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Bifurcation of the Kolmogorov flow with an external friction

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1 Introduction

Through my graduate school days studying under professor Sadao Miyatake, I have considered some bifurcation problems about the Kolmogorov flow. The Kolmogorov flow means a plane periodic flow of an incompressible fluid under the action of a spatially periodic external force. Since proposed in 1959, it has been conceived of only as a convenient object for theoretical investigations. But twenty years later, the flow was realized physically as a laboratory model by Bondarenko and his group (see its outline in [2] and Obkuhov[9]). The results of their experiments were found to be in good qualitative agreement with the previous theories described in Meshalkin and Sinai[8] and Iudovich[4], but in some cases, probably because they could only create a thin layer, there were some serious disagreement caused by a friction on the bottom of the channel. Then, they asserted that they should understand the influence of the friction in order to investigate a motion in a thin layer and built an updated model of the Kolmogorov flow with an external friction.

The corresponding equations in stationary case take the form:

$$(1.1)\qquad \begin{cases} uu_x + vu_y = -P_x + \nu\Delta u - \kappa u + \gamma \sin y, \\ uv_x + vv_y = -P_y + \nu\Delta v - \kappa v, \\ u_x + v_y = 0, \qquad \text{in } R^2, \end{cases}$$

where $u = u(x, y)$ and $v = v(x, y)$ are the velocity components, $P = P(x, y)$ is the pressure, $\nu > 0$ is the kinematic viscosity, γ is the intensity of the external force $(\gamma \sin y, 0)$, Δ is the two-dimensional Laplace operator, and κ is the coefficient of external friction which can be defined by the formula $\kappa \equiv 2\nu/h^2$ with h, the depth of the fluid layer. Let the system of solutions $V(x,y) = {}^t(u(x,y), v(x,y))$ and $P(x,y)$ satisfy

$$(1.2) \qquad \begin{cases} V(x,y) = V(x+2\pi/\alpha, y) = V(x, y+2\pi), \\ P(x,y) = P(x+2\pi/\alpha, y) = P(x, y+2\pi), \\ \iint_D V(x,y)dxdy = 0, \quad \iint_D P(x,y)dxdy = 0, \end{cases}$$

where $D = \{(x,y) \ : \ |x| \leq \pi/\alpha, \ |y| \leq \pi\}$.

Introducing the stream function $\psi(x,y)$, we represent the velocity as $(u,v) = (\psi_y, -\psi_x)$. The pressure is known to be determined by the velocity. Then, eliminating P and replacing ψ with $\gamma\nu^{-1}\psi$, we reduce the problem (1.1-2) to:

$$(1.3) \qquad \lambda J(\Delta\psi, \psi) = \nu\Delta^2\psi - \zeta\Delta\psi + \cos y, \quad J(f,g) \equiv f_x g_y - f_y g_x,$$

$$(1.4) \qquad \begin{cases} \psi(x,y) = \psi(x+2\pi/\alpha, y) = \psi(x, y+2\pi), \\ \iint_D \psi(x,y)dxdy = 0, \end{cases}$$

where $\lambda \equiv \gamma/\nu^2$ and $\zeta \equiv \kappa/\nu = 2/h^2$.

We can see that $\psi_0(x,y) \equiv -(1+\zeta)^{-1}\cos y$ satisfies (1.3-4) for any $\lambda > 0$ and $\zeta \geq 0$. We call this a basic solution. The velocity field of the basic solution is given by $(u_0, v_0) = (\gamma\nu^{-1}(1+\zeta)^{-1}\sin y, 0)$, which represents a shear flow parallel to the x-axis.

We would like to search solutions in the form $\psi = \psi_0 + \varphi$. From (1.3), we have

$$(1.5) \qquad f(\lambda, \varphi) \equiv \left\{\Delta^2 - \zeta\Delta - \lambda(1+\zeta)^{-1}\sin y(\Delta + I)\partial_x\right\}\varphi - \lambda J(\Delta\varphi, \varphi) = 0,$$

where I is the identity operator. $\varphi = 0$ corresponds to the basic solution for all λ and ζ. We consider φ in the Sobolev space X satisfying (1.4) such as $X \equiv H^4(D)/R$ with the inner product defined by

$$(\varphi, \varphi)_X \equiv (\Delta^2\varphi, \Delta^2\varphi)_{L^2} < \infty, \quad \varphi \in X.$$

The symbol $/R$ implies that only those functions with zero spatial mean are collected.

Theorem 1 *We fix $\alpha \in (0,1)$ and $\zeta \in [0,\infty)$. Let $r \in N$ satisfy $r\alpha < 1 \leq (r+1)\alpha$. Then there exists $\lambda = \lambda_k$ where $k \in K_\alpha \equiv \{\pm 1, \cdots, \pm r\}$, and in a neighborhood of $(\lambda_k, 0)$ there exists one parameter family of solution of (1.5) except the basic solution:*

$$(\lambda, \varphi) = (\mu(s), \varphi(s)), \qquad |s| < 1,$$

where $\mu(0) = \lambda_k$, $\varphi(0) = 0$ and $\mu_s(0) = 0$. Moreover, $\mu_{ss}(0) > 0$ is obtained for each $\zeta \geq 0$ when $k\alpha$ is close to one, which leads that this bifurcation is supercritical.

The problem is reduced the same one studied in [7] if $\zeta = 0$. As for this case where there's no external friction, professor Sadao Miyatake and myself have examined the bifurcation curves of solutions to the problem with a symmetric condition $\varphi(x,y) = \varphi(-x,-y)$ in order to use Crandall-Rabinowitz bifurcation theorem which requires $\dim\ker f_\varphi(\lambda_0, 0) = 1$. However, in this time we first remove the symmetric condition for the velocity, then obtain the similar result as seen in [7].

2 Guideline of the proof

2.1 Linearlized equations

First, we solve the linearized equation and obtain the function $\lambda = \lambda(\beta, \zeta)$ defined on $\beta \in (0,1)$ and $\zeta \in [0,\infty)$. The linearized eigenvalue problem for fixed α and ζ is

$$f_\varphi(\lambda, 0)\varphi = \left\{\Delta^2 - \zeta\Delta - \lambda(1+\zeta)^{-1}\sin y(\Delta + I)\partial_x\right\}\varphi = 0, \tag{2.1}$$

where λ is called eigenvalue if (2.1) has a solution $\varphi \neq 0$.

$\varphi \in X$ is expanded in the Fourier series:

$$\varphi = \sum_{m,n} c_{m,n} e^{i(m\alpha x + ny)}, \quad \sum_{m,n}(m^2\alpha^2 + n^2)^4 |c_{m,n}|^2 < +\infty, \quad c_{0,0} = 0,$$

where the summation is taken over all the pairs of integers but $(m,n) = (0,0)$. $c_{0,0} = 0$ follows from $\iint_D \varphi dxdy = 0$.

For each integer m, the coefficients $c_{m,n}$ satisfy the infinite system of linear equations:

$$(m^2\alpha^2 + n^2)(m^2\alpha^2 + n^2 + \zeta)c_{m,n} + \tfrac{\lambda m\alpha}{2(1+\zeta)}\{m^2\alpha^2 + (n-1)^2 - 1\}c_{m,n-1}$$
$$-\tfrac{\lambda m\alpha}{2(1+\zeta)}\{m^2\alpha^2 + (n+1)^2 - 1\}c_{m,n+1} = 0, \quad n = 0, \pm 1, \pm 2, \cdots.$$

We see $c_{0,n} = 0$ for any integer n. For $m \neq 0$, we put

$$a_{m,n} \equiv \frac{2(1+\zeta)(m^2\alpha^2 + n^2)(m^2\alpha^2 + n^2 + \zeta)}{\lambda m\alpha(m^2\alpha^2 + n^2 - 1)}, \qquad b_{m,n} \equiv (m^2\alpha^2 + n^2 - 1)c_{m,n},$$

then the above equations are simply described by

$$a_{m,n}b_{m,n} + b_{m,n-1} - b_{m,n+1} = 0, \quad n = 0, \pm 1, \pm 2, \cdots. \tag{2.2}$$

We remark that the set of solutions $\{b_{m,n}\}$ is one dimensional. Let us seek non-trivial solutions of the system (2.2) such that $b_{m,n} \to 0$ as $|n| \to \infty$ for each $m \neq 0$. In order to find these $b_{m,n}$, we need to solve the following equation:

$$-\frac{a_{m,0}}{2} = \frac{1}{a_{m,1}}\Big| + \frac{1}{a_{m,2}}\Big| + \cdots. \tag{2.3}$$

We may restrict ourselves to the case where $m > 0$, since for negative m the argument is similar because of $a_{m,n} = -a_{-m,n}$. We omit m and put $\beta \equiv m\alpha$ and $a_n \equiv a_{m,n}$ simply. Denoting the right hand side of (2.3) by $G(\lambda, \beta, \zeta)$, we rewrite (2.3) as

$$\frac{(1+\zeta)\beta(\beta^2+\zeta)}{\lambda(1-\beta^2)} = G(\lambda, \beta, \zeta). \tag{2.3'}$$

We state properties of (2.3′) in the following proposition (the proof is written in [12]).

Proposition 1 *For the solutions of* (2.3′), *we obtain the following results:*

(1) (2.3′) *has no positive solution if* $\beta > 1$ *and* $\zeta \geq 0$.

(2) *If* $0 < \beta < 1$, *there exists a continuous function* $\lambda(\beta, \zeta)$ *such that:*

  (i) (2.3′) *has a solution if and only if* $\lambda = \lambda(\beta, \zeta)$;

  (ii) *For fixed* $\zeta > 0$, $\lim_{\beta\to 0} \lambda(\beta, \zeta) = \lim_{\beta \to 1} \lambda(\beta, \zeta) = +\infty$ *and for* $\zeta = 0$, *it holds* $\lim_{\beta\to 0} \lambda(\beta, 0) = \sqrt{2}$ *and* $\lim_{\beta\to 1} \lambda(\beta, 0) = +\infty$;

  (iii) *For fixed* $\beta \in (0, 1)$, $\lambda(\beta, \zeta)$ *is a strictly monotone increasing function of* $\zeta > 0$.

Because of this difference between $\zeta > 0$ and $\zeta = 0$, Bondarenko and his groups created an updated model with an external friction.

From (2) of Proposition 1, (2.3) has a solution $\lambda = \lambda(\beta, \zeta) \equiv \lambda_k$ only if $\beta \equiv k\alpha \in (0, 1)$. Then, integer k is restricted as follows:

$$k \in K_\alpha \equiv \{1, 2, \cdots, r\ ;\ r \in N, r\alpha < 1 \leq (r+1)\alpha\}.$$

Then, we take a solution $b_{k,n}$ for $k \in K_\alpha$ defined by

$$b_{k,n} \equiv \begin{cases} \prod_{i=1}^{n} \rho_{k,i} & \text{for} \quad n > 0, \\ 1 & \text{for} \quad n = 0, \\ (-1)^n \prod_{i=1}^{-n} \rho_{k,i} & \text{for} \quad n < 0, \end{cases} \tag{2.4}$$

where

$$\rho_{k,i} = \frac{-1 \rfloor}{\lfloor a_{k,i}} + \frac{1 \rfloor}{\lfloor a_{k,i+1}} + \cdots, \qquad a_{k,i} = a_{k,i}(\lambda_k), \quad i \geq 1.$$

Let us consider the case where $m < 0$ and $|m| \in K_\alpha$. As we note $a_{m,n} = -a_{-m,n}$, we obtain that $b_{-k,n} = (-1)^n b_{k,n}$ for $k \in K_\alpha$ also satisfy (2.2). Therefore, the set of the non-trivial solutions of (2.1) is given as follows:

$$\text{(2.5)} \qquad \ker f_\varphi(\lambda_k, 0) = \left\{\varphi^{(k)} = t_1\varphi_k + t_2\varphi_{-k} \; ; \; t_1, t_2 \in R\right\},$$

where $\varphi_k \equiv \sum_{n=-\infty}^{+\infty} c_{k,n} e^{i(k\alpha x + ny)}$, $c_{k,n} = (k^2\alpha^2 + n^2 - 1)^{-1} b_{k,n}$. We see that φ_{-k} is equal to $\bar{\varphi}_k$, the conjugate function of φ_k, since we have $c_{-k,n} = (-1)^n c_{k,n} = c_{k,-n}$ due to $b_{-k,n} = (-1)^n b_{k,n} = b_{k,-n}$. Moreover, using Euler's formula, we can rewrite (2.5):

$$\text{(2.5}'\text{)} \qquad \ker f_\varphi(\lambda_k, 0) = \left\{\varphi^{(k)} = s_1\varphi_{k,1} + s_2\varphi_{k,2} \; ; \; s_1, s_2 \in R\right\},$$

where $\varphi_{k,1} \equiv \sum_{n=-\infty}^{\infty} c_{k,n} \cos(k\alpha x + ny)$ and $\varphi_{k,2} \equiv \sum_{n=-\infty}^{\infty} c_{k,n} \sin(k\alpha x + ny)$.

Similarly, let us seek non-trivial solutions Φ of the conjugate equation of (2.1):

$$\text{(2.6)} \qquad f_\varphi^*(\lambda, 0)\Phi = \left\{\Delta^2 - \zeta\Delta + \lambda(1+\zeta)^{-1}(\Delta + I)\sin y \partial_x\right\}\Phi = 0,$$

in the form $\Phi(x, y) = \sum_{m,n} d_{m,n} e^{i(m\alpha x + ny)}$. f_φ is a bounded operator from H_0^ℓ to $H_0^{\ell-4}$ where $\varphi \in H_0^\ell$ means $\varphi(x, y) = \sum_{m,n} c_{m,n} e^{i(m\alpha x + ny)}$ with $c_{0,0} = 0$ and $\sum_{m,n}(m^2 + n^2)^\ell c_{m,n}^2 < \infty$. And we have the following relation of $d_{m,n}$ for each integer m:

$$a_{m,n} d_{m,n} - d_{m,n-1} + d_{m,n+1} = 0.$$

Putting $b'_{m,n} \equiv (-1)^n d_{m,n}$, we have also

$$a_{m,n} b'_{m,n} + b'_{m,n-1} - b'_{m,n+1} = 0,$$

which is the same form as (2.2). Applying the same argument as that in (2.2), we obtain the non-trivial solutions of (2.6) if $\lambda = \lambda_k$ $k \in K$:

$$\text{(2.7)} \qquad \ker f_\varphi^*(\lambda_k, 0) = \left\{\Phi^{(k)} = t_1\Phi_k + t_2\Phi_{-k} \; ; \; t_1, t_2 \in R\right\},$$

where $\Phi_k = \sum_{n=-\infty}^{\infty} d_{k,n} e^{i(k\alpha x + ny)}$, $d_{k,n} = (-1)^n b_{k,n}$ and $b_{k,n}$ are given by (2.4). Note that each $\Phi^{(k)} \in \ker f_\varphi^*(\lambda_k, 0)$ is smooth function. We rewrite $\Phi^{(k)} \in \ker f_\varphi^*(\lambda_k, 0)$ as

$$\text{(2.7}'\text{)} \qquad \ker f_\varphi(\lambda_k, 0)^* = \left\{\Phi^{(k)} = s_1\Phi_{k,1} + s_2\Phi_{k,2} \; ; \; s_1, s_2 \in R\right\},$$

where $\Phi_{k,1} \equiv \sum_{n=-\infty}^{\infty} d_{k,n} \cos(k\alpha x + ny)$ and $\Phi_{k,2} \equiv \sum_{n=-\infty}^{\infty} d_{k,n} \sin(k\alpha x + ny)$.

We remark that the both $\ker f_\varphi(\lambda_k, 0)$ and $\ker f_\varphi^*(\lambda_k, 0)$ are two dimensional spaces.

2.2 Existence of bifurcation points

For $\alpha \in (0,1)$ and $\zeta \in [0,\infty)$, (2.1) has non-trivial solutions if and only if λ is equal to the values λ_k given in the previous section. Using the method of Ljapunov-Schmidt, we prove that $\lambda = \lambda_k$ is the bifurcation point of (1.5).

Assume $\varphi \in X$ and $\omega \in Y \equiv L_0^2$ where $g \in L_0^2$ means $g \in L^2$ and $\iint_D g\,dxdy = 0$. We decompose them orthogonally by:

$$\begin{aligned} \varphi &= \varphi_1 + \varphi_2, \quad \varphi_1 \in X_1, \quad \varphi_2 \in X_2, \\ \omega &= \omega_1 + \omega_2, \quad \omega_1 \in Y_1, \quad \omega_2 \in Y_2. \end{aligned}$$

X_i and Y_i $(i = 1,2)$ are defined as follows: $X_1 = \ker f_\varphi(\lambda_k, 0)$, X_2 is the orthogonal complement of X_1. Y_2 is the range of $f_\varphi(\lambda_k, 0)$ and Y_1 is the orthogonal complement of Y_2.

According to Section 2, $X_1 = \ker f_\varphi(\lambda_k, 0)$ and $\ker f_\varphi^*(\lambda_k, 0)$ are two dimensional space. We also see $\dim Y_1$ is two, namely, we verify

$$(3.1) \qquad Y_1 = \ker f_\varphi^*(\lambda_k, 0).$$

In fact, put $T \equiv f_\varphi(\lambda_k, 0)$ and $T^* \equiv f_\varphi^*(\lambda_k, 0)$, then $\omega_1 \in Y_1$ satisfies $(\omega_1, T\psi)_{L^2} = 0$ for $\psi \in X$. Hence we have $T^*\omega_1 = 0$ in the sense of distribution. Although ω_1 belongs to L_0^2 space and $\ker T^*$ is subspace of $X = H_0^4$, we can see that this ω_1 is smooth enough to belong to $\ker T^*$ by the hypo-ellipticity as follows. From (2.6), we write $T^* \equiv \Delta^2 + T^{(3)}$. Then $T^*\omega_1 = 0$ implies $\Delta^2\omega_1 = -T^{(3)}\omega_1$. Since $\omega_1 \in Y_1$, the right hand-side of this equation belongs to $H_0^{(-3)}$, namely, the Fourier expansion coefficients of ω_1 satisfy $\sum(m^2+n^2)^{-3}c_{m,n}^2 < \infty$. Then the left hand-side belongs to $H_0^{(-3)}$, which implies $\omega_1 \in H_0^1$. Repeating this several times, we see that ω_1 is sufficiently smooth.

We denote the projection to Y_1 of Y by P. Then, $Q \equiv I - P$ is the projection to Y_2. Corresponding to the above decomposition, we have the system of the following two equations which is equivalent to (1.5):

$$\begin{cases} Qf(\lambda, \varphi_1 + \varphi_2) = 0 \quad \text{in } Y_2, & \cdots (3.2) \\ Pf(\lambda, \varphi_1 + \varphi_2) = 0 \quad \text{in } Y_1. & \cdots (3.3) \end{cases}$$

Hereafter, we seek the solution (λ, φ) of this system, depending on one parameter $s \in (-1,1)$ as follows: $(\lambda, \varphi) = (\mu(s), \varphi_1(s) + \varphi_2(s))$. We suppose that $\mu(s) \in R$, $\varphi_1(s) \in X_1$ and $\varphi_2(s) \in X_2$ satisfy $\mu(0) = \lambda_k$. We put $\varphi_1(s) = s\varphi^{(k)}$ where $\varphi^{(k)}$ is a non-trivial solution of (2.1) given in (2.5). Then we look for $\lambda = \mu(s)$ and $\varphi_2(s)$.

First, let us consider (3.2). We put $Qf(\lambda, \varphi_1 + \varphi_2) \equiv g(\tau, \varphi_2)$ with $\tau \equiv (\lambda, s)$ for fixed $\alpha \in (0,1)$ and $\zeta \in [0, \infty)$. Note that $g(\tau_k, 0) = 0$ for $\tau_k \equiv (\lambda_k, 0)$ since $f(\lambda, 0) = 0$. By definition we see that $g_{\varphi_2}(\tau_k, 0) = Qf_\varphi(\lambda_k, 0)$ is a bijective mapping from X_2 to Y_2. Then from the implicit function theorem, there exists a function $\psi(\tau)$ which satisfies $g(\tau, \psi(\tau)) = 0$ and $\psi(\tau_k) = 0$ in the neighborhood of $(\tau_k, 0)$. We shall determine $\psi = \psi(\tau)$ more precisely. From (3.2), with $\varphi_1 = s\varphi^{(k)}$ and $\varphi_2 = \psi$, ψ satisfies the following equation:

$$H[\psi] - \tilde{L}[s\varphi^{(k)} + \psi] - \lambda J(\Delta(s\varphi^{(k)} + \psi), s\varphi^{(k)} + \psi) = 0,$$

where $H \equiv Qf_\varphi(\lambda_k, 0)$, $\tilde{L} \equiv (\lambda - \lambda_k)(1+\zeta)^{-1} \sin y (\Delta + I)\partial_x$. Since H is a bijective mapping from X_2 to Y_2, it holds that

$$\psi - H^{-1}\tilde{L}[s\varphi^{(k)} + \psi] - \lambda H^{-1} J(\Delta(s\varphi^{(k)} + \psi), s\varphi^{(k)} + \psi) = 0.$$

We define a sequence of functions $\{\psi_n\}$ $(n = 0, 1, 2, \cdots)$ as follows:

$$\psi_0 = 0, \quad \psi_n \equiv H^{-1}\tilde{L}[s\varphi^{(k)} + \psi_{n-1}] - \lambda H^{-1} J(\Delta(s\varphi^{(k)} + \psi_{n-1}), s\varphi^{(k)} + \psi_{n-1}).$$

Let us show that $\{\psi_n\}$ is a Cauchy sequence in the neighborhood of $s = 0$. In fact, since the non-linear term becomes $O(s^2)$, it can be omitted. Choosing λ such as $|\lambda - \lambda_k| \le 4^{-1}\|H^{-1}\|^{-1}$, we have $\|\psi_1\| = O(s)$ and $\|\psi_2 - \psi_1\| \le 2^{-1}\|\psi_1\|$. Similarly, it holds that $\|\psi_{n+1} - \psi_n\| \le 2^{-n}\|\psi_1\|$. Then $\{\psi_n\}$ is a Cauchy sequence and converges to a limit $\psi = \psi(\lambda, s)$ which belongs to X_2 satisfying $\psi(\lambda, 0) = 0$ and

(3.4) $$\psi = H^{-1}\tilde{L}[s\varphi^{(k)} + \psi] - \lambda H^{-1} J(\Delta(s\varphi^{(k)} + \psi), s\varphi^{(k)} + \psi)$$

for small s.

In order to show that λ_k is a bifurcation point, we have to prove the existence of the solution $\mu(s)$ of (3.3) satisfying $\mu(0) = \lambda_k$. Substituting $\varphi_2 = \psi(\tau)$ into the left hand side of (3.3) and defining

$$Pf(\lambda, s\varphi^{(k)} + \psi(\lambda, s)) \equiv h(\lambda, s),$$

we denote

$$\chi(\lambda, s) \equiv \begin{cases} \{h(\lambda, s) - h(\lambda, 0)\}/s, & \text{for } s \ne 0, \\ h_s(\lambda, 0), & \text{for } s = 0. \end{cases}$$

Note that $h(\lambda, 0) = 0$ holds and the continuity of χ follows from that of h_s. The reason why we define $\chi(\lambda, s)$ is that we cannot apply the implicit function theorem to

$h(\lambda, s)$. Remark that $h_\lambda(\lambda, 0) = 0$ holds from $\psi(\lambda, 0) = 0$ for all λ. From $h_s(\lambda, s) = Pf_\varphi(\lambda, s\varphi^{(k)} + \psi(\lambda, s))[\varphi^{(k)} + \psi_s(\lambda, s)]$, it holds that $h_s(\lambda, 0) = Pf_\varphi(\lambda, 0)[\varphi^{(k)} + \psi_s(\lambda, 0)]$. Now we verify $\psi_s(\lambda_k, 0) = 0$. Differentiating $Qf(\lambda, s\varphi^{(k)} + \psi(\lambda, s)) = 0$ by s and putting $(\lambda, s) = (\lambda_k, 0)$, we have $Qf_\varphi(\lambda_k, 0)[\psi_s(\lambda_k, 0)] = 0$. Since $Qf_\varphi(\lambda_k, 0)$ is a bijective mapping from X_2 to Y_2, $\psi_s(\lambda_k, 0) = 0$ holds.

$\chi(\lambda, s) = 0$ is equivalent to the following equations:

$$\chi^{(1)}(\lambda, s) \equiv (\chi(\lambda, s), \Phi_{k,1})_{L^2} = 0, \tag{3.5}$$

$$\chi^{(2)}(\lambda, s) \equiv (\chi(\lambda, s), \Phi_{k,2})_{L^2} = 0, \tag{3.6}$$

where $\Phi_{k,i} \in Y_1 = \ker f_\varphi^*(\lambda_k, 0)$ $(i = 1, 2)$. First, we seek a solution λ of (3.5) putting $\varphi^{(k)} = t_1\varphi_{k,1} + t_2\varphi_{k,2}$ for $(t_1, t_2) \neq (0, 0)$. Differentiating (3.5) by λ, then we have

$$\begin{aligned}
\chi_\lambda^{(1)}(\lambda_k, 0) &= \left(\lim_{\Delta\lambda \to 0} \frac{\chi(\lambda_k + \Delta\lambda, 0) - \chi(\lambda_k, 0)}{\Delta\lambda}, \Phi_{k,1}\right)_{L^2} \\
&= (Pf_{\varphi\lambda}(\lambda_k, 0)[\varphi^{(k)}], \Phi_{k,1})_{L^2} = (f_{\varphi\lambda}(\lambda_k, 0)[\varphi^{(k)}], P^*\Phi_{k,1})_{L^2} \\
&= (f_{\varphi\lambda}(\lambda_k, 0)[\varphi^{(k)}], P\Phi_{k,1})_{L^2} \\
&= t_1(-(1+\zeta)^{-1} \sin y(\Delta + I)\partial_x\varphi_{k,1}, \Phi_{k,1})_{L^2}.
\end{aligned}$$

We show

$$(-(1+\zeta)^{-1} \sin y(\Delta + I)\partial_x\varphi_{k,1}, \Phi_{k,1})_{L^2} > 0. \tag{3.7}$$

Since $\varphi_{k,1}$ is a solution of (2.1), we have

$$-(1+\zeta)^{-1} \sin y(\Delta + I)\partial_x\varphi_{k,1} = \lambda_k^{-1}(\zeta)(-\Delta^2 + \zeta\Delta)\varphi_{k,1}.$$

Using $\varphi_{k,1} = \sum_n c_{k,n} \cos(k\alpha x + ny)$ and $\Phi_{k,1} = \sum_n d_{k,n} \cos(k\alpha x + ny) = \sum_n (-1)^n (k^2\alpha^2 + n^2 - 1)c_{k,n} \cos(k\alpha x + ny)$, we obtain

$$((-\Delta^2 + \zeta\Delta)\varphi_{k,1}, \Phi_{k,1})_{L^2} \equiv \frac{1}{2}|D| \sum_n (-1)^{n+1} \tilde{c}_{k,n},$$

where $\tilde{c}_{k,n} \equiv (k^2\alpha^2 + n^2)(k^2\alpha^2 + n^2 + \zeta)(k^2\alpha^2 + n^2 - 1)c_{k,n}^2$. Meanwhile, we can verify $\sum_n \tilde{c}_{k,n} = 0$ (seen in Iudovich[4]). In fact, from $f_\varphi(\lambda_k, 0)\varphi_{k,1} = 0$, multiplying this equation $(\Delta + I)\varphi_{k,1}$ and integrating over the rectangle D, we obtain

$$\begin{aligned}
0 = &\iint_D (\Delta + I)\varphi_{k,1}(\Delta^2 - \zeta\Delta)\varphi_{k,1} dxdy \\
&- \lambda_k(1+\zeta)^{-1} \iint_D (\Delta + I)\varphi_{k,1} \sin y(\Delta + I)\partial_x\varphi_{k,1} dxdy,
\end{aligned}$$

and see that the second term vanishes. Then, we have

$$\iint_D (\Delta + I)\varphi_{k,1}(\Delta^2 - \zeta\Delta)\varphi_{k,1}dxdy = \frac{-1}{2}|D|\sum_n \tilde{c}_{k,n} = 0.$$

From $\sum_n \tilde{c}_{k,n} = 0$ and $\tilde{c}_{k,-n} = \tilde{c}_{k,n}$, we obtain (3.7) since it holds

$$\begin{aligned}\sum_n (-1)^{n+1}\tilde{c}_{k,n} &= -\tilde{c}_{k,0} + 2\sum_{m=1,3,5,\cdots}\tilde{c}_{k,m} - 2\sum_{m=2,4,6,\cdots}\tilde{c}_{k,m} \\ &= 4\sum_{m=1,3,5,\cdots}\tilde{c}_{k,m} > 0.\end{aligned}$$

As a result, we have $\chi^{(1)}_\lambda(\lambda_k, 0) \neq 0$ if $t_1 \neq 0$. From the implicit function theorem, there exists a function $\lambda = \mu(s)$ satisfying $\chi^{(1)}(\mu(s), s) = 0$ and $\mu(0) = \lambda_k$.

Next, we suppose the question whether $\lambda = \mu(s)$ satisfies (3.6). Since $h_s(\lambda_k, 0) = 0$ holds from $h_s(\lambda, 0) = Pf_\varphi(\lambda, 0)[\varphi^{(k)} + \psi_s(\lambda, 0)]$ and $\psi_s(\lambda_k, 0) = 0$, we can see $\chi^{(2)}(\lambda_k, 0) = (h_s(\lambda_k, 0), \Phi_{k,2})_{L^2} = 0$. As for $s \neq 0$, it holds

$$\begin{aligned} s\chi^{(2)}(\lambda, s) &= (h(\lambda, s), \Phi_{k,2})_{L^2} \\ &= (Pf(\lambda, s\varphi^{(k)} + \psi(\lambda, s)), \Phi_{k,2})_{L^2} \\ &= (f(\lambda, s\varphi^{(k)} + \psi(\lambda, s)), \Phi_{k,2})_{L^2}. \end{aligned}$$

Then we have the following formula:

$$\begin{aligned} & s\chi^{(2)}(\mu(s), s) = (f(\mu(s), s\varphi^{(k)} + \psi(\mu(s), s)), \Phi_{k,2})_{L^2} \\ = & \Big(\{\Delta^2 - \zeta\Delta - \mu(s)\sin y(\Delta + I)\partial_x\}[s\varphi^{(k)} + \psi(\mu(s), s)], \Phi_{k,2}\Big)_{L^2} \\ & -\mu(s)\Big(J(\Delta(s\varphi^{(k)} + \psi(\mu(s), s)), s\varphi^{(k)} + \psi(\mu(s), s)), \Phi_{k,2}\Big)_{L^2}. \end{aligned}$$

The question is how we choose $\varphi^{(k)}$. From (3.4), if $\varphi^{(k)}$ is represented as a liner combination of $\varphi_{k,1}$ and $\varphi_{k,2}$, $\psi(\mu(s), s)$ is expanded by both sine and cosine functions. In this case, we cannot expect in general that the above formula goes to zero. However, if we put $\varphi^{(k)} = \varphi_{k,1}$, $\psi(\mu(s), s)$ is expanded by cosine only. As a result, the inner-product with $\Phi_{k,2}$ becomes zero and, hence, $\mu(s)$ satisfies (3.6). Thus, we obtain the former part of Theorem 1.

2.3 Properties of the Bifurcation curve

We shall consider the convex property of $\lambda = \mu(s)$ with regard to s. Putting $T \equiv f_\varphi(\lambda_k, 0)$ and $\tilde{\lambda}(s) \equiv \mu(s) - \lambda_k$, we rewrite $f(\mu(s), \varphi(s)) = 0$ as

$$T\varphi(s) = \frac{\tilde{\lambda}(s)}{1+\zeta} \sin y(\Delta + I)\partial_x \varphi(s) + \mu(s) J(\Delta\varphi(s), \varphi(s)), \tag{4.1}$$

where $\varphi(s) \equiv s\varphi_{k,1} + \psi(\mu(s), s)$. Let us differentiate (4.1) by s:

$$\begin{aligned} T\varphi_s(s) &= \frac{\tilde{\lambda}_s(s)}{1+\zeta} \sin y(\Delta + I)\partial_x \varphi(s) + \frac{\tilde{\lambda}(s)}{1+\zeta} \sin y(\Delta + I)\partial_x \varphi_s(s) \\ &\quad + \mu_s(s) J(\Delta\varphi(s), \varphi(s)) + \mu(s) J(\Delta\varphi(s), \varphi(s))_s; \\ T\varphi_{ss}(s) &= \frac{\tilde{\lambda}_{ss}(s)}{1+\zeta} \sin y(\Delta + I)\partial_x \varphi(s) + \frac{2\tilde{\lambda}_s(s)}{1+\zeta} \sin y(\Delta + I)\partial_x \varphi_s(s) \\ &\quad + \frac{\tilde{\lambda}(s)}{1+\zeta} \sin y(\Delta + I)\partial_x \varphi_{ss}(s) + \mu_{ss}(s) J(\Delta\varphi(s), \varphi(s)) \\ &\quad + 2\mu_s(s) J(\Delta\varphi(s), \varphi(s))_s + \mu(s) J(\Delta\varphi(s), \varphi(s))_{ss}; \\ \varphi_s(s) &= \varphi_{k,1} + \psi_\lambda(\mu(s), s)\mu_s(s) + \psi_s(\mu(s), s). \end{aligned}$$

Putting $s = 0$, we have

$$T\varphi_{ss}(0) = \frac{2\mu_s(0)}{1+\zeta} \sin y(\Delta + I)\partial_x \varphi_{k,1} + 2\lambda_k J(\Delta\varphi_{k,1}, \varphi_{k,1}). \tag{4.2}$$

If we take the L^2 inner-product with $\Phi_{k,1} \in \ker T^*$, (4.2) becomes

$$0 = \frac{2\mu_s(0)}{1+\zeta} (\sin y(\Delta + I)\partial_x \varphi_{k,1}, \Phi_{k,1})_{L^2} + 2\lambda_k (J(\Delta\varphi_{k,1}, \varphi_{k,1}), \Phi_{k,1})_{L^2},$$

and from $T\varphi_{k,1} = 0$, we obtain

$$0 = \frac{2\mu_s(0)}{\lambda_k} ((\Delta^2 - \zeta\Delta)\varphi_{k,1}, \Phi_{k,1})_{L^2} + 2\lambda_k (J(\Delta\varphi_{k,1}, \varphi_{k,1}), \Phi_{k,1})_{L^2}.$$

Since the Fourier coefficients of $J(\Delta\varphi_{k,1}, \varphi_{k,1})$ consist of a linear combination of $\cos ny$ and $\cos(2k\alpha x + ny)$, we have $(J(\Delta\varphi_{k,1}, \varphi_{k,1}), \Phi_{k,1})_{L^2} = 0$. Also, from the proof of (3.7), we have

$$((\Delta^2 - \zeta\Delta)\varphi_{k,1}, \Phi_{k,1})_{L^2} < 0. \tag{4.3}$$

Therefore, we obtain $\mu_s(0) = 0$.

Differentiating (4.1) once more and putting $s = 0$, we have

$$\begin{aligned} T\varphi_{sss}(0) &= 3\mu_{ss}(0)(1+\zeta)^{-1}\sin y(\Delta + I)\partial_x\varphi_{k,1} \\ &\quad +3\lambda_k\left\{J(\Delta\varphi_{ss}(0),\varphi_{k,1}) + J(\Delta\varphi_{k,1},\varphi_{ss}(0))\right\} \\ &= 3\mu_{ss}(0)\lambda_k^{-1}(\Delta^2 - \zeta\Delta)\varphi_{k,1} \\ &\quad +3\lambda_k\left\{J(\Delta\varphi_{ss}(0),\varphi_{k,1}) + J(\Delta\varphi_{k,1},\varphi_{ss}(0))\right\}, \end{aligned}$$

and taking the L^2 inner-product with $\Phi_{k,1} \in \ker T^*$,

$$\begin{aligned} 0 &= (T\varphi_{sss}(0), \Phi_{k,1})_{L^2} \\ &= 3\mu_{ss}(0)\lambda_k^{-1}((\Delta^2 - \zeta\Delta)\varphi_{k,1}, \Phi_{k,1})_{L^2} \\ &\quad +3\lambda_k(J(\Delta\varphi_{ss}(0),\varphi_{k,1}) + J(\Delta\varphi_{k,1},\varphi_{ss}(0)), \Phi_{k,1})_{L^2} \end{aligned}$$

holds. Then we have

$$\mu_{ss}(0) = \frac{-\lambda_k^2}{((\Delta^2 - \zeta\Delta)\varphi_{k,1}, \Phi_{k,1})_{L^2}}\Big(J(\Delta\varphi_{ss},\varphi_{k,1}) + J(\Delta\varphi_{k,1},\varphi_{ss}), \Phi_{k,1}\Big)_{L^2}.$$

Let us determine the sign of $\mu_{ss}(0)$. From (4.3), this sign is equal to that of

$$\iint_D \left\{J(\Delta\varphi_{ss},\varphi_{k,1}) + J(\Delta\varphi_{k,1},\varphi_{ss})\right\}\Phi_{k,1}dxdy. \tag{4.4}$$

Here $\varphi_{ss} \equiv \varphi_{ss}(0) = \psi_{ss}(\lambda_k, 0)$ is obtained by

$$T\varphi_{ss} = 2\lambda_k J(\Delta\varphi_{k,1}, \varphi_{k,1}). \tag{4.5}$$

The right-hand side of (4.5) consists of two terms extended respectively by $\cos \ell y$ and $\cos(2k\alpha x + \ell y)$.

We have the following proposition:

Proposition 2 *The solution of* (4.5) *takes the following form:*

$$\varphi_{ss} = {}^t\boldsymbol{w}^{(0)}\boldsymbol{\Lambda c}(0) + {}^t\boldsymbol{w}^{(2k)}\boldsymbol{DEc}(2k\alpha) \equiv Z_1 + Z_2, \tag{4.6}$$

$$Z_1 \equiv {}^t\boldsymbol{w}^{(0)}\boldsymbol{\Lambda c}(0), \qquad Z_2 \equiv {}^t\boldsymbol{w}^{(2k)}\boldsymbol{DEc}(2k\alpha).$$

Here $\mathbf{c}(0)$, $\mathbf{c}(2k\alpha)$, $\boldsymbol{w}^{(0)}$ *and* $\boldsymbol{w}^{(2k)}$ *are column vectors with the following* ℓ*-th components:*

$$(\mathbf{c}(0))_\ell = \cos \ell y, \quad (\mathbf{c}(2k\alpha))_\ell = \cos(2k\alpha x + \ell y),$$
$$(\boldsymbol{w}^{(0)})_\ell = \lambda_k k\alpha\ell\, {}^t\boldsymbol{\varphi}^{(k)} \boldsymbol{K} S^\ell \boldsymbol{\varphi}^{(k)},$$
$$(\boldsymbol{w}^{(2k)})_\ell = \lambda_k k\alpha\, {}^t\boldsymbol{\varphi}^{(k)} \boldsymbol{K}(2\boldsymbol{N} - \ell \boldsymbol{I}) R S^\ell \boldsymbol{\varphi}^{(k)},$$

where $\boldsymbol{\varphi}^{(k)}$ *is a column vector corresponding to the Fourier coefficients of* $\varphi_{k,1}$ *with* n*-th component* $\varphi_n = (k^2\alpha^2 + n^2 - 1)^{-1} b_{k,n}$ *(*$b_{k,n}$ *is defined by (2.6)),* $\boldsymbol{K}$ *and* $\boldsymbol{N}$ *are diagonal matrices with* n*-th elements* $-k_n \equiv -(k^2\alpha^2 + n^2)$ *and* n *respectively.* S^ℓ *and* R *are matrices with* (i,j) *elements as follows:*

$$(S^\ell)_{i,j} = \begin{cases} 1 & \text{for } j - i = \ell, \\ 0 & \text{otherwise,} \end{cases} \qquad (R)_{i,j} = \begin{cases} 1 & \text{for } i + j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$\boldsymbol{\Lambda}$ *and* $\boldsymbol{E}$ *are diagonal matrices with* n*-th elements*

$$\boldsymbol{\Lambda}_n = \begin{cases} (n^4 + \zeta n^2)^{-1} & \text{for } n \neq 0, \\ 0 & \text{for } n = 0, \end{cases} \qquad \boldsymbol{E}_n = \frac{1+\zeta}{\lambda_k k\alpha(4k^2\alpha^2 + n^2 - 1)},$$

and $\boldsymbol{D} = (\cdots \boldsymbol{d}^{(m)} \cdots)$ *is a matrix where* $\boldsymbol{d}^{(m)}$ *are column vectors with* n*-th component* $\boldsymbol{d}_n^{(m)}$ *as follows:*

$$\boldsymbol{d}_n^{(m)} = \begin{cases} \left(\Pi_{i=m+1}^{n} \eta_i^+\right) N_{m+1}^{-1} & \text{for } n > m, \\ N_{m+1}^{-1} & \text{for } n = m, \\ \left(\Pi_{i=n+1}^{m} \eta_i^-\right)^{-1} N_{m+1}^{-1} & \text{for } n < m, \end{cases}$$

where

$$\begin{aligned} \eta_n^+ &\equiv \frac{1 \rfloor}{a'_n} + \frac{1 \rfloor}{a'_{n+1}} + \cdots, \\ \eta_n^- &\equiv -a'_{n-1} + \frac{-1 \rfloor}{a'_{n-2}} + \cdots, \\ N_{m+1} &\equiv \eta_{m+1}^+ - \eta_{m+1}^-, \\ a'_n &\equiv \frac{(1+\zeta)(4k^2\alpha^2 + n^2)(4k^2\alpha^2 + n^2 + \zeta)}{\lambda_k k\alpha(4k^2\alpha^2 + n^2 - 1)}. \end{aligned}$$

We can prove Proposition 2 in the same way to Section 3.2 of [7].

Substituting (4.6) into (4.4), we have

$$\iint_D \{J(\Delta\varphi_{ss}(0), \varphi_{k,1}) + J(\Delta\varphi_{k,1}, \varphi_{ss}(0))\}\Phi_{k,1}dxdy \equiv D_1 + D_2,$$

$$D_1 \equiv \iint_D \{J(\Delta Z_1, \varphi_{k,1}) + J(\Delta\varphi_{k,1}, Z_1)\}\Phi_{k,1}dxdy,$$

$$D_2 \equiv \iint_D \{J(\Delta Z_2, \varphi_{k,1}) + J(\Delta\varphi_{k,1}, Z_2)\}\Phi_{k,1}dxdy.$$

As for D_1 and D_2, we obtain the following proposition.

Proposition 3 *For each fixed $\zeta \geq 0$, $D_1 > |D_2|$ holds if $k\alpha$ close to one.*

The proof is given in my current preprint [12], which is based on the previous paper (Section 4 and 5 of [7]). This proposition means that $\mu_{ss}(0) > 0$ holds if $k\alpha \in (0,1)$ is sufficiently close to one. Thus, Theorem 1 is proved.

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