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On Uniqueness of The Solutions of The Obstacle Problem

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Abstract

In [2], R. Fehlmann and F. P. Gardiner studied an extremal problem for a topologically finite Riemann surface and established the slit mapping theorem by showing existence of a quadratic differential which associated with the solution of the extremal problem. In this article, we give a condition for non-uniqueness of such slit mappings, by using deformation of a Riemann surface using the foliation structure of the differential associated with the solution.

1 Introduction

Suppose S is a *finite bordered Riemann surface* with the border Γ. In other words, the boundary Γ consists of a finite number of simple closed curves, and the double of S with respect to the border Γ is of finite analytic type. Let $T(S)$ be the Teichmüller space of the interior S° of S. Let $A(S)$ be the set of integrable holomorphic quadratic differential φ on S with the properties that $\varphi = \varphi(z)dz^2$ is real along the border Γ.

Definition 1.1 *Let E be a compact subset of S° which satisfies that $S \setminus E$ is of finitely connected and of the same genus as S. We say that E with these properties is an* allowable *subset of S.*

Next fix an element $\varphi \in A(S)$. If each component of an allowable E is a horizontal arc of φ or a union of a finite number of horizontal arcs and critical points of φ, we say that E is an allowable slit with respect to φ.

Let E be an allowable subset of S. Let $\mathfrak{F}(S,E)$ be the family of pairs (g, S_g), where g is a conformal map of $S \setminus E$ into another Riemann surface S_g such that g maps the border Γ onto the border of S_g and the puncture of S onto the puncture of S_g. In particular, $(g, S_g) \in \mathfrak{F}(S,E)$ induces an isomorphism ι_g from the fundamental group $\pi_1(S)$ of S onto $\pi_1(S_g)$. Let f be a quasiconformal map of S onto S_g which induces the same isomorphism ι_g, and μ the Beltrami differential of f. We denote f also by f^μ and the Teichmüller (equivalence) class of f^μ in $T(S)$ by $[(\mu; g, S_g)]$.

Let $\mathfrak{S}(S)$ be the family of simple closed curves in S° which is homotopic neither to a point of S nor to a puncture on S. Let $\mathfrak{S}[S]$ be the set of homotopy class of an element of $\mathfrak{S}(S)$. For $\varphi \in A(S)$ and $\gamma \in \mathfrak{S}(S)$, we denote the height of γ with respect to φ by $h_\varphi(\gamma)$, and that the height of homotopy class $[\gamma]$ by $h_\varphi[\gamma]$. For the details, see for instance [4].

Now it is known (cf. [4]) that, for every $(f, S_f) \in \mathfrak{F}(S, E)$ and $\varphi \in A(S) \setminus \{0\}$, there is a holomorphic quadratic differential φ_f on S_f whose heights on S_f are equal to the corresponding heights of φ on S. Fehlmann and Gardiner posed the *extremal problem for* (S, φ, E), of maximizing

$$M_f = \|\varphi_f\|_{L^1(S_f)} = \int_{S_f} |\varphi_f|$$

in $\mathfrak{F}(S, E)$, and showed the following result.

Theorem 1.2 (Fehlmann-Gardiner) *Suppose that S is a finite bordered Riemann surface, and that $\varphi \in A(S) \setminus \{0\}$. Let E be an allowable subset of S. Then there exists a point $[(\mu; g, S_g)] \in T(S)$ associated with an element $(g, S_g) \in \mathfrak{F}(S, E)$ such that M_g attains the maximum*

$$M = \max_{(f, S_f) \in \mathfrak{F}(S, E)} M_f.$$

Moeover, for this point $[(\mu; g, S_g)] \in T(S)$, $E_g = S_g \setminus g(S \setminus E)$ is an allowable slit with respect to φ_g.

The point $[(\mu; g, S_g)] \in T(S)$ in Theorem 1.2 is called an *extremal point* of the extremal problem for (S, φ, E), the map g an *extremal slit mapping* associated with it, and the associated differential φ_g the *structure differential* for g.

We show in this note the following theorem which gives a condition for extremal points, and hence extremal slit mappings, not to be unique.

Theorem 1.3 *Suppose R is a finite bordered Riemann surface, and that $\psi \in A(R) \setminus \{0\}$. Let E_ψ be an allowable slit of R with respect to ψ such that*

1. *there is a component of E_ψ which contains a zero point p_0 of ψ of order $m \geq 3$ and at least two of horizontal arcs ℓ_1, ℓ_2 with an end point at p_0, and that*
2. *each of the angles between ℓ_1, ℓ_2 are larger than $\frac{2\pi}{m+2}$.*

Then, there is a finite bordered Riemann surface $\tilde{R}$, a pair $(h, \tilde{R}) \in \mathfrak{F}(R, E_\psi)$, and a holomorphic quadratic differential $\tilde{\psi} \in A(\tilde{R}) \setminus \{0\}$, such that *(i)* $E_{\tilde{\psi}} = \tilde{R} \setminus h(R \setminus E_{\psi})$ *is an allowable slit of* $\tilde{R}$ *with respect to* $\tilde{\psi}$*,*

(ii) the heights of $\tilde{\psi}$ *on* $\tilde{R}$ *is the same as the corresponding heights of* ψ *on* R*, and*

(iii) the point $[(\mu; h, \tilde{R})] \in T(R)$ *is different from the origin* $[(0; id, R)]$ *of* $T(R)$.

We call the conditions 1. and 2. for E_{ψ} in the Theorem 1.3 the *refolding conditions*, and the point p_0 a *refolding point.*

Corollary 1.4 *Suppose* S *is a finite bordered Riemann surface and that* $\varphi \in A(S) \setminus \{0\}$. *Let* E *be an allowable subset of* S*, and* $[(\mu; g, S_g)] \in T(S)$ *the extremal point of the extremal problem for* (S, φ, E). *If the allowable slit* E_g *of* S_g *with respect to the structure differential* φ_g *satisfies the refolding conditions , then there exists another extremal point of the extremal problem for* (S, φ, E) *different from* $[(\mu; g, S_g)]$.

Proof. Take the triple (S_g, φ_g, E_g) as the triple (R, ψ, E_{ψ}) in the Theorem 1.3. Then we obtain a finite bordered Riemann surface $\tilde{R}$, a pair $(h, \tilde{R}) \in \mathfrak{F}(S_g, E_g)$, and a holomorphic quadratic differential $\tilde{\psi} \in A(\tilde{R}) \setminus \{0\}$ such that

(i) $E_{\tilde{\psi}}$ is an allowable slit of $\tilde{R}$ with respect to $\tilde{\psi}$,

(ii) the heights of $\tilde{\psi}$ on $\tilde{R}$ is the same as the corresponding heights of φ_g on S_g (and hence of φ on S), and

(iii) the point $[(\mu; h, \tilde{R})] \in T(S_g)$ is different from the origin $[(0; id, S_g)]$ of $T(S_g)$.

Then, we know (cf. [2]) that, from (i) and (ii), the point $[(\mu; h \circ g, \tilde{R})] \in T(S)$ is an extremal point of the extremal problem for (S, φ, E). By (iii), the point $[(\mu; g, S_g)]$ is different from the point $[(\mu; h \circ g, \tilde{R})]$. Thus we have the assertion. ∎

2 Example

In this section we give an example of the triple (S, φ, E) which satisfies the assumptions of Corollary 1.4.

First take three copies M_1, M_2, M_3 of a rectangle

$$M = \{z = x + iy \in \mathbb{C} \mid |x| \leq 2, |y| \leq 1\},$$

and let z_j be the coordinate corresponding to z on each M_j. Next on each M_j, identify two pair of parallel sides under the translations

$$z_j \to z_j + 4, \qquad z_j \to z_j + 2i.$$

Then we obtain three copies T_1, T_2, T_3 of a torus T. And the quadratic differential dz^2 on M induces the holomorphic quadatic differential φ_0 on T.

Cut M_j along the segment

$$I_j = \{z_j = x_j + iy_j \mid -1 \le x_j \le 0, y_j = 0\},$$

and connect them cyclically. More precisely, we paste the upper edge I_1^+ of the slit I_1 and the lower edge I_2^- of the slit I_2, the upper edge I_2^+ of the slit I_2 and the lower edge I_3^- of the slit I_3, and the upper edge I_3^+ of the slit I_3 and the lower edge I_1^- of the slit I_1. Then we obtain a compact Riemann surface S of genus three.

Now let Π be the natural projection from S to the torus T, and φ the pull-back of φ_0 by Π. Finaly, let E be a subset of S, consisting of the arcs ℓ_1 and ℓ_2, where each ℓ_i is one on M_i corresponding to

$$\{z \mid 0 \le x \le 1, y = 0\}.$$

Now we consider the extremal problem for (S, φ, E). Then the set E is an allowable slit of S with respect to φ. Hence we know the identical mapping of S gives the extremal slit map associated with the extremal problem for this triple. Moreover, we can easily see that E satisfies the refolding conditions.

Thus the assumptions in Corollary 1.4 are satisfied, and as a consequence, the extremal points of the extremal problem for (S, φ, E) are not uniquely determined in $T(S)$.

3 Proof of theorem 1.3

Assume that a component J of E_ψ contains a refolding point p_0 of ψ of order $m \ge 3$ and horizontal arcs ℓ_1 and ℓ_2, one of whose end point is p_0 and an angle between ℓ_1 and ℓ_2 is

$$\frac{2k\pi}{m+2} \qquad \left(2 \le k \le \frac{m+2}{2}\right).$$

Here the arcs ℓ_1, ℓ_2 are segments on the real axis with an endpoint at the origin with respect to the natural parameter $\zeta = \zeta_\psi$ induced from ψ.

We take a subarc $\kappa_j \subset \ell_j$ such that p_0 is an endpoint of each κ_j and that ψ has no zeros on $\kappa_j \setminus \{p_0\}$. Let p_j be the other endpoint of κ_j for each j. Also set $K = \kappa_1 \bigcup \kappa_2$.

Now, cut R along κ_1 and κ_2. For each j, let κ_j^+ and κ_j^-, respectively, the right-side and the left-side edge of the slit κ_j, with respect to the orientation which corresponds to moving along the slit from p_0 to p_j. Assume that κ_1^- and κ_2^+, resp. κ_1^+ and κ_2^-, makes the angle

$$\frac{2k\pi}{m+2}, \quad \text{resp.} \quad \left(1 - \frac{k}{m+2}\right) 2\pi.$$

Paste κ_1^- and κ_2^+ so that points having the same absolute value withrespect to ζ are identified. By the same way, paste κ_1^+ and κ_2^-. Then we obtain a finite bordered Riemann surface $\tilde{R}$ and the natural conformal embedding $h : R\backslash K \to \tilde{R}$. This pair $(h, \tilde{R})$ is an element of a family $\mathfrak{F}(R, K) \subset \mathfrak{F}(R, E_\psi)$.

Moreover, from the construction we can extend ψ restricted on $R \setminus K$ naturally to a holomorphic quadratic differential $\tilde{\psi}$ on $\tilde{R}$, and $E_{\tilde{\psi}} = \tilde{R} \setminus h(R \setminus E_\psi)$ is allowable slit of $\tilde{R}$ with respect to $\tilde{\psi}$.

Now let f^μ be a quasiconformal map from R onto $\tilde{R}$, which is a representation of the point $[(\mu; h, \tilde{R})] \in T(R)$.

Lemma 3.1

$$h_{\tilde{\psi}}[\tilde{\gamma}] = h_\psi[(f^\mu)^{-1}(\tilde{\gamma})]$$

for every $[\tilde{\gamma}] \in \mathfrak{S}[\tilde{R}]$.

Proof. We say that a simple closed curve $\tilde{\beta}$ on $\tilde{R}$ is a $\tilde{\psi}$-polygon, if $\tilde{\beta}$ is a union of a finite number of horizontal arcs and vertical arcs of $\tilde{\psi}$. Note that for every $[\tilde{\gamma}] \in \mathfrak{S}[\tilde{R}]$

$$h_{\tilde{\psi}}[\tilde{\gamma}] = \inf_{\tilde{\beta}} h_{\tilde{\psi}}(\tilde{\beta}),$$

where the infimum is taken over all $\tilde{\psi}$-polygons $\tilde{\beta}$ homotopic to $\tilde{\gamma}$ on $\tilde{R}$.

Now we can deform the pre-image $h^{-1}(\tilde{\beta})$ of such a $\tilde{\psi}$-polygon $\tilde{\beta}$ to a ψ-polygon β such that β is homotopic to $h^{-1}(\tilde{\beta})$ on R and

$$h_\psi(\beta) = h_{\tilde{\psi}}(\tilde{\beta}).$$

Hence we conclude that

$$h_\psi[(f^\mu)^{-1}(\gamma)] \leq h_\psi(\beta) = h_{\tilde{\psi}}(\tilde{\beta})$$

for every $\tilde{\psi}$-polygon $\tilde{\beta}$ which is homotopic to $\tilde{\gamma}$, which in turn implies that

$$h_\psi[(f^\mu)^{-1}(\tilde{\gamma})] \leq h_{\tilde{\psi}}[\tilde{\gamma}]$$

for ever $[\tilde{\gamma}] \in \mathfrak{S}[\tilde{R}]$.

On the other hand, we can similary see as above that

$$h_{\tilde{\psi}}[f^\mu(\gamma)] \leq h_\psi[\gamma]$$

for every $[\gamma] \in \mathfrak{S}[R]$. Thus we have the assertion. ∎

From Lemma 3.1, we see that the holomorphic quadratic differential $\tilde{\psi} \in A(\tilde{R}) \setminus \{0\}$ satisfies the condition (ii). Moreover, by definition, $E_{\tilde{\psi}}$ is an allowable slit of $\tilde{R}$ with respect to $\tilde{\psi}$, and

$$\tilde{\psi} \circ h(h')^2 = \psi \text{ on } R \setminus E_\psi.$$

Lemma 3.2 *The point* $[(\mu; h, \tilde{R})] \in T(R)$ *is different from the origin* $[(0; id, R)]$ *of* $T(R)$.

Proof. Assume that

$$[(\mu; h, \tilde{R})] = [(0; id, R)].$$

Then there would exist a conformal map $\iota : R \to \tilde{R}$ such that the induced isomorphism $(\iota)_* : \pi_1(R) \to \pi_1(\tilde{R})$ is the same as the one induced by h.

Fix a $[\gamma] \in \mathfrak{S}[R]$ arbitrarily. Then Lemma 3.1 gives that

$$h_{\tilde{\psi}}[\iota(\gamma)] = h_\psi[\gamma].$$

Since $h_{\tilde{\psi}\circ\iota(\iota')^2}[\gamma] = h_{\tilde{\psi}}[\iota(\gamma)]$, we obtain

$$h_{\tilde{\psi}\circ\iota(\iota')^2}[\gamma] = h_\psi[\gamma]$$

for every $[\gamma] \in \mathfrak{S}[R]$. Hence the heights mapping theorem implies that $\tilde{\psi} \circ \iota(\iota')^2 = \psi$ on R. In particular, the map ι maps the zeros of ψ to zeros of $\tilde{\psi}$ including multiplicities.

Now from the construction, the zero p_0 of order $m \geq 3$ breaks into two zeros $\tilde{q}_1$ and $\tilde{q}_2$ of $\tilde{\psi}$ of order $k-2$ and $m-k$, respectively, with $2 \leq k \leq (m+2)/2$. And the endpoints p_1 of κ_1 and p_2 of κ_2 gather to a zero $\tilde{q}$ of $\tilde{\psi}$ on $\tilde{R}$ of order 2.

Set $\tilde{K} = \tilde{R} \setminus h(R \setminus K)$. Then all zeros $\tilde{q}, \tilde{q}_1$ and $\tilde{q}_2$ of $\tilde{\psi}$ on $\tilde{K}$ have the orders strictly less than m. Hence we see that

$$\iota(p_0) \notin \tilde{K}.$$

Since the conformal embedding h maps $R \setminus K$ onto $\tilde{R} \setminus \tilde{K}$, $h^{-1} \circ \iota(p_0)$ is well defined and $h^{-1} \circ \iota(p_0) \notin K$. In particular,

$$h^{-1} \circ \iota(p_0) \neq p_0.$$

Next assume that, for a positive integer n,

$$(h^{-1} \circ \iota)^n(p_0) \neq (h^{-1} \circ \iota)^k(p_0)$$

for every k with $0 \leq k \leq n-1$. Then, $\iota \circ (h^{-1} \circ \iota)^n(p_0) \notin \tilde{K}$, for the zero $\iota \circ (h^{-1} \circ \iota)^n(p_0)$ of $\tilde{\psi}$ is of order m. Hence similarly as above, $(h^{-1} \circ \iota)^{n+1}(p_0) \notin K$. In particular,

$$(h^{-1} \circ \iota)^{n+1}(p_0) \neq p_0.$$

Also by the assumption,

$$(h^{-1} \circ \iota)^{n+1}(p_0) \neq (h^{-1} \circ \iota)^k(p_0)$$

for every k with $1 \leq k \leq n$.

Thus by the induction, we conclude that, for every positive integer n, we have

$$(h^{-1} \circ \iota)^n(p_0) \neq (h^{-1} \circ \iota)^k(p_0)$$

for every k with $0 \leq k \leq n-1$, which imples that ψ has infinitely many distinct zeros. This is absord, and we have shown that

$$[(\mu; h, \tilde{R})] \neq [(0; id, R)].$$

■

Remark As the example in Section 2, if one can see the widths of ψ on R and that of $\tilde{\psi}$ on $\tilde{R}$, it is easy to show the claim of Lemma 3.2. Because if $[(\mu; h, \tilde{R})] = [(0; \mathrm{id}, R)]$ in $T(R)$, then from Lemma 3.1 and the heights theorem we can see that widths of ψ on R is equal to the corresponding widths of $\tilde{\psi}$ on $\tilde{R}$. For example, in the case of Section 2 we denote by $\tilde{S}$ a Riemann surface obtained from S by deformation and denote by $\tilde{\varphi}$ an integrable holomorphic quadratic differential whose heights is the same as the corresponding heights of φ on S. Let $\gamma \in \mathfrak{S}(S)$ be rounding I_1 on M_1. Then the width of $[\gamma] \in \mathfrak{S}[S]$ is equal to 2. On the other hand, for this $[\gamma] \in \mathfrak{S}[S]$ the corresponding width of $\tilde{\varphi}$ on $\tilde{S}$ is equal to 4. Therefore the deformation actually change the surface in $T(S)$.

Thus we have completed the proof of Theorem 1.3.

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