V^*$ holds for any positive operator $A \geq 0$ and positive number $\alpha > 0$.

$$\begin{aligned}
TS &= U|T||S^*|V \\
&= UW\big||T||S^*|\big|V \qquad \text{by (2.1)} \\
&= UW(|S^*||T|^2|S^*|)^{\frac{1}{2}}V \\
&= UWV(|S|V^*|T|^2V|S|)^{\frac{1}{2}}V^*V \\
&= UWV(S^*T^*TS)^{\frac{1}{2}} \\
&= UWV|TS|.
\end{aligned}$$

(ii) Secondly, we shall show that $N(TS) = N(UWV)$.

$$N(TS) = N(|T||S^*|V) = N(WV)$$

holds since $N(|T||S^*|) = N(W)$, and

$$\begin{aligned} N(WV) &\subseteq N(UWV) \\ &= N(|T|WV) \qquad \text{since } N(U) = N(|T|) \\ &\subseteq N(|S^*||T|WV) \\ &= N(W^*WV) \qquad \text{since } N(|S^*||T|) = N(W^*) \\ &= N(WV), \end{aligned}$$

so that $N(WV) = N(UWV)$ holds. Hence we have $N(TS) = N(WV) = N(UWV)$.
(iii) Lastly, we shall prove that UWV is a partial isometry. By (ii), we obtain that

$$N(UWV)^{\perp} = N(TS)^{\perp} = N(|TS|)^{\perp} = \overline{R(|TS|)}.$$

For any $x \in R(|TS|)$, there exists $y \in H$ such that $x = |TS|y$, so that we have

$$\begin{aligned} \|UWVx\| &= \|UWV|TS|y\| \\ &= \|TSy\| \qquad \text{by (i)} \\ &= \||TS|y\| \\ &= \|x\|. \end{aligned}$$

Hence we obtain $\|UWVx\| = \|x\|$ for all $x \in \overline{R(|TS|)} = N(UWV)^{\perp}$, that is, UWV is a partial isometry. □

Proof of Theorem 2.2. We have only to replace T and S with $|T|^{\frac{1}{2}} = U^*U|T|^{\frac{1}{2}}$ and $U|T|^{\frac{1}{2}}$ respectively in Theorem 2.1 since $U^*UW = W$. □

Next, we shall notice the form $TS = UV|TS|$ in Theorem 2.A, and obtain an equivalent condition to that $TS = UV|TS|$ becomes the polar decomposition, which is an extension of Theorem 2.A.

Theorem 2.3. *Let $T = U|T|$ and $S = V|S|$ be the polar decompositions. Then $|T||S^*| = |S^*||T|$ if and only if*

$$TS = UV|TS|$$

is the polar decomposition.

We can regard Theorem 2.3 as a bridge between the polar decomposition of the product and a kind of commutativity between two operators. We remark that in case T and S are partial isometries, Theorem 2.3 was already pointed out in [7, Lemma 2].

In order to give a proof of Theorem 2.3, we use the following lemmas.

Lemma 2.B ([6]). *If $T^2 = T$ and $\|T\| \leq 1$, then T is a projection.*

Lemma 2.4. *Let A and B be positive operators. Then the following assertions are equivalent:*

(i) $AB = BA$.

(ii) $AB = P_{N(A)^\perp}P_{N(B)^\perp}|AB|$ *is the polar decomposition, where P_M denotes the projection onto a closed subspace M.*

Proof. Since $A = P_{N(A)^\perp}A$ and $B = P_{N(B)^\perp}B$ are the polar decompositions, (i) $\Longrightarrow$ (ii) is obvious by Theorem 2.A.

We shall prove (ii) $\Longrightarrow$ (i). By (ii), $P_{N(A)^\perp}P_{N(B)^\perp}$ is a partial isometry. Then we have

$$\begin{aligned} P_{N(A)^\perp}P_{N(B)^\perp} &= P_{N(A)^\perp}P_{N(B)^\perp}(P_{N(A)^\perp}P_{N(B)^\perp})^*P_{N(A)^\perp}P_{N(B)^\perp} \\ &= P_{N(A)^\perp}P_{N(B)^\perp}P_{N(B)^\perp}P_{N(A)^\perp}P_{N(A)^\perp}P_{N(B)^\perp} \\ &= (P_{N(A)^\perp}P_{N(B)^\perp})^2. \end{aligned}$$

Since $\|P_{N(A)^\perp}P_{N(B)^\perp}\| \leq 1$ holds, we obtain that $P_{N(A)^\perp}P_{N(B)^\perp}$ is the projection onto $N(P_{N(A)^\perp}P_{N(B)^\perp})^\perp = N(AB)^\perp$ by Lemma 2.B. Hence we have

$$AB = P_{N(A)^\perp}P_{N(B)^\perp}|AB| = |AB| \geq 0,$$

i.e., $AB = BA$. □

Proof of Theorem 2.3. First, we shall prove "only if" part. By the condition $|T||S^*| = |S^*||T|$,

$$|T||S^*| = U^*UVV^*||T||S^*||$$

is the polar decomposition by Lemma 2.4. Hence we obtain that

$$TS = UU^*UVV^*V|TS| = UV|TS|$$

is the polar decomposition by Theorem 2.1.

Next, we shall prove "if" part. Let $|T||S^*| = W||T||S^*||$ be the polar decomposition. Then by Theorem 2.1,

$$TS = UWV|TS|$$

is also the polar decomposition. Hence we have $UV = UWV$ by the uniqueness of the polar decomposition. Since $N(W) = N(|T||S^*|) \supseteq N(|S^*|) = N(VV^*)$ and $H = N(VV^*) \oplus N(VV^*)^\perp$ hold, we have

$$WVV^* = W.$$

On the other hand, since $N(W^*) = N(|S^*||T|) \supseteq N(|T|) = N(U^*U)$ and $H = N(U^*U) \oplus N(U^*U)^{\perp}$ hold, we have $W^*U^*U = W^*$, that is,

$$U^*UW = W.$$

Hence by $UWV = UV$,

$$W = U^*UWVV^* = U^*UVV^*,$$

and we obtain the following polar decomposition

$$|T||S^*| = U^*UVV^*\big||T||S^*|\big|.$$

Therefore $|T||S^*| = |S^*||T|$ by Lemma 2.4. □

3 Properties and characterizations of binormal and centered operators

3.1 Binormal operators

We shall show a characterization of binormal operators via the polar decomposition and the Aluthge transformation.

Theorem 3.1. *Let $T = U|T|$ be the polar decomposition. Then the following assertions are equivalent:*

(i) *T is binormal.*

(ii) *$\tilde{T} = \tilde{U}|\tilde{T}|$ is the polar decomposition.*

(iii) *$T^2 = U^2|T^2|$ is the polar decomposition.*

In particular, if T is a partial isometry, then T is binormal if and only if $\tilde{T}$ or T^2 is a partial isometry.

We remark that $\tilde{U} = |U|^{\frac{1}{2}}U|U|^{\frac{1}{2}} = U^*UU$ for a partial isometry U, and (i) $\Longrightarrow$ (iii) has been already pointed out by Furuta [5].

Proof. (i) $\Longleftrightarrow$ (ii). Since $|T|^{\frac{1}{2}} = U^*U|T|^{\frac{1}{2}}$ and $U|T|^{\frac{1}{2}} = U|T|^{\frac{1}{2}}$ are the polar decompositions, we obtain that

$$\tilde{T} = |T|^{\frac{1}{2}} \cdot U|T|^{\frac{1}{2}} = U^*UU|\tilde{T}| = \tilde{U}|\tilde{T}|$$

is the polar decomposition if and only if

$$[|T|^{\frac{1}{2}}, |T^*|^{\frac{1}{2}}] = [|T|^{\frac{1}{2}}, |(U|T|^{\frac{1}{2}})^*|] = 0$$

by Theorem 2.3.
(i) $\Longleftrightarrow$ (iii). Since $T = U|T|$ is the polar decomposition, we obtain that

$$T^2 = T \cdot T = U^2|T^2|$$

is the polar decomposition if and only if

$$[|T|, |T^*|] = 0$$

by Theorem 2.3. $\square$

Campbell [3] gave an example of a binormal operator T such that T^2 is not binormal, and Furuta [5] showed an equivalent condition to that T^2 is binormal when T is binormal.

Theorem 3.A **([5]).** *Let $T = U|T|$ be the polar decomposition of a binormal operator T. Then T^2 is binormal if and only if the following properties hold:*

(i) $[(U^2)^*U^2, U^2(U^2)^*] = 0$. (ii) $[U^2(U^2)^*, U^*|T||T^*|U] = 0$.

(iii) $[(U^2)^*U^2, U|T||T^*|U^*] = 0$. (iv) $[U^*|T||T^*|U, U|T||T^*|U^*] = 0$.

On the other hand, most results on the Aluthge transformation $\tilde{T}$ show that it has better properties than T, see [1] for example. But there exists a binormal operator T such that $\tilde{T}$ is not binormal, for example $T = \begin{pmatrix} 0 & 0 & 5 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \end{pmatrix}$. Here we shall show an equivalent condition to that $\tilde{T}$ is binormal for a binormal operator T.

Theorem 3.2. *Let $T = U|T|$ be the polar decomposition of a binormal operator T. Then the following assertions are equivalent:*

(i) *$\tilde{T}$ is binormal.*

(ii) $[U^2|T|(U^2)^*, |T|] = 0$.

In order to give a proof of Theorem 3.2, we shall prepare the following lemma which is a modification of [4, Theorem 2].

Lemma 3.B. *Let $A, B \geq 0$ and $[A, B] = 0$. Then*

$$[P_{N(A)^\perp}, P_{N(B)^\perp}] = [P_{N(A)^\perp}, B] = [A, P_{N(B)^\perp}] = 0.$$

Proof of Theorem 3.2. T is binormal if and only if

$$[U|T|U^*, |T|] = 0. \tag{3.1}$$

Then we obtain

$$[U|T|U^*, U^2|T|(U^2)^*] = 0 \tag{3.2}$$

since

$$\begin{aligned} U^2|T|(U^2)^* \cdot U|T|U^* &= U \cdot U|T|U^* \cdot |T| \cdot U^* \\ &= U \cdot |T| \cdot U|T|U^* \cdot U^* \qquad \text{by (3.1)} \\ &= U|T|U^* \cdot U^2|T|(U^2)^*. \end{aligned}$$

Therefore we have

$$\begin{aligned} |\tilde{T}|^2|\tilde{T}^*|^2 &= |T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}} \cdot |T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}} \\ &= U^* \cdot U|T|^{\frac{1}{2}}U^* \cdot |T| \cdot U|T|U^* \cdot U^2|T|(U^2)^* \cdot U|T|^{\frac{1}{2}}U^* \cdot U \\ &= U^*\{|T| \cdot U^2|T|(U^2)^*\}U|T|^2U^*U \qquad \text{by (3.1) and (3.2)} \\ &= U^*\{|T| \cdot U^2|T|(U^2)^*\}U|T|^2 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} |\tilde{T}^*|^2|\tilde{T}|^2 &= |T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}} \cdot |T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}} \\ &= U^* \cdot U|T|^{\frac{1}{2}}U^* \cdot U^2|T|(U^2)^* \cdot U|T|U^* \cdot |T| \cdot U|T|^{\frac{1}{2}}U^* \cdot U \\ &= U^*\{U^2|T|(U^2)^* \cdot |T|\}U|T|^2U^*U \qquad \text{by (3.1) and (3.2)} \\ &= U^*\{U^2|T|(U^2)^* \cdot |T|\}U|T|^2. \end{aligned} \tag{3.4}$$

Proof of (ii) $\Longrightarrow$ (i). By (3.3) and (3.4), we have (i).
Proof of (i) $\Longrightarrow$ (ii). By Lemma 3.B,

$$[|T|, |T^*|] = [U^*U, U|T|U^*] = [|T|, UU^*] = [U^*U, UU^*] = 0. \tag{3.5}$$

Since $\tilde{T}$ is binormal, we have

$$\begin{aligned} \{U^2|T|(U^2)^* \cdot |T|\}U|T|^2 &= UU^*\{U^2|T|(U^2)^* \cdot |T|\}U|T|^2 \\ &= UU^*\{|T| \cdot U^2|T|(U^2)^*\}U|T|^2 \qquad \text{by (3.3) and (3.4)} \\ &= \{|T|UU^* \cdot U^2|T|(U^2)^*\}U|T|^2 \qquad \text{by (3.5)} \\ &= \{|T| \cdot U^2|T|(U^2)^*\}U|T|^2, \end{aligned}$$

that is, $U^2|T|(U^2)^* \cdot |T| = |T| \cdot U^2|T|(U^2)^*$ on

$$\overline{R(U|T|^2)} = N(|T|^2U^*)^\perp = N(UU^*)^\perp = R(UU^*).$$

In other words,

$$U^2|T|(U^2)^* \cdot |T| \cdot UU^* = |T| \cdot U^2|T|(U^2)^* \cdot UU^*. \tag{3.6}$$

Hence we have

$$\begin{aligned} U^2|T|(U^2)^* \cdot |T| &= U^2|T|(U^2)^* \cdot UU^* \cdot |T| \\ &= U^2|T|(U^2)^* \cdot |T| \cdot UU^* \qquad \text{by (3.5)} \\ &= |T| \cdot U^2|T|(U^2)^* \cdot UU^* \qquad \text{by (3.6)} \\ &= |T| \cdot U^2|T|(U^2)^*. \end{aligned}$$

□

Next we shall show the following result on the binormality of $\tilde{T}_n$.

Theorem 3.3. *Let $T = U|T|$ be the polar decomposition. Then for each non-negative integer n, the following assertions are equivalent:*

(i) $\widetilde{T_k}$ *is binormal for all $k = 0, 1, 2, \ldots, n$.*

(ii) $[U^k|T|(U^k)^*, |T|] = 0$ *for all $k = 1, 2, \ldots, n+1$.*

We prepare the following lemmas in order to give a proof of Theorem 3.3.

Lemma 3.4. *Let $T = U|T|$ be the polar decomposition. For each natural number n, if*

$$[U^k|T|(U^k)^*, |T|] = 0 \quad \textit{for all } k = 1, 2, \ldots, n,$$

then the following properties hold:

(i) $U^k|T|^\alpha(U^k)^* = \{U^k|T|(U^k)^*\}^\alpha$ *for any $\alpha > 0$ and all $k = 1, 2, \ldots, n+1$.*

(ii) $[U^k|T|^\alpha(U^k)^*, |T|] = [U^k|T|^\alpha(U^k)^*, U^*U] = 0$ *for any $\alpha > 0$ and all $k = 1, 2, \ldots, n$.*

(iii) $U^s|T|^\alpha(U^s)^*U^t = U^s|T|^\alpha(U^{s-t})^*$ *and* $(U^t)^*U^s|T|^\alpha(U^s)^* = U^{s-t}|T|^\alpha(U^s)^*$ *for any $\alpha > 0$ and all natural numbers s and t such that $1 \le t \le s \le n+1$.*

(iv) $(U^s)^*|T|^\alpha U^s(U^t)^* = (U^s)^*|T|^\alpha U^{s-t}$ *and* $U^t(U^s)^*|T|^\alpha U^s = (U^{s-t})^*|T|^\alpha U^s$ *for any $\alpha > 0$ and all natural numbers s and t such that $1 \le t \le s \le n$.*

(v) $[(U^k)^*|T^*|^\alpha U^k, |T^*|] = [(U^{k-1})^*|T|^\alpha U^{k-1}, U|T|U^*] = 0$ *for any $\alpha > 0$ and all $k = 1, 2, \ldots, n$.*

(vi) $[U^s|T|^\alpha(U^s)^*, U^t|T|^\alpha(U^t)^*] = 0$ *for any $\alpha > 0$ and all natural numbers s and t such that $s, t \in [1, n+1]$.*

(vii) $U^{n+1}|\tilde{T}|(U^{n+1})^* = U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot U^n|T|^{\frac{1}{2}}(U^n)^*$.

Proof. (i) We have only to prove that $[U^k|T|(U^k)^*, |T|] = 0$ for all $k = 1, 2, \ldots, n$ implies $U^{n+1}|T|^\alpha(U^{n+1})^* = \{U^{n+1}|T|(U^{n+1})^*\}^\alpha$ by induction on n. We remark that

$$[U^k|T|(U^k)^*, U^*U] = 0 \quad \text{for all } k = 1, 2, \ldots, n \tag{3.7}$$

by Lemma 3.B and the assumption. In case $n = 1$,

$$\begin{aligned} U^2|T|^\alpha(U^2)^* &= U(U|T|U^*)^\alpha U^* \\ &= U(U^*U)^{2\alpha}(U|T|U^*)^\alpha U^* \\ &= U(U^*U \cdot U|T|U^* \cdot U^*U)^\alpha U^* \quad \text{by (3.7)} \\ &= (UU^*UU|T|U^*U^*UU^*)^\alpha \\ &= \{U^2|T|(U^2)^*\}^\alpha. \end{aligned}$$

Assume that (i) holds for some natural number n. We shall show that it holds for $n+1$.

$$\begin{aligned} U^{n+2}|T|^\alpha(U^{n+2})^* &= U\left\{U^{n+1}|T|^\alpha(U^{n+1})^*\right\}U^* \\ &= U\left\{U^{n+1}|T|(U^{n+1})^*\right\}^\alpha U^* \quad \text{by the inductive hypothesis} \\ &= U(U^*U)^{2\alpha}\left\{U^{n+1}|T|(U^{n+1})^*\right\}^\alpha U^* \\ &= U\left\{U^*U \cdot U^{n+1}|T|(U^{n+1})^* \cdot U^*U\right\}^\alpha U^* \quad \text{by (3.7)} \\ &= \left\{UU^*UU^{n+1}|T|(U^{n+1})^*U^*UU^*\right\}^\alpha \\ &= \left\{U^{n+2}|T|(U^{n+2})^*\right\}^\alpha. \end{aligned}$$

(ii) By the assumption, (i) and Lemma 3.B, we have (ii).

(iii) By using (ii) repeatedly, we have

$$\begin{aligned} U^s|T|^\alpha(U^s)^*U^t &= U\left\{U^{s-1}|T|^\alpha(U^{s-1})^* \cdot U^*U\right\}U^{t-1} \\ &= U\left\{U^*U \cdot U^{s-1}|T|^\alpha(U^{s-1})^*\right\}U^{t-1} \quad \text{by (ii)} \\ &= U^2\left\{U^{s-2}|T|^\alpha(U^{s-2})^* \cdot U^*U\right\}U^{t-2} \\ &= U^2\left\{U^*U \cdot U^{s-2}|T|^\alpha(U^{s-2})^*\right\}U^{t-2} \quad \text{by (ii)} \\ &= U^3\left\{U^{s-3}|T|^\alpha(U^{s-3})^* \cdot U^*U\right\}U^{t-3} \\ &= \cdots \\ &= U^t\left\{U^{s-t}|T|^\alpha(U^{s-t})^* \cdot U^*U\right\} \\ &= U^t\left\{U^*U \cdot U^{s-t}|T|^\alpha(U^{s-t})^*\right\} \quad \text{by (ii)} \\ &= U^t \cdot U^{s-t}|T|^\alpha(U^{s-t})^* \\ &= U^s|T|(U^{s-t})^*, \end{aligned}$$

so that $U^s|T|^\alpha(U^s)^*U^t = U^s|T|^\alpha(U^{s-t})^*$ and $(U^t)^*U^s|T|^\alpha(U^s)^* = U^{s-t}|T|^\alpha(U^s)^*$.

(iv) Since $T^* = U^*|T^*|$ is polar decomposition of T^*,

$$(U^{s+1})^*|T^*|^\alpha U^{s+1}(U^t)^* = (U^{s+1})^*|T^*|^\alpha U^{s+1-t}$$
$$\text{and} \quad U^t(U^{s+1})^*|T^*|^\alpha U^{s+1} = (U^{s+1-t})^*|T^*|^\alpha U^{s+1}$$

for any $\alpha > 0$ and all natural numbers s and t such that $1 \leq t \leq s \leq n$ by (iii), so that we have

$$(U^s)^*|T|^\alpha U^s(U^t)^* = (U^s)^*|T|^\alpha U^{s-t} \quad \text{and} \quad U^t(U^s)^*|T|^\alpha U^s = (U^{s-t})^*|T|^\alpha U^s.$$

(v) Since $|T^*| = U|T|U^*$, we easily obtain

$$[(U^k)^*|T^*|^\alpha U^k, |T^*|] = [(U^{k-1})^*|T|^\alpha U^{k-1}, U|T|U^*]$$

for any $\alpha > 0$ and all $k = 1, 2, \ldots, n$, and also we have

$$\begin{aligned}
(U^{k-1})^*|T|^\alpha U^{k-1} \cdot U|T|U^* &= (U^{k-1})^*|T|^\alpha \cdot U^k|T|U^* \\
&= (U^{k-1})^* \left\{|T|^\alpha \cdot U^k|T|(U^k)^*\right\} U^{k-1} && \text{by (iii)} \\
&= (U^{k-1})^* \left\{U^k|T|(U^k)^* \cdot |T|^\alpha\right\} U^{k-1} && \text{by the assumption} \\
&= U|T| \cdot (U^k)^*|T|^\alpha U^{k-1} && \text{by (iii)} \\
&= U|T|U^* \cdot (U^{k-1})^*|T|^\alpha U^{k-1}
\end{aligned}$$

for any $\alpha > 0$ and all $k = 1, 2, \ldots, n$.

(vi) We may assume $t < s$.

$$\begin{aligned}
U^s|T|^\alpha(U^s)^* \cdot U^t|T|^\alpha(U^t)^* &= U^s|T|^\alpha(U^{s-t})^* \cdot |T|^\alpha(U^t)^* && \text{by (iii)} \\
&= U^t\left\{U^{s-t}|T|^\alpha(U^{s-t})^* \cdot |T|^\alpha\right\}(U^t)^* \\
&= U^t\left\{|T|^\alpha \cdot U^{s-t}|T|^\alpha(U^{s-t})^*\right\}(U^t)^* && \text{by (ii)} \\
&= U^t|T|^\alpha \cdot U^{s-t}|T|^\alpha(U^s)^* \\
&= U^t|T|^\alpha(U^t)^* \cdot U^s|T|^\alpha(U^s)^* && \text{by (iii).}
\end{aligned}$$

(vii) By using (ii) and (iii), we have

$$\begin{aligned}
U^{n+1}|\tilde{T}|(U^{n+1})^* &= U^{n+1}(|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})^{\frac{1}{2}}(U^{n+1})^* \\
&= U^n(U|T|^{\frac{1}{2}}U^* \cdot |T| \cdot U|T|^{\frac{1}{2}}U^*)^{\frac{1}{2}}(U^n)^* \\
&= U^n \cdot U|T|^{\frac{1}{2}}U^* \cdot |T|^{\frac{1}{2}} \cdot (U^n)^* && \text{by (ii)} \\
&= U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot U^n|T|^{\frac{1}{2}}(U^n)^* && \text{by (iii).}
\end{aligned}$$

□

Lemma 3.5. *Let $T = U|T|$ be the polar decomposition and n be a natural number. If*

$$[U^k|T|(U^k)^*, |T|] = 0 \quad \text{for all } k = 1, 2, \ldots, n,$$

then the following assertions are equivalent:

(i) $[U^{n+1}|T|(U^{n+1})^*, |T|] = 0$.

(ii) $[U^n|\tilde{T}|(U^n)^*, |\tilde{T}|] = 0$.

Proof. First, we remark that $[U|T|^{\frac{1}{2}}U^*, |T|] = 0$ by (ii) of Lemma 3.4, then we have

$$\begin{aligned} |\tilde{T}| &= (|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})^{\frac{1}{2}} = U^*(U|T|^{\frac{1}{2}}U^* \cdot |T| \cdot U|T|^{\frac{1}{2}}U^*)^{\frac{1}{2}}U \\ &= U^* \cdot |T|^{\frac{1}{2}} \cdot U|T|^{\frac{1}{2}}U^* \cdot U = U^*|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}. \end{aligned} \tag{3.8}$$

In case $n = 1$. Since $[U|T|^{\frac{1}{2}}U^*, |T|] = 0$, we have

$$\begin{aligned} U|\tilde{T}|U^* &= U(|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})^{\frac{1}{2}}U^* = (U|T|^{\frac{1}{2}}U^* \cdot |T| \cdot U|T|^{\frac{1}{2}}U^*)^{\frac{1}{2}} \\ &= (|T|^{\frac{1}{2}} \cdot U|T|U^* \cdot |T|^{\frac{1}{2}})^{\frac{1}{2}} = |(\tilde{T})^*|. \end{aligned}$$

Hence $[U|\tilde{T}|U^*, |\tilde{T}|] = [|(\tilde{T})^*|, |\tilde{T}|]$, i.e., $\tilde{T}$ is binormal, so that this case just coincides with Theorem 3.2.

Next, we shall prove that Lemma 3.5 holds for each natural number n such that $n \geq 2$. Suppose that $[U^k|T|(U^k)^*, |T|] = 0$ for all $k = 1, 2, \ldots, n$. Then we have

$$\begin{aligned} &U^n|\tilde{T}|(U^n)^* \cdot |\tilde{T}| \\ &= \left\{U^n|T|^{\frac{1}{2}}(U^n)^* \cdot U^{n-1}|T|^{\frac{1}{2}}(U^{n-1})^*\right\} \cdot \left\{U^*|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\right\} \\ &\qquad\qquad \text{by (3.8) and (vii) of Lemma 3.4} \\ &= U^n|T|^{\frac{1}{2}}(U^n)^* \cdot U^{n-1}|T|^{\frac{1}{2}}(U^{n-1})^* \cdot (U^*U)^2U^*|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \\ &= U^*U \cdot U^n|T|^{\frac{1}{2}}(U^n)^* \cdot U^*U \cdot U^{n-1}|T|^{\frac{1}{2}}(U^{n-1})^* \cdot U^*|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \\ &\qquad\qquad \text{by (ii) of Lemma 3.4} \\ &= U^* \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot U^n|T|^{\frac{1}{2}}(U^n)^* \cdot |T|^{\frac{1}{2}} \cdot U|T|^{\frac{1}{2}} \\ &= U^*\left\{U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot |T|^{\frac{1}{2}}\right\}U^n|T|^{\frac{1}{2}}(U^n)^*U|T|^{\frac{1}{2}} \quad \text{by (ii) of Lemma 3.4} \\ &= U^*\left\{U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot |T|^{\frac{1}{2}}\right\}U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &|\tilde{T}| \cdot U^n|\tilde{T}|(U^n)^* \\ &= \left\{U^*|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\right\} \cdot \left\{U^n|T|^{\frac{1}{2}}(U^n)^* \cdot U^{n-1}|T|^{\frac{1}{2}}(U^{n-1})^*\right\} \\ &\qquad\qquad \text{by (3.8) and (vii) of Lemma 3.4} \\ &= U^*|T|^{\frac{1}{2}}U \cdot U^n|T|^{\frac{1}{2}}(U^n)^* \cdot U^{n-1}|T|^{\frac{1}{2}}(U^{n-1})^* \cdot |T|^{\frac{1}{2}} \quad \text{by (ii) of Lemma 3.4} \\ &= U^*|T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^*U \cdot U^{n-1}|T|^{\frac{1}{2}}(U^{n-1})^* \cdot |T|^{\frac{1}{2}} \quad \text{by (iii) of Lemma 3.4} \\ &= U^*\left\{|T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^*\right\}U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}}. \end{aligned} \tag{3.10}$$

Proof of (i) $\Longrightarrow$ (ii). Since $[U^{n+1}|T|(U^{n+1})^*, |T|] = 0$, we have $[U^n|\tilde{T}|(U^n)^*, |\tilde{T}|] = 0$, that is, (ii) holds for n by (3.9) and (3.10).

Proof of (ii) $\Longrightarrow$ (i). Assume $[U^n|\tilde{T}|(U^n)^*, |\tilde{T}|] = 0$. Then we have

$$\begin{aligned}
&\left\{U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot |T|^{\frac{1}{2}}\right\} U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}} \\
&= UU^* \left\{U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot |T|^{\frac{1}{2}}\right\} U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}} \\
&= UU^* \left\{|T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^*\right\} U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}} && \text{by (3.9) and (3.10)} \\
&= \left\{|T|^{\frac{1}{2}} \cdot UU^* \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^*\right\} U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}} && \text{by (3.5)} \\
&= \left\{|T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^*\right\} U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}}.
\end{aligned}$$

This is equivalent to

$$U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot |T|^{\frac{1}{2}} = |T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^*$$

on $\overline{R(U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}})}$. Since $N(U) = N(|T|)$, we obtain

$$\begin{aligned}
\overline{R(U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}})} &= N(|T|^{\frac{1}{2}}U^{n-1}|T|^{\frac{1}{2}}(U^n)^*)^\perp = N(U^n|T|^{\frac{1}{2}}(U^n)^*)^\perp \\
&= N(|T|^{\frac{1}{4}}(U^n)^*)^\perp = N(U(U^n)^*)^\perp = N(U^n(U^n)^*)^\perp = \overline{R(U^n(U^n)^*)}.
\end{aligned}$$

Therefore we have

$$U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot |T|^{\frac{1}{2}} \cdot U^n(U^n)^* = |T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot U^n(U^n)^*, \tag{3.11}$$

so that

$$\begin{aligned}
U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot |T|^{\frac{1}{2}} &= U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot |T|^{\frac{1}{2}} \cdot U^n(U^n)^* && \text{by (iv) of Lemma 3.4} \\
&= |T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot U^n(U^n)^*, && \text{by (3.11)} \\
&= |T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}U^* \cdot (U^n)^* && \text{by (iii) of Lemma 3.4} \\
&= |T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^*,
\end{aligned}$$

that is, (i) holds for n. □

Proof of Theorem 3.3. We shall prove Theorem 3.3 by induction on n. We remark that if $[U^k|T|(U^k)^*, |T|] = 0$ for all $k = 1, 2, \ldots, n+1$, then we have

$$[U^{n+1}|\tilde{T}|(U^{n+1})^*, |T|] = [U^{n+1}|\tilde{T}|(U^{n+1})^*, U^*U] = 0 \tag{3.12}$$

by (ii) and (vii) of Lemma 3.4.

In case $n = 1$, we have already shown in Theorem 3.2. Suppose that Theorem 3.3 holds for some natural number n. (i) holds for $n+1$ if and only if

$$\widetilde{T_k} \text{ is binormal for all } k = 0, 1, 2, \ldots, n+1.$$

By putting $S = \tilde{T}$, it is equivalent to

$$T \text{ and } \widetilde{S_k} \text{ are binormal for all } k = 0, 1, 2, \ldots, n. \tag{3.13}$$

Since $S = U^*UU|S|$ is the polar decomposition by Theorem 3.1, (3.13) is equivalent to

$$\begin{aligned} &T \text{ is binormal and} \\ &[(U^*UU)^k|S|\{(U^*UU)^k\}^*, |S|] = [U^*U \cdot U^k|\tilde{T}|(U^k)^* \cdot U^*U, |\tilde{T}|] = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } k = 1, 2, \ldots, n+1 \end{aligned} \tag{3.14}$$

by the inductive hypothesis. On the other hand, if we assume (i) or (ii), then

$$[U^k|T|(U^k)^*, |T|] = 0 \text{ for all } k = 1, 2, \ldots, n+1$$

by the inductive hypothesis, so that (3.14) is equivalent to

$$T \text{ is binormal and } [U^k|\tilde{T}|(U^k)^*, |\tilde{T}|] = 0 \text{ for all } k = 1, 2, \ldots, n+1 \tag{3.15}$$

by (3.12) and $U^*U|\tilde{T}| = U^*U(|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})^{\frac{1}{2}} = |\tilde{T}|$. Moreover Lemma 3.5 assures that (3.15) is equivalent to

$$[U^k|T|(U^k)^*, |T|] = 0 \text{ for all } k = 1, 2, \ldots, n+2,$$

i.e., (ii) holds for $n+1$. Hence the proof is complete. □

3.2 Centered operators

Morrel-Muhly [9] showed the following properties and characterization of centered operators.

Theorem 3.C ([9]). *Let $T = U|T|$ be the polar decomposition of a centered operator T. Then the following assertions hold:*

(i) *U^n is a partial isometry for all natural number n.*

(ii) *Operators $\{(U^n)^*|T|U^n\}_{n=1}^{\infty}$ commute with one another.*

(iii) *$T^n = U^n\{|T| \cdot U^*|T|U \cdots (U^{n-1})^*|T|U^{n-1}\}$ is the polar decomposition for all natural number n.*

Theorem 3.D ([9]). *Let $T = U|T|$ be the polar decomposition and U be unitary. Then T is centered if and only if operators*

$$\{(U^n)^*|T|U^n\}_{n=-\infty}^{\infty}$$

commute with one another.

We shall show the following characterization of centered operators which is an extension of (ii) of Theorem 3.C and Theorem 3.D.

Theorem 3.6. *Let $T = U|T|$ be the polar decomposition. Then the following assertions are mutually equivalent;*

(i) T *is centered.*

(ii) $[|T^n|, |(T^m)^*|] = 0$ *for all natural numbers n and m.*

(iii) $[|T^n|, |T^*|] = 0$ *for all natural number n.*

(iv) *Operators* $\{(U^n)^*|T|U^n, U^n|T|(U^n)^*, |T|\}_{n=1}^{\infty}$ *commute with one another.*

(v) $[U^n|T|(U^n)^*, |T|] = 0$ *for all natural number n.*

(vi) $\tilde{T}_n$ *is binormal for all non-negative integer n.*

In order to give a proof of Theorem 3.6, we shall prepare the following lemmas.

Lemma 3.7. *Let T be the polar decomposition. For each natural numbers n and m, if*

$$[U^k|T|(U^k)^*, |T|] = 0 \quad \text{for all } k = 0, 1, 2, \ldots, m+n-2, \tag{3.16}$$

then the following assertions are equivalent:

(i) $[U^m|T|(U^m)^*, |T^n|] = 0$.

(ii) $[U^{m+n-1}|T|(U^{m+n-1})^*, |T|] = 0$.

Proof. We shall prove Lemma 3.7 by induction on n. The case $n = 1$ is obvious. Assume that it holds for some natural number n and each natural number m. Then we shall prove that it holds for $n+1$ and each natural number m.

Let m be a natural number and suppose that (3.16) holds for $n+1$, i.e.,

$$[U^k|T|(U^k)^*, |T|] = 0 \quad \text{for all } k = 0, 1, 2, \ldots, m+n-1. \tag{3.17}$$

Then

$$[U|T|U^*, |T^n|] = [UU^*, |T^n|] = 0. \tag{3.18}$$

holds by the inductive assumption and Lemma 3.B, so that we have

$$\begin{aligned} |T^{n+1}|^2 &= |T|U^*|T^n|^2U|T| \\ &= U^* \cdot U|T|U^* \cdot |T^n|^2 \cdot U|T| \\ &= U^* \cdot |T^n|^2 \cdot U|T|U^* \cdot U|T| \quad \text{by (3.18)} \\ &= U^*|T^n|^2U|T|^2. \end{aligned} \tag{3.19}$$

Therefore we have

$$\begin{aligned} |T^{n+1}|^2 \cdot U^m|T|(U^m)^* &= U^*|T^n|^2U|T|^2 \cdot U^m|T|(U^m)^* \quad \text{by (3.19)} \\ &= U^*|T^n|^2U \cdot U^m|T|(U^m)^* \cdot |T|^2 \quad \text{by (3.17)} \qquad (3.20) \\ &= U^*\left\{|T^n|^2 \cdot U^{m+1}|T|(U^{m+1})^*\right\}U|T|^2 \end{aligned}$$

and

$$\begin{aligned} U^m|T|(U^m)^* \cdot |T^{n+1}|^2 &= U^m|T|(U^m)^* \cdot U^*|T^n|^2U|T|^2 \quad \text{by (3.19)} \\ &= U^*\left\{U^{m+1}|T|(U^{m+1})^* \cdot |T^n|^2\right\}U|T|^2 \qquad (3.21) \\ &\qquad \text{by (iii) of Lemma 3.4.} \end{aligned}$$

Proof of (ii) $\Longrightarrow$ (i). Assume that (ii) holds for $n+1$. Since

$$[U^{m+(n+1)-1}|T|(U^{m+(n+1)-1})^*, |T|] = [U^{(m+1)+n-1}|T|(U^{(m+1)+n-1})^*, |T|] = 0,$$

we have $[U^{m+1}|T|(U^{m+1})^*, |T^n|] = 0$ by the inductive assumption. Hence we obtain

$$[|T^{n+1}|, U^m|T|(U^m)^*] = 0,$$

that is, (i) holds for $n+1$ by (3.20) and (3.21).

Proof of (i) $\Longrightarrow$ (ii). Assume that (i) holds for $n+1$. Then we have

$$\begin{aligned} U^{m+1}|T|(U^{m+1})^* \cdot |T^n|^2 \cdot U|T|^2 &= UU^*\left\{U^{m+1}|T|(U^{m+1})^* \cdot |T^n|^2\right\}U|T|^2 \\ &= UU^*\left\{|T^n|^2 \cdot U^{m+1}|T|(U^{m+1})^*\right\}U|T|^2 \\ &\qquad \text{by (3.20) and (3.21)} \\ &= |T^n|^2 \cdot UU^* \cdot U^{m+1}|T|(U^{m+1})^* \cdot U|T|^2 \quad \text{by (3.18)} \\ &= |T^n|^2 \cdot U^{m+1}|T|(U^{m+1})^* \cdot U|T|^2, \end{aligned}$$

that is,

$$U^{m+1}|T|(U^{m+1})^* \cdot |T^n|^2 = |T^n|^2 \cdot U^{m+1}|T|(U^{m+1})^*$$

holds on $\overline{R(U|T|^2)} = N(|T|^2U^*)^\perp = N(UU^*)^\perp = R(UU^*)$. Then we have

$$U^{m+1}|T|(U^{m+1})^* \cdot |T^n|^2 \cdot UU^* = |T^n|^2 \cdot U^{m+1}|T|(U^{m+1})^* \cdot UU^*, \qquad (3.22)$$

so that we obtain

$$\begin{aligned} U^{m+1}|T|(U^{m+1})^* \cdot |T^n|^2 &= U^{m+1}|T|(U^{m+1})^* \cdot UU^* \cdot |T^n|^2 \\ &= U^{m+1}|T|(U^{m+1})^* \cdot |T^n|^2 \cdot UU^* \qquad \text{by (3.18)} \\ &= |T^n|^2 \cdot U^{m+1}|T|(U^{m+1})^* \cdot UU^* \qquad \text{by (3.22)} \\ &= |T^n|^2 \cdot U^{m+1}|T|(U^{m+1})^*. \end{aligned}$$

Hence we have

$$[U^{(m+1)+n-1}|T|(U^{(m+1)+n-1})^*, |T|] = [U^{m+(n+1)-1}|T|(U^{m+(n+1)-1})^*, |T|] = 0,$$

that is, (ii) holds for $n+1$ by the inductive assumption. Hence the proof is complete. $\square$

Lemma 3.8. *Let $T = U|T|$ be the polar decomposition. For each natural number n, if*

$$[U^k|T|(U^k)^*, |T|] = 0 \quad \textit{for all } k = 0, 1, 2, \ldots, n-1,$$

then

$$|(T^n)^*| = U|T|U^* \cdot U^2|T|(U^2)^* \cdots U^n|T|(U^n)^*.$$

Proof. We shall prove Lemma 3.8 by induction on n. It is obvious the case $n = 1$. Assume that Lemma 3.8 holds for some natural number n. Then we shall show that it holds for $n+1$. By the inductive assumption, we have

$$|(T^n)^*| = U|T|U^* \cdot U^2|T|(U^2)^* \cdots U^n|T|(U^n)^*. \tag{3.23}$$

Then we obtain

$$\begin{aligned}
|(T^{n+1})^*| &= (U|T| \cdot |(T^n)^*|^2 \cdot |T|U^*)^{\frac{1}{2}} \\
&= \left\{U|T| \left(U|T|U^* \cdot U^2|T|(U^2)^* \cdots U^n|T|(U^n)^*\right)^2 |T|U^*\right\}^{\frac{1}{2}} \quad \text{by (3.23)} \\
&= \{U|T| \cdot U|T|^2U^* \cdot U^2|T|^2(U^2)^* \cdots U^n|T|^2(U^n)^* \cdot |T|U^*\}^{\frac{1}{2}} \\
&\qquad \text{by (vi) of Lemma 3.4} \\
&= \{U|T|(U^*U)^{n+1} \cdot U|T|^2U^* \cdot U^2|T|^2(U^2)^* \cdots U^n|T|^2(U^n)^* \cdot |T|U^*\}^{\frac{1}{2}} \\
&= \{U|T| \cdot U^*U \cdot U|T|^2U^* \cdot U^*U \cdot U^2|T|^2(U^2)^* \cdot U^*U \\
&\qquad \cdots U^*U \cdot U^n|T|^2(U^n)^* \cdot U^*U \cdot |T|U^*\}^{\frac{1}{2}} \quad \text{by (ii) of Lemma 3.4} \\
&= \{U|T|U^* \cdot U^2|T|^2(U^2)^* \cdot U^3|T|^2(U^3)^* \cdots U^{n+1}|T|^2(U^{n+1})^* \cdot U|T|U^*\}^{\frac{1}{2}} \\
&= U|T|U^* \cdot U^2|T|(U^2)^* \cdot U^3|T|(U^3)^* \cdots U^{n+1}|T|(U^{n+1})^* \\
&\qquad \text{by (vi) of Lemma 3.4.} \quad \square
\end{aligned}$$

Proof of Theorem 3.6. (i) $\Longrightarrow$ (ii), (ii) $\Longrightarrow$ (iii) and (iv) $\Longrightarrow$ (v) are obvious, and (v) $\Longleftrightarrow$ (vi) follows from Theorem 3.3. Then we have only to prove (iii) $\Longrightarrow$ (v), (v) $\Longrightarrow$ (iv), (v) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (i).

Proof of (iii) $\Longrightarrow$ (v). Firstly $[U|T|U^*, |T|] = 0$ and $[U|T|U^*, |T^2|] = 0$ ensures $[U^2|T|(U^2)^*, |T|] = 0$ by Lemma 3.7. Secondly $[U^k|T|(U^k)^*, |T|] = 0$ for $k = 1, 2$ and $[U|T|U^*, |T^3|] = 0$ ensures $[U^3|T|(U^3)^*, |T|] = 0$ by Lemma 3.7. By repeating this method, we have (v).

Proof of (v) $\Longrightarrow$ (iv). (v) implies

$$[(U^{n-1})^*|T|(U^{n-1}), U|T|U^*] = 0 \tag{3.24}$$

by (v) of Lemma 3.4. Then we have

$$\begin{aligned}(U^n)^*|T|U^n \cdot |T| &= U^* \left\{(U^{n-1})^*|T|U^{n-1} \cdot U|T|U^*\right\} U \\ &= U^* \left\{U|T|U^* \cdot (U^{n-1})^*|T|U^{n-1}\right\} U \quad \text{by (3.24)} \\ &= |T| \cdot (U^n)^*|T|U^n,\end{aligned}$$

that is, $[(U^n)^*|T|U^n, |T|] = 0$ holds for all natural number n. Moreover we obtain

$$\begin{aligned}(U^n)^*|T|U^n \cdot U^m|T|(U^m)^* &= (U^n)^* \left\{|T| \cdot U^{n+m}|T|(U^{n+m})^*\right\} U^n \quad \text{by (iii) of Lemma 3.4} \\ &= (U^n)^* \left\{U^{n+m}|T|(U^{n+m})^* \cdot |T|\right\} U^n \quad \text{by (v)} \\ &= U^m|T|(U^m)^* \cdot (U^n)^*|T|U^n \quad \text{by (iii) of Lemma 3.4,}\end{aligned}$$

that is, $[(U^n)^*|T|U^n, U^m|T|(U^m)^*] = 0$ holds for all natural numbers n and m. Hence we have (iv).

Proof of (v) $\Longrightarrow$ (ii). By (v) and Lemma 3.8, we have

$$|(T^m)^*| = U|T|U^* \cdot U^2|T|(U^2)^* \cdots U^m|T|(U^m)^* \quad \text{for all natural number } m, \tag{3.25}$$

and also by (v) and Lemma 3.7, we have

$$[U^m|T|(U^m)^*, |T^n|] = 0 \quad \text{for all natural numbers } m \text{ and } n. \tag{3.26}$$

Hence we obtain (ii) by (3.25) and (3.26).

Proof of (ii) $\Longrightarrow$ (i). For $s > t$, we have

$$\begin{aligned}|T^s|^2|T^t|^2 &= (T^t)^* \cdot |T^{s-t}|^2 \cdot |(T^t)^*|^2 \cdot T^t \\ &= (T^t)^* \cdot |(T^t)^*|^2 \cdot |T^{s-t}|^2 \cdot T^t \quad \text{by (ii)} \\ &= |T^t|^2|T^s|^2\end{aligned}$$

and

$$\begin{aligned}|(T^s)^*|^2|(T^t)^*|^2 &= T^t \cdot |(T^{s-t})^*|^2 \cdot |T^t|^2 \cdot (T^t)^* \\ &= T^t \cdot |T^t|^2 \cdot |(T^{s-t})^*|^2 \cdot (T^t)^* \quad \text{by (ii)} \\ &= |(T^t)^*|^2|(T^s)^*|^2,\end{aligned}$$

so that we have (i). □

Remark. As a parallel result to (vi) of Theorem 3.6, one might consider that if T^n is binormal for all natural number n, then T is centered. The converse obviously holds by the definition of centered operators. But there exists an operator T such that T^n is binormal for all natural number n and T is not centered, for example $T = \begin{pmatrix} 0 & & & \\ A & 0 & & \\ & I & 0 & \\ & & A & 0 \end{pmatrix}$.

We shall show a characterization of centered operators via the polar decomposition and the Aluthge transformation. We remark that Theorem 3.9 corresponds to Theorem 3.1 for binormal operators, and is related to (iii) of Theorem 3.C.

Theorem 3.9. *Let $T = U|T|$ be the polar decomposition. Then the following assertions are equivalent:*

(i) *T is centered.*

(ii) *$\widetilde{T_n} = \widetilde{U_n}|\widetilde{T_n}|$ is the polar decomposition for all non-negative integer n.*

(iii) *$T^n = U^n|T^n|$ is the polar decomposition for all natural number n.*

In particular, if T is a partial isometry, then T is centered if and only if $\widetilde{T_n}$ or T^n is a partial isometry for all natural number n.

Proof. (i) $\Longleftrightarrow$ (ii). Assume that T is a centered operator, then $\widetilde{T_n}$ is binormal for all non-negative integer n by Theorem 3.6. Then we have that

$$\widetilde{T} = \widetilde{U}|\widetilde{T}| \text{ is the polar decomposition} \tag{3.27}$$

by Theorem 3.1 since T is binormal. Next since $\widetilde{T}$ is binormal and (3.27) holds, we have that

$$\widetilde{T_2} = \widetilde{U_2}|\widetilde{T_2}| \text{ is the polar decomposition}$$

by Theorem 3.1. Repeating this method, we have that $\widetilde{T_n} = \widetilde{U_n}|\widetilde{T_n}|$ is the polar decomposition for all non-negative integer n.

Conversely, assume that $\widetilde{T_n} = \widetilde{U_n}|\widetilde{T_n}|$ is the polar decomposition for all non-negative integer n. Then we obtain that $\widetilde{T_n}$ is binormal for all non-negative integer n by Theorem 3.1. Hence T is centered by Theorem 3.6.

(i) $\Longleftrightarrow$ (iii). Assume that T is a centered operator, then we have that

$$T^2 = U^2|T^2| \text{ is the polar decomposition} \tag{3.28}$$

by $[|T|, |T^*|] = 0$ and Theorem 2.3. Next by (3.28) and $[|T^2|, |T^*|] = 0$, we have that

$$T^3 = U^3|T^3| \text{ is the polar decomposition}$$

by Theorem 2.3. Repeating this method, we have that $T^n = U^n|T^n|$ is the polar decomposition for all natural number n.

Conversely, assume that $T^n = U^n|T^n|$ is the polar decomposition for all natural number n. Then we obtain that

$$T^{n-1} = U^{n-1}|T^{n-1}| \text{ and } T = U|T|$$

are the polar decompositions. By Theorem 2.3, we have

$$[|T^{n-1}|, |T^*|] = 0$$

for all natural number $n > 1$. Hence T is centered by Theorem 3.6. □

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