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Asymptotic decay toward the rarefaction waves of solutions for viscous conservation laws in one-dimensional half space

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1 Introduction

We consider the initial-boundary value problem for scalar viscous conservation laws in one-dimensional half space $\mathbb{R}_+ := (0, \infty)$:

$$\begin{cases} u_t + f(u)_x = u_{xx}, \quad x \in \mathbb{R}_+, \ t > 0, \\ u(0,t) = u_-, \quad t > 0, \\ u(x,0) = u_0(x) = \begin{cases} = u_-, & x = 0, \\ \to u_+, & x \to \infty, \end{cases} \end{cases} \tag{1}$$

where f is a smooth function and $u_\pm$ are constants. We consider this problem under the following assumptions:

$$f'' \geq {}^\exists \alpha > 0, \ \ u_* \leq u_- < u_+ \ \ (f'(u_*) = 0). \tag{2}$$

Under these conditions, it was already shown in [5] that the solutions of (1) converge to the corresponding rarefaction waves as $t \to \infty$. The main purpose of the present research is to obtain the convergence rate.

The rarefaction wave $r(x,t)$ is given as a weak solution of the Riemann problem for the corresponding hyperbolic conservation laws on the whole space:

$$\begin{cases} r_t + f(r)_x = 0, \quad x \in \mathbb{R}, \ t > -1, \\ r(x,-1) = r_0^{\mathrm{R}}(x) := \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases} \end{cases} \tag{3}$$

Here note that $r(x,t)$ is a continuous function for $t \geq 0$. $r(x,t)$ is expressed explicitly for $t > -1$ by

$$r(x,t) = \begin{cases} u_-, & x \leq f'(u_-)(t+1), \\ (f')^{-1}(\dfrac{x}{t+1}), & f'(u_-)(t+1) \leq x \leq f'(u_+)(t+1), \\ u_+, & f'(u_+)(t+1) \leq x. \end{cases}$$

For the half space problem (1), it is shown by Liu, Matsumura and Nishihara [5] that the asymptotic states of the solutions of (1) are classified into the following three cases according to the signatures of $f'(u_\pm)$: (a) $f'(u_-) < f'(u_+) \le 0$, (b) $f'(u_-) < 0 < f'(u_+)$ and (c) $0 \le f'(u_-) < f'(u_+)$. In the case (a), the solutions of (1) converge to stationary solutions. In the case (b), the asymptotic states are superpositions of stationary solutions and rarefaction waves. And the case (c) yields rarefaction waves.

The main purpose of the present paper is to obtain the convergence rate for the case (c). Here note that the convergence rate for the case (a) is also considered in [5] and that the case (b) should be considered. The main theorem of the present paper is stated as follows.

Theorem 1.1 *Suppose that (2) hold. Let $u_0 - u_+ \in (H^1 \cap L^1)(\mathbb{R}_+)$ and $u_0(0) = u_-$. Then the initial-boundary value problem (1) has a unique global solution $u(x,t)$. Moreover, $u(x,t)$ satisfies the following estimates:*

$$\|u(t) - r(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}} \log(2+t),$$
$$\|u(t) - r(t)\|_{L^\infty} \le C(1+t)^{-\frac{1}{2}} \log^3(2+t),$$

where C is a positive constant depending only on u_0.

2 Smooth approximation and reformulation of the problem

First, we derive the smooth approximation of the rarefaction wave $r(x,t)$ by employing the idea of Hattori and Nishihara [1]. We define $\tilde{w}(x,t)$ as a solution of the following Cauchy problem:

$$\begin{cases} \tilde{w}_t + \tilde{w}\tilde{w}_x = \tilde{w}_{xx}, & x \in \mathbb{R},\ t > -1, \\ \tilde{w}(x,-1) = w_0^{\mathrm{R}}(x), & x \in \mathbb{R}, \end{cases} \tag{4}$$

where the initial data $w_0^{\mathrm{R}}(x)$ is defined by

$$w_0^{\mathrm{R}}(x) := \begin{cases} f'(u_-), & x < 0, \\ f'(u_+), & x > 0, \end{cases} \quad (\text{if } f'(u_-) > 0),$$
$$w_0^{\mathrm{R}}(x) := \begin{cases} -f'(u_+), & x < 0, \\ f'(u_+), & x > 0, \end{cases} \quad (\text{if } f'(u_-) = 0).$$

Because (4) is the Burgers equation, we can get the explicit formula of $\tilde{w}(x,t)$ by using the Hopf-Cole transformation. Successively, by using this $\tilde{w}(w,t)$, we define a smooth approximation $w(x,t)$ of the rarefaction wave $r(x,t)$ as

$$w(x,t) := (f')^{-1}(\tilde{w}(x,t)). \tag{5}$$

Substituting (5) to (4), we have the equation of $w(x,t)$:

$$\begin{cases} w_t + f(w)_x = w_{xx} + \dfrac{f'''(w)}{f''(w)} w_x^2, & x \in \mathbb{R},\ t > 0, \\ w(x,0) = w_0(x) := (f')^{-1}(\tilde{w}(x,0)), & x \in \mathbb{R}. \end{cases} \tag{6}$$

Here we summarize the well-known results for the smooth approximation $w(x,t)$ in Lemma 2.1. This lemma is proved by the direct computations of the explicit formula of $\tilde{w}(x,t)$. For details, readers refer to [1, 4].

Lemma 2.1 *For $1 \le p \le \infty$ and $t \ge 0$, $w(x,t)$ satisfies the followings.*

(i) $0 \le w(0,t) - u_- \le Ce^{-c(1+t)}$ *for* $f'(u_-) > 0$ *and* $w(0,t) = u_-$ *for* $f'(u_-) = 0$.

(ii) $|w_x(0,t)| \le Ce^{-c(1+t)}$, $|w_{xx}(0,t)| \le Ce^{-c(1+t)}$.

(iii) $\|w(t) - r(t)\|_{L^p} \le C(1+t)^{-\frac{1}{2}+\frac{1}{2p}}$.

(iv) $\|w_x(t)\|_{L^p} \le C(1+t)^{-1+\frac{1}{p}}$, $\|w_{xx}(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}+\frac{1}{2p}}$.

(v) $w_x(x,t) > 0$ *for* $x \in \mathbb{R}$.

We can see that $w(x,t)$ does not satisfy the boundary condition in (1). So we need to modify $w(x,t)$ around the boundary. Our modified smooth approximation $W(x,t)$ is defined as

$$W(x,t) := w(x,t) - \psi(x,t), \tag{7}$$

where

$$\psi(x,t) := (w(0,t) - u_-)e^{-x}. \tag{8}$$

By virtue of this modification, $W(x,t)$ satisfies the boundary condition $W(0,t) = u_-$. By using this $W(x,t)$, we define the perturbation $v(x,t)$ from the modified smooth approximation $W(x,t)$ as

$$v(x,t) := u(x,t) - W(x,t).$$

From (1), (6) and (7), we have the equation of $v(x,t)$:

$$\begin{cases} v_t + (f(W+v) - f(W))_x = v_{xx} + R(x,t), & x \in \mathbb{R}_+,\ t > 0, \\ v(0,t) = 0, & t > 0, \\ v(x,0) = v_0(x) := u_0(x) - W_0(x), & x \in \mathbb{R}_+, \end{cases} \tag{9}$$

where $R(x,t)$ is defined as

$$R(x,t) := -\frac{f'''(w)}{f''(w)} w_x^2 + \psi_t + (f(W+\psi) - f(W))_x - \psi_{xx}.$$

From Lemma 2.1, we can see that $R(x,t)$ satisfies

$$\|R(t)\|_{L^p} \le C(1+t)^{-2+\frac{1}{p}}.$$

Making use of a standard iteration method, it is shown that the equation (9) has a unique solution locally in time in the space

$$X_M(0,T) = \left\{ v \in C^0([0,T]; H^1(\mathbb{R}_+)) \;\middle|\; v_x \in L^2(0,T; H^1(\mathbb{R}_+)) \text{ and } \sup_{0\le t\le T} \|v(t)\|_{H^1} \le M \right\}$$

for positive constants T and M.

Proposition 2.2 (Local existence) *Suppose that* $v_0 \in H^1(\mathbb{R}_+)$ *and* $v_0(0) = 0$. *For any* $M > 0$ *with* $\|v_0\|_{H^1} \le M$, *there exists a positive time* T *depending on* M *such that the equation (9) has a unique solution* $v \in X_{2M}(0,T)$.

3 Apriori estimate and decay estimate

In this section, we show the a priori estimate and decay estimate of $v(x,t)$. By virtue of the modification in Section 2, the outline of the proof of these estimates is similar to that of the full space problem. Therefore, in this paper, we omit the proof of the following propositions and theorems. For details, readers refer to [2, 3, 5, 6].

Proposition 3.1 (A priori estimate) *Suppose that* $v \in X_M(0,T)$ *is a solution of (9) for some positive constants T and M. Then there exists a positive constants C independent of T, such that* $v(x,t)$ *satisfies the estimate*

$$\|v(t)\|_{H^1}^2 + \int_0^t \|\sqrt{W_x(\tau)}v(\tau)\|_{L^2}^2 + \|v_x(\tau)\|_{H^1}^2 \, d\tau \le C(\|v_0\|_{H^1}^2 + 1). \tag{10}$$

The combination of Proposition 2.2 and Proposition 3.1 proves the global existence theorem.

Theorem 3.2 (Global existence) *Suppose that* $v_0 \in H^1(\mathbb{R}_+)$ *and* $v_0(0) = 0$. *Then there exists a unique global solution* $v(x,t)$ *of (9) satisfying*

$$v \in C^0([0,\infty); H^1(\mathbb{R}_+)), \quad v_x \in L^2(0,\infty; H^1(\mathbb{R}_+))$$

and the estimate (10).

In order to derive the decay estimate, we employ the L^1-estimate of v.

Proposition 3.3 (L^1-estimate) *Suppose that* $v_0 \in (H^1 \cap L^1)(\mathbb{R}_+)$. *Then the solution* $v(x,t)$ *of (9) satisfies the estimate*

$$\|v(t)\|_{L^1} \leq \|v_0\|_{L^1} + C\log(1+t). \tag{11}$$

Finally, we obtain the decay estimate of v. The following theorem is proved by using L^1-estimate and H^1-estimate.

Theorem 3.4 (Decay estimate) *Suppose that* $v_0 \in (H^1 \cap L^1)(\mathbb{R}_+)$. *Then the solution* $v(x,t)$ *of (9) satisfies*

$$(1+t)^{\frac{1}{2}+\varepsilon}\|v(t)\|_{L^2}^2 + \int_0^t (1+\tau)^{\frac{1}{2}+\varepsilon}\left\{\|\sqrt{W_x}v(\tau)\|_{L^2}^2 + \|v_x(\tau)\|_{L^2}^2\right\}d\tau \leq C(1+t)^{\varepsilon}\log^2(2+t), \tag{12}$$

$$(1+t)^{\frac{3}{2}+\varepsilon}\|v_x(t)\|_{L^2}^2 + \int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon}\left\{\|\sqrt{W_x}v_x(\tau)\|_{L^2}^2 + \|v_{xx}(\tau)\|_{L^2}^2 + f'(u_-)v_x(0,\tau)^2\right\}d\tau \leq C(1+t)^{\varepsilon}\log^{10}(2+t) \tag{13}$$

for arbitrary constant $\varepsilon \in (0, \frac{1}{2})$.

The combination of Lemma 2.1 and Theorem 3.4 immediately proves Theorem 1.1.

References

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