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A NEW SCHEME FOR SINGULAR PERTURBATION PROBLEMS IN CONSERVATION LAWS WITH PRESENCE OF SHOCK WAVES

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ABSTRACT. In this paper we systematically use a new scheme to study a singular perturbation of an $n\times n$ strictly hyperbolic conservation laws so that one could clearly reveal the basic structure of the new scheme without too much technical complication.

1. INTRODUCTION

A new scheme is introduced in [15] to study both the hydrodynamic limit for Boltzmann equation and the zero dissipation limit for compressible Navier-Stokes equation. These problems are closely related to singular perturbations of genuinely nonlinear hyperbolic conservation laws

$$\alpha_t + F(\alpha)_x = 0,\ \alpha\in\mathbb{R}^n. \tag{1.1}$$

The singular perturbation problems are also related to the problems like, relaxation system with stiff source term, the convergence of finite difference scheme, etc..

When the solution α is smooth, one can use the Hilbert expansion, [6], to study the singular perturbation problem. The condition that α is smooth is a technical assumption in the consideration so that the Hilbert expansion works. However, the condition is too strong to exclude shock wave which is a generic nonlinear wave pattern of (1.1). When shock wave shows up in the solution α, the singular perturbation problems becomes more interesting compared to the case that α is smooth. It is because that the hyperbolic system (1.1) itself is not well-posed when α contains shock waves.

The new scheme introduced in [15] imposes an initial-boundary value problem for a linearized Euler equation around a shock wave to generalize the Hilbert expansion so that one still can obtain a formal expansion series. It also integrates the local nonlinear internal layers and the formal expansion series so that conservation laws are still valid for the approximate solutions obtained by the new scheme.

Though the problem (1.2) had been extensively studied by [5], [14], [1], the main purpose of this paper is to abstract the essence of the new scheme and to illustrate how it works. The viscous conservation laws with an artificial viscosity are suitable for these purposes. So, we consider

$$\alpha^\kappa_t + F(\alpha^\kappa)_x = \kappa\,\alpha^\kappa_{xx},\ \ \kappa>0,\ \alpha^\kappa\in\mathbb{R}^n \tag{1.2}$$

with assumptions on the invscid solution α

(1) α is periodic in space with a period $\mathbb{T}$.
(2) α is piecewise smooth for $t\in[0,t_0]$.
(3) α contains only one shock wave in the domain $[0,\mathbb{T}]\times[0,t_0]$.
(4) The total variation of α is sufficiently small in $[0,\mathbb{T}]$ for any $t\in[0,t_0]$.

Theorem 1.1 (Main Theorem). *Under the above assumptions on α, there exist $\kappa_0>0$ and $\tau_0>0$ such that for any $\kappa\in(0,\kappa_0)$ there is solution α^κ of (1.2) so that*

$$w\text{-}lim_{\kappa\to 0+}\alpha^\kappa(x,t)=\alpha(x,t)\ \textit{for any}\ t\in(0,\tau_0).$$

In Section 5, we will give a precise pointwise estimate of $\alpha^\kappa - \alpha$. The estimates of $\alpha^\kappa - \alpha$ will yield the convergence to a weak solution; and the Main theorem follows. Here, the estimate is based on the following components

- A formal asymptotic expansion (Hilbert Expansion).
- An initial-boundary value problem of a linearized Euler equation.
- Existence of shock layers.
- Stability of shock layers.
- Energy estimates.

The first three components are for the new scheme to construct an approximate solution; and the last two components are for estimating $\alpha - \alpha^\kappa$. Note that the last component, energy estimates, is not absolutely necessary. It could be replaced by other type time-asymptotic analysis such as the pointwise estimates in [8], [14]. Here, the energy estimates essentially originated from [4].

When the solution α is smooth, the Hilbert expansion is applied to obtain the convergence of a finite difference scheme for hyperbolic conservation in [10], the hydrodynamic limit of Boltzmann equation in [2], [12], [13], the hydrodynamic limit of Broadwell model in [3].

When the solution α contains shock waves under the condition that α is piecewise smooth, a matching inner and outer expansion method is used to study the singular perturbation problem in [11], [4]. The approach in [14] is a prototype of the new scheme for studying (1.2). Recently, by using center manifold one can obtain the zero dissipation limit of α^κ with a general setting on α by [1].

2. Preliminary

A Formal Asymptotic Expansion.
One can formally expand the solution α^κ as follows

$$\alpha^\kappa = \alpha + \kappa\alpha_1 + \kappa^2\alpha_2 + \kappa^3\alpha_3 + \cdots . \tag{2.1}$$

The equations for α, α_1, α_2, $\cdots$ are given as follows

$$\begin{aligned}
&\partial_t\alpha + \partial_x F(\alpha) = 0,\\
&\partial_t\alpha_1 + \partial_x F'(\alpha)\alpha_1 = \partial_x \mathbb{S}_1,\ \ \mathbb{S}_1 \equiv \partial_x\alpha,\\
&\partial_t\alpha_2 + \partial_x F'(\alpha)\alpha_2 = \partial_x \mathbb{S}_2,\ \ \mathbb{S}_2 \equiv \partial_x\alpha_1 - \frac{1}{2}F''(\alpha)(\alpha_1,\alpha_1),\\
&\partial_t\alpha_3 + \partial_x F'(\alpha)\alpha_3 = \partial_x \mathbb{S}_3,\ \ \mathbb{S}_3 \equiv \partial_x\alpha_2 - F''(\alpha)(\alpha_2,\alpha_1) - \frac{1}{6}F'''(\alpha)(\alpha_1,\alpha_1,\alpha_1),\\
&\ \vdots
\end{aligned}$$

Notions for Strictly Hyperbolic Conservation Laws.
The matrix $F'(\alpha)$ is an $n\times n$ matrix with n distinct real eigenvalues $\lambda_i(\alpha)$, $1\le i\le n$,

$$\begin{cases}
F'(\alpha)\boldsymbol{r}_i(\alpha) = \lambda_i(\alpha)\,\boldsymbol{r}_i(\alpha),\\
\boldsymbol{l}_i(\alpha)F'(\alpha) = \lambda_i(\alpha)\,\boldsymbol{l}_i(\alpha),\\
\lambda_1(\alpha) < \lambda_2(\alpha) < \cdots < \lambda_n(\alpha),\\
|\boldsymbol{r}_i| \equiv 1,\\
(\boldsymbol{l}_i,\boldsymbol{r}_j) = \delta^i_j;
\end{cases}$$

and

$$\nabla\lambda_i(\alpha)\cdot\boldsymbol{r}_i(\alpha) \neq 0 \text{ for } \alpha\in\mathbb{R}^n, i = 1,2,\cdots,n$$

for the genuinely nonlinearity of $F(\alpha)$.

Notions for Shock Waves.
A shock wave is a discontinuity (α_-, α_+) in α satisfying the Rankine-Hugoniot condition

$$\sigma(\alpha_- - \alpha_+) = F(\alpha_-) - F(\alpha_+), \ \sigma \in \mathbb{R}$$

and the Lax's entropy condition, [7],

$$\begin{cases} \lambda_p(\alpha_+) < \sigma < \lambda_p(\alpha_-), \\ \lambda_{p-1}(\alpha_-) < \sigma < \lambda_{p+1}(\alpha_+). \end{cases}$$

Such a shock wave is called a p-**shock wave**; and σ is the speed of shock wave (α_-, α_+).

Notions for Shock Layers.
A shock layer connecting a p-shock wave (α_-, α_+) is a travelling wave solution of (1.2) satisfying

$$\begin{cases} \alpha^\kappa(x,t) = U(\frac{x-\sigma t}{\kappa}), \\ \lim_{\xi \to \pm\infty} U(\xi) = \alpha_\pm. \end{cases}$$

When $|\alpha_- - \alpha_+|$ is sufficiently small, the function $\lambda_p(U(\xi))$ is strictly monotone decreasing, [9],

$$\partial_\xi \lambda_p(U(\xi)) < 0.$$

Notions for Shock Locations, Shock Waves, Shock Layers for (1.2).
Denote $x = s(t)$ the location of shock wave of $(\alpha(s(t)-,t), \alpha(s(t)+,t))$, and denote $U(\xi,t)$ a shock layer connecting the shock wave $(\alpha(s(t)-,t), \alpha(s(t)+,t))$ with a normalized condition

$$U^1(0,t) = \frac{1}{2}(1,0,\cdots,0)\cdot(\alpha(s(t)-,0) + \alpha(s(t)+,0)).$$

The strength of the shock in the time domain $[0, t_0]$ is assumed to satisfy

$$\begin{cases} \epsilon \equiv \min_{t\in[0,t_0]} |\alpha(s(t)-,t) - \alpha(s(t)+,t)| \ll 1, \\ \dfrac{\max_{t\in[0,t_0]} |\alpha(s(t)-,t) - \alpha(s(t)+,t)|}{\min_{t\in[0,t_0]} |\alpha(s(t)-,t) - \alpha(s(t)+,t)|} = O(1). \end{cases}$$

Assumption. *The shock wave $(\alpha(s(t)-,t), \alpha(s(t)+,t))$ is assumed to be a p-**shock wave** for $t \in [0, t_0]$.*

A Microscopic Coordinates

$$\begin{cases} \kappa x' = x - s(t), \\ \kappa t' = t. \end{cases}$$

We still use the same notations to denote the conjugate functions as follows

$$\begin{cases} \alpha(x',t') & \equiv & \alpha(x,t) \\ \alpha^\kappa(x',t') & \equiv & \alpha^\kappa(x,t) \\ \{x'=0\} & \equiv & \{x=s(t)\} \\ U(\xi,t') & \equiv & U(\xi,t) \\ s(t') & \equiv & s(t)/\kappa \\ (\alpha(0-,t'),\alpha(0+,t')) & \equiv & (\alpha(s(t)-,t),\alpha(s(t)+,t)). \end{cases}$$

The system for α^κ in the microscopic system is

$$\partial_{t'}\alpha^\kappa + \partial_{x'}\{-s'(t')\,\alpha^\kappa + F(\alpha^\kappa)\} - \partial^2_{x'}\alpha^\kappa = 0;$$

and the system for $U(\xi,t')$ is

$$\begin{cases} -s'(t')\,\partial_\xi U + \partial_\xi F(U) = \partial^2_\xi U, \\ \lim\limits_{\xi\to\pm\infty} U(\xi,t') = \lim\limits_{\eta\to\pm 0}\alpha(\eta,t'). \end{cases}$$

The Asymptotic Expansion in the Microscopic Coordinate System
We rewrite the expansion in (2.1) as follows

$$\begin{aligned} (2.2)\qquad & \alpha^\kappa(x',t') = \alpha(x',t') + \alpha_1(x',t') + \alpha_2(x',t') + \cdots, \\ & \partial_{t'}\alpha - s'\partial_{x'}\alpha + \partial_{x'}F(\alpha) = 0, \\ & [\partial_{t'} - s'\partial_{x'} + \partial_{x'}F'(\alpha)]\alpha_1 = \partial_{x'}\mathbb{S}_1,\ \mathbb{S}_1 \equiv \partial_{x'}\alpha, \\ & [\partial_{t'} - s'\partial_{x'} + \partial_{x'}F'(\alpha)]\alpha_2 = \partial_{x'}\mathbb{S}_2,\ \mathbb{S}_2 \equiv \partial_{x'}\alpha_1 - \frac{1}{2}F''(\alpha)(\alpha_1,\alpha_1), \\ & [\partial_{t'} - s'\partial_{x'} + \partial_{x'}F'(\alpha)]\alpha_3 = \partial_{x'}\mathbb{S}_3,\ \mathbb{S}_3 \equiv \partial_{x'}\alpha_2 - F''(\alpha)(\alpha_2,\alpha_1) - \frac{1}{6}F'''(\alpha)(\alpha_1,\alpha_1,\alpha_1), \end{aligned}$$

3. Linearized Conservation Laws

From the notions in (2.2), we need to consider inhomogeneous linearized Euler equation. We consider the linear homogeneous problem first

$$\partial_{t'}V - s'(t')\,\partial_{x'}V + \partial_{x'}F'(\alpha)V = 0. \tag{3.1}$$

Since α contains shock waves, the linear problem (3.1) is not well-posed. We will treat this problem as an initial-boundary value problem in order to properly impose conservation laws.
First we decompose the solution $V(x',t')$ as

$$V(x',t') = \sum_{i=1}^{n} V^i(x',t')\boldsymbol{r}_i(\alpha(x',t')) \text{ at } x' \neq 0.$$

The flux enters the shock at $(0,t')$ is given by

$$\sum_{i=p}^{n}(\lambda_i(\alpha) - s'(t'))V^i\boldsymbol{r}_i(\alpha)\Big|_{x'=0-} - \sum_{j=1}^{p}(\lambda_j(\alpha) - s'(t'))V^j\boldsymbol{r}_j(\alpha)\Big|_{x'=0+}$$

The flux created at the shock is of the form

$$-\sum_{i=1}^{p-1}(\lambda_i(\alpha) - s')V^i\boldsymbol{r}_i(\alpha)\Big|_{x'=0-} + \sum_{j=p+1}^{n}(\lambda_j(\alpha) - s')V^j\boldsymbol{r}_j(\alpha)\Big|_{x'=0+} - S\,\vec{\boldsymbol{s}}(t'),$$

where $\vec{s}(t') \equiv (\alpha(0+,t') - \alpha(0-,t'))$. From the consideration of conservation laws, we impose the boundary condition for V at $x' = 0-$ and $x' = 0+$:

$$(3.2)\quad \sum_{i=1}^{p-1}(\lambda_i(\alpha) - s')V^i \boldsymbol{r}_i(\alpha)\Big|_{x'=0-} - \sum_{j=p+1}^{n}(\lambda_j(\alpha) - s')V^j \boldsymbol{r}_j(\alpha)\Big|_{x'=0+} + S\,\vec{s}(t')$$
$$= -\sum_{i=p}^{n}(\lambda_i(\alpha) - s')V^i \boldsymbol{r}_i(\alpha)\Big|_{x'=0-} + \sum_{j=1}^{p}(\lambda_j(\alpha) - s')V^j \boldsymbol{r}_j(\alpha)\Big|_{x'=0+}.$$

This initial boundary value problem is well-posed.

Remark 3.1. The determination of the flux entering and creating at the shock is simply a consequence of Lax's entropy condition. ∎

Let $e_i \in C^\infty[0, \boldsymbol{t}_0/\kappa]$, $i \neq p$, $\mathsf{I} \in C^\infty(\mathbb{R}/\frac{\mathbb{T}}{\kappa}\mathbb{Z})$, and $\mathfrak{S} \in C^\infty((\mathbb{R}/\frac{\mathbb{T}}{\kappa}\mathbb{Z} \sim \{\bar{0}\}) \times [0, \boldsymbol{t}_0/\kappa])$ be $n+1$ given functions. We consider the following initial-boundary value problem

$$(\mathrm{P})\quad \begin{cases} \partial_{t'} V - s'(t')\,\partial_{x'} V + \partial_{x'} F'(\alpha) V = \mathfrak{S}, \\ \displaystyle\sum_{i=1}^{p-1}[(\lambda_i(\alpha) - s')V^i + e_i]\boldsymbol{r}_i(\alpha)\Big|_{x'=0-} - \sum_{i=p+1}^{n}[(\lambda_i(\alpha) - s')V^i + e_i]\boldsymbol{r}_i(\alpha)\Big|_{x'=0+} \\ +S\,\vec{s}(t') \\ \displaystyle= \sum_{i=p}^{n} -(\lambda_i(\alpha) - s')V^i \boldsymbol{r}_i(\alpha)\Big|_{x'=0-} + \sum_{j=1}^{p}(\lambda_j(\alpha) - s')V^j \boldsymbol{r}_j(\alpha)\Big|_{x'=0+}, \\ V(x', 0) = \mathsf{I}(x'). \end{cases}$$

Denote the solution operator Ξ

$$V \equiv \Xi[\mathsf{I}, \{e_i\}_{i\neq p}, \mathfrak{S}].$$

The function S depends on the values $V(0\pm, t')$ and $\{e_j\}_{j\neq p}$. Due to the dependency on $V(0\pm, t')$, we denote S as follows to relate its dependency on I, $\{e_i\}_{i\neq p}$, and $\mathfrak{S}$,

$$S \equiv \Psi[\mathsf{I}, \{e_i\}_{i\neq p}, \mathfrak{S}].$$

The solution V will satisfy that

$$\frac{d}{dt'}\int_{-\frac{\mathbb{T}}{2\kappa}}^{\frac{\mathbb{T}}{2\kappa}} V(x', t')\,dx' = \int_{-\frac{\mathbb{T}}{2\kappa}}^{\frac{\mathbb{T}}{2\kappa}} \mathfrak{S}(x', t')dx' - \sum_{i<p} e_i \boldsymbol{r}_i(\alpha(0-, t')) - \sum_{i>p} e_i \boldsymbol{r}_i(\alpha(0+, t')) - S(t')\vec{s}(t').$$

In the construction of an approximate solution, there is no particular condition imposed on the initial data. However, we will need to select initial data properly so that there is no singularity due the inconsistency of the initial data and the boundary data.
We denote $\mathfrak{B}$ is such a scheme to choose a suitable initial data consistent with the boundary data and the external source term so that the resulted solution does not contain singularities.

$$\mathsf{I} \equiv \mathfrak{B}[\{e_i\}_{i\neq p}, \mathfrak{S}].$$

4. The Scheme

Denote $\mathsf{A}_0(x',t')$ as follows

$$\mathsf{A}_0(x',t') = ch_-(x')[\alpha(x',t') - \alpha(0-,t')] + ch_+(x')[\alpha(x',t') - \alpha(0+,t')] + U(x',t'),$$

where

$$\begin{cases} ch_\pm \in C^\infty(\mathbb{R}), \\ ch'_- \leq 0, \\ ch'_+ \geq 0, \\ ch_-(x') = 0 \text{ if } x' \geq 0, \\ ch_+(x') = 0 \text{ if } x' \leq 0, \\ ch_-(x') = 1 \text{ if } x' \leq -1, \\ ch_+(x') = 1 \text{ if } x' \geq 1. \end{cases}$$

Denote

$$\begin{aligned} &\sum_{i=1}^{p-1} e^i_{1,co}\boldsymbol{r}_i(\alpha(0-,t')) + \sum_{i=p+1}^{n} e^i_{1,co}\boldsymbol{r}_i(\alpha(0+,t')) + e^p_{1,co}\vec{\boldsymbol{s}} \\ &\qquad \equiv \int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} (\partial_{t'}\mathsf{A}_0 - s'\partial_{x'}\mathsf{A}_0 + \partial_{x'}F(\mathsf{A}_0) - \partial^2_{x'}\mathsf{A}_0)\, dx', \\ &\sum_{i=1}^{p-1} e^i_{1,ext}\boldsymbol{r}_i(\alpha(0-,t')) + \sum_{i=p+1}^{n} e^i_{1,ext}\boldsymbol{r}_i(\alpha(0+,t')) + e^p_{1,ext}\vec{\boldsymbol{s}} \equiv \mathbb{S}_1|^{x'=0+}_{x'=0-}, \\ &e^i_1 \equiv e^i_{1,ext} + e^i_{1,co} \text{ for } i = 1,\cdots,n \end{aligned}$$

The updated A_1 is given as follows

$$\begin{cases} \mathsf{I}_1 \equiv \mathfrak{B}[\{e^i_1\}_{i\neq p}, \partial_{x'}\mathbb{S}_1], \\ \mathsf{J}_1 \equiv \Xi[\mathsf{I}_1, \{e^i_1\}_{i\neq p}, \partial_{x'}\mathbb{S}_1], \\ S_1 \equiv \Psi[\mathsf{I}_1, \{e^i_1\}_{i\neq p}, \partial_{x'}\mathbb{S}_1], \\ \delta_1(t') \equiv \int_0^{t'} e^p_1(r) - S_1(r)dr, \\ \mathsf{A}_1 \equiv \mathsf{A}_0 + U(\xi - \delta_1, t') - U(\xi, t') + \int_{\mathbb{R}} \mathsf{J}_1(y) ch_0(x'-y)dy, \\ \mathsf{R}_1 \equiv \partial_{t'}\mathsf{A}_1 - s'\,\partial_{x'}\mathsf{A}_1 + \partial_{x'}F(\mathsf{A}_1) - \partial^2_{x'}\mathsf{A}_1, \end{cases}$$

where ch_0 satisfies

$$\begin{cases} ch_0 \in C^\infty_c(\mathbb{R}), \\ ch_0(x') = ch_0(-x'), \\ ch_0 \geq 0, \\ \int_{\mathbb{R}} ch_0(x')\, dx' = 1. \end{cases}$$

The updated function A_j for $j \geq 2$ is given by the following

$$\begin{cases} \sum_{i=1}^{p-1} e_j^i r_i(\alpha(0-,t')) + \sum_{i=p+1}^{n} e_j^i r_i(\alpha(0+,t')) + e_j^p \vec{s} \equiv \mathbb{S}_j|_{x'=0-}^{x'=0+}, \\ \mathsf{I}_j \equiv \mathfrak{B}[\{e_j^i\}_{i\neq p}, \partial_{x'}\mathbb{S}_j], \\ \mathsf{J}_j \equiv \Xi[\mathsf{I}_j, \{e_j^i\}_{i\neq p}, \partial_{x'}\mathbb{S}_j], \\ S_j \equiv \Psi[\mathsf{I}_j, \{e_j^i\}_{i\neq p}, \partial_{x'}\mathbb{S}_j], \\ \delta_j \equiv \delta_{j-1} + \int_0^{t'} -S_j(r) + e_j^p dr, \\ \mathsf{A}_j \equiv \mathsf{A}_{j-1} + U(x' - \delta_j, t') - U(x' - \delta_{j-1}, t') + \int_{\mathbb{R}} \mathsf{J}_j(y) ch_0(x' - y) dy, \\ \mathsf{R}_j \equiv \partial_{t'}\mathsf{A}_j - s'\,\partial_{x'}\mathsf{A}_j + \partial_{x'}F(\mathsf{A}_j) - \partial_{x'}^2\mathsf{A}_j. \end{cases}$$

This scheme was made to assure that A_i are smooth all $i \geq 0$ and that for $i \geq 1$

$$\frac{d}{dt'}\int_{-\frac{\mathbb{T}}{2\kappa}}^{\frac{\mathbb{T}}{2\kappa}} \mathsf{A}_i(x',t')\, dx' = 0(mod)e^{-O(1)\epsilon|\mathbb{T}|/\kappa}.$$

This yields

$$\int_{-\frac{\mathbb{T}}{2\kappa}}^{\frac{\mathbb{T}}{2\kappa}} \mathsf{R}_i(x',t')\, dx' = 0(mod)e^{-O(1)\epsilon|\mathbb{T}|/\kappa} \text{ for } i \geq 1.$$

When x' is away from the shock wave, the sequence $\{\mathsf{A}_i\}_{i\geq 0}$ essentially is a partial sum of an asymptotic expansion thus the truncation error R_i is of the order κ^{i+2}. Thus, one can easily conclude that

$$\mathsf{R}_i(x',t') = O(1)\left[\kappa^{i+2} + \kappa\, e^{-O(1)\epsilon|x'|}\right] \text{ for } x' \in [-\frac{\mathbb{T}}{2\kappa}, \frac{\mathbb{T}}{2\kappa}].$$

5. Error Estimates of A_1

The scheme constructs approximate solutions to (1.2). Next, we need assure that there is a solution of (1.2) close to A_1 for $t' \leq O(1)1/\kappa$.

Consider an initial value problem

$$\begin{cases} \partial_{t'}\alpha^\kappa - s'\,\partial_{x'}\alpha^\kappa + \partial_{x'}F(\alpha^\kappa) - \partial_{x'}^2\alpha^\kappa = 0, \\ \alpha^\kappa(x',0) = \mathsf{A}_1(x',0). \end{cases}$$

Take

$$\begin{cases} \mathsf{V} \equiv \alpha^\kappa - \mathsf{A}_1, \\ \sum_{i=1}^{n} \mathsf{V}^i r_i(\mathsf{A}_1) \equiv \mathsf{V}, \\ \mathsf{W}(x',t') \equiv \int_{-\frac{\mathbb{T}}{2\kappa}}^{x'} \mathsf{V}(y,t')\, dy, \\ \sum_{i=1}^{n} \mathsf{W}^i r_i(\mathsf{A}_1) \equiv \mathsf{W}. \end{cases}$$

The systems for V, W, and W^i are

(5.1)
$$\partial_{t'}\mathsf{V} - s'\partial_{x'}\mathsf{V} + \partial_{x'}F'(\mathsf{A}_1)\mathsf{V} + \partial_{x'}\mathsf{N}[\mathsf{V}] - \partial_{x'}^2\mathsf{V} = -\mathsf{R}_1,$$
$$\partial_{t'}\mathsf{W} - s'\partial_{x'}\mathsf{W} + F'(\mathsf{A}_1)\mathsf{W}_{x'} + \mathsf{N}[\mathsf{V}] - F(\mathsf{A}_1) - \partial_{x'}^2\mathsf{W} = -\mathfrak{R}_1,$$

$$\partial_{t'}\mathsf{W}^i + (\lambda_i(\mathsf{A}) - s')\partial_{x'}\mathsf{W}^i - \partial^2_{x'}\mathsf{W}^i = O(1)\,|(\lambda_i(\mathsf{A}_1) - s')|\cdot|\mathsf{W}|\cdot(|\partial_{x'}\boldsymbol{r}_1| + |\partial_{x'}\boldsymbol{r}_2|)$$
$$+ O(1)|\mathsf{W}|\cdot|\partial^2_{x'}\mathsf{A}_1| + |\partial_{x'}\mathsf{W}|\cdot|\partial_{x'}\mathsf{A}_1| + O(1)|\mathsf{N}[\mathsf{V}]| + |\partial_{t'}\mathsf{A}_1| + |\mathfrak{R}|, \tag{5.2}$$

where

$$\mathsf{N}[\mathsf{V}] \equiv F(\mathsf{A}_1 + \mathsf{V}) - F(\mathsf{A}_1) - F'(\mathsf{A}_1)\mathsf{V} = O(1)|\mathsf{V}|^2,$$
$$\mathsf{R}_1 = O(1)\kappa\left[\kappa^2 + e^{-O(1)\epsilon|x'|}\right] \text{ for } |x'| \le \frac{\mathbb{T}}{2\kappa},$$
$$\mathfrak{R}_1 = O(1)\kappa\left[\kappa + \frac{1}{\epsilon}e^{-O(1)\epsilon|x'|}\right] \text{ for } |x'| \le \frac{\mathbb{T}}{2\kappa}.$$

We consider $\sum_{i=1}^n \int_0^\tau \int_{-\frac{\mathbb{T}}{2\kappa}}^{\frac{\mathbb{T}}{2\kappa}} \mathsf{W}^i\cdot(5.2)dx'dt'$ to yield that

$$\frac{1}{2}\int_{-\frac{\mathbb{T}}{2\kappa}}^{\frac{\mathbb{T}}{2\kappa}} |\mathsf{W}|^2\,dx'\Bigg|_{t'=0}^{t'=\tau} + \int_0^\tau \int_{-\frac{\mathbb{T}}{2\kappa}}^{\frac{\mathbb{T}}{2\kappa}} \sum_{i=1}^n \left(-\frac{1}{2}(\partial_{x'}\lambda_i(\mathsf{A}_1))|\mathsf{W}^i|^2 + |\mathsf{W}^i_{x'}|^2\right) dx'dt'$$
$$\le O(1)\int_0^\tau \int_{-\frac{\mathbb{T}}{2\kappa}}^{\frac{\mathbb{T}}{2\kappa}} \left|\sum_{i=1}^n (\lambda_i(\mathsf{A}_i) - s')\mathsf{W}^i|\mathsf{W}||\partial_{x'}\mathsf{A}_1|\right| dx'dt'$$
$$+ O(1)\int_0^\tau \int_{-\frac{\mathbb{T}}{2\kappa}}^{\frac{\mathbb{T}}{2\kappa}} |\mathsf{W}|^2(|\partial^2_{x'}\mathsf{A}_1| + \partial_{t'}\mathsf{A}_1|) + |\mathsf{W}||\mathsf{V}|^2 + |\mathsf{W}||\mathfrak{R}_1|dx'dt'. \tag{5.3}$$

Since the shock is a p-shock wave, from the monotonicity of $\lambda_p(U)$ we have

$$|\partial_{x'}\lambda_p(\mathsf{A}_1)| \le -\partial_{x'}\lambda_p(\mathsf{A}_1) + O(1)\kappa. \tag{5.4}$$

From the construction of the approximate solution A_1 we have that

$$\begin{cases} \|\partial_{t'}\mathsf{A}_1\|_\infty = O(1)\ \kappa \\ |\partial^2_{x'}\mathsf{A}_1| \le O(1)\ \epsilon\ |\partial_{x'}\lambda_p(\mathsf{A}_1)| + O(1)\ \kappa^2 \end{cases} \tag{5.5}$$

for any $\tau \le O(1)t_0/\kappa$, where t_0 is a given small number which is independent of κ.

Now we impose a smallness assumption for $\tau \le t_0/\kappa$

$$\sup_{\substack{x'\in\mathbb{R}\\ 0\le i\le n}} |\partial^i_{x'}\mathsf{W}|(x',\tau) \le O(1)\ \epsilon. \tag{5.6}$$

From (5.4), (5.5), and (5.6), the estimate (5.3) becomes

$$\frac{1}{2}\int_{-\frac{\mathbb{T}}{2\kappa}}^{\frac{\mathbb{T}}{2\kappa}} |\mathsf{W}|^2\,dx'\Bigg|_{t'=0}^{t'=\tau} + \int_0^\tau \int_{-\frac{\mathbb{T}}{2\kappa}}^{\frac{\mathbb{T}}{2\kappa}} \frac{1}{4}|\partial_{x'}\lambda_p(\mathsf{A}_1)||\mathsf{W}^p|^2 + \sum_{i=1}^n |\mathsf{W}^i_{x'}|^2\,dx'dt'$$
$$\le O(1)\int_0^\tau \int_{-\frac{\mathbb{T}}{2\kappa}}^{\frac{\mathbb{T}}{2\kappa}} \sum_{j\ne p} |\partial_{x'}\mathsf{A}_1||\mathsf{W}^j|^2 + \epsilon|\mathsf{V}|^2 + \kappa|\mathsf{W}|^2 + |\mathsf{W}||\mathfrak{R}_1|dx'dt'. \tag{5.7}$$

Transversal Wave Estimates

When $j \ne p$,

$$\int_0^\tau \int_{-\frac{\mathbb{T}}{2\kappa}}^{\frac{\mathbb{T}}{2\kappa}} -(\lambda_p(\mathsf{A}_1) - s')_{x'}|\mathsf{W}^j|^2dx'dt' \tag{5.8}$$

$$= \int_0^\tau \int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} \frac{2(\lambda_p(\mathsf{A}_1)-s')}{(\lambda_j(\mathsf{A}_1)-s')}(\lambda_1(\mathsf{A}_1)-s')\mathsf{W}^j\partial_{x'}\mathsf{W}^j dx'dt'$$

$$= \int_0^\tau \int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} \frac{2(\lambda_p(\mathsf{A}_1)-s')}{(\lambda_j(\mathsf{A}_1)-s')}\mathsf{W}^j\left[(-\partial_{t'}+\partial_{x'}^2)\mathsf{W}^j + O(1)\left(|\partial_{x'}^2\mathsf{A}_1||\mathsf{W}|+|\partial_{x'}\mathsf{A}_1||\mathsf{V}|+|\mathsf{V}|^2+|\mathfrak{R}_1|\right)\right] dx'dt'$$

$$= O(1)\,\epsilon \int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} |\mathsf{W}^j|^2 dx'\Big|_{\tau=t'}$$

$$+O(1)\int_0^\tau \int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} |\partial_{x'}^2\mathsf{A}_1||\mathsf{W}^j|^2 + |\partial_{x'}\mathsf{A}_1||\mathsf{W}^j||\mathsf{W}^j_{x'}|dx'dt'$$

$$+O(1)\epsilon\int_0^\tau \int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} \mathsf{W}^j\left[|\partial_{x'}^2\mathsf{A}_1||\mathsf{W}|+|\partial_{x'}\mathsf{A}_1||\mathsf{V}|+|\mathsf{V}|^2+|\mathfrak{R}_1|\right] dx'dt'.$$

Note that we have used the condition $\|\lambda_p(\mathsf{A}_1)-s'\|_\infty = O(1)\ \epsilon$ in the above transervsal wave estimates.

Lemma 5.1. *For any given functions h_1 and h_2, it follows*

$$\int_0^\tau \int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} h_1h_2\,dx'dt' \le \frac{5}{6}\int_0^\tau \int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} h_1(x',t')^{\frac{6}{5}}dx'dt' + \frac{4}{6}\sup_{\bar\tau\in[0,\tau]}\|h_2(\cdot,\bar\tau)\|^4_{L^2(-\frac{T}{2\kappa},\frac{T}{2\kappa}])}\int_0^\tau \int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} |\partial_{x'}h_2(x',t')|^2dx'dt'. \tag{5.9}$$

Combine (5.7), (5.8), and Lemma 5.1 together with the smallness assumption, then there exist $C \gg 1$

$$\frac{1}{4}\int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} |\mathsf{W}|^2\,dx'\Big|_{t'=\tau} + \int_0^\tau \int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} \frac{1}{4}|\partial_{x'}\lambda_p(\mathsf{A}_1)|(|\mathsf{W}^p|^2 + C\sum_{j\neq p}|\mathsf{W}^j|^2) + \frac{1}{2}\sum_{i=1}^n|\mathsf{W}^i_{x'}|^2\,dx'dt' \le O(1)\int_0^\tau \int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} \epsilon|\mathsf{V}|^2 + |\mathfrak{R}_1|^{6/5}dx'dt' \tag{5.10}$$

when $\tau\kappa \ll 1$.

Consider $\int_0^\tau\int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}}(\partial_{x'}^5\mathsf{V})(\partial_{x'}^5(5.1))dx'dt'$. It results in

$$\frac{1}{2}\int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} |\partial_{x'}^5\mathsf{V}|^2dx'\Big|_{t'=\tau} + \int_0^\tau \int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} |\partial_{x'}^6\mathsf{V}|^2\,dx'dt' \le O(1)\epsilon\int_0^\tau \int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} \sum_{j=0}^5|\partial_{x'}^j\mathsf{V}|^2\,dx'dt' + O(1)\int_0^\tau \int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} \frac{|\mathsf{R}_1|^2}{\epsilon}\,dx'dt'. \tag{5.11}$$

From (5.10), (5.11), and Sobolev's interpolation theorem it follows

$$\frac{1}{8}\int_0^\tau \int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} |\mathsf{W}|^2 + |\partial_{x'}^5\mathsf{V}|^2dx'\Big|_{t'=\tau} + \frac{1}{4}\int_0^\tau \int_{-\frac{T}{2\kappa}}^{\frac{T}{2\kappa}} |\partial_{x'}\lambda_p(\mathsf{A}_1)|(|\mathsf{W}^2|^2 + C\sum_{j\neq p}|\mathsf{W}^j|^2) + |\mathsf{V}|^2 + |\partial_{x'}^6\mathsf{V}|^2dx'dt' \tag{5.12}$$

$$\leq O(1)\left(\frac{\tau\kappa^{\frac{6}{5}}}{\epsilon^{\frac{11}{5}}}+\frac{\tau\kappa^2}{\epsilon^2}\right).$$

This justifies the smallness assumption (5.6) when $\tau\kappa$ is sufficiently small. Thus, by Sobolev's embedding theorem, one has that for $t' \leq \tau_0\kappa^{-1}$ with $\tau_0 \ll 1$

$$\|\mathsf{V}(\cdot,t')\|_\infty \leq O(1)\tau_0\left(\frac{\kappa^{1/5}}{\epsilon^{\frac{11}{5}}}+\frac{\kappa}{\epsilon^2}\right).$$

Thus,

$$w\text{-}\lim_{\kappa\to 0+}\alpha^\kappa = w\text{-}\lim_{\kappa\to 0+}(\mathsf{A}_1+\mathsf{V}) = w\text{-}\lim_{\kappa\to 0+}\mathsf{A}_1 + w\text{-}\lim_{\kappa\to 0+}\mathsf{V} = \alpha+0=\alpha.$$

The main theorem follows.

Here, we can have a stronger version of convergence estimate as follows for $x \in [-\mathbb{T}/2, \mathbb{T}/2]$ and for $t \in (0,\tau_0)$

$$|\alpha^\kappa - \alpha|(x,t) \leq O(1)[\tau_0\kappa^{1/5}/\epsilon^{11/5} + e^{-O(1)\epsilon|x-s(t)|/\kappa}].$$

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