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Modified Elastic Wave Equations on Riemannian and Kähler Manifolds

東京大学大学院数理科学研究科 安富 義泰 (Yoshiyasu YASUTOMI)
Graduate School of Mathematical Sciences,
The University of Tokyo
3-8-1 Komaba, Meguro, Tokyo, 153-8914 JAPAN.

We introduce some geometrically invariant systems of differential equations on any Riemannian manifolds and also on any Kähler manifolds, which are natural extensions of the elastic wave equations on $\mathbb{R}^3$. Further we prove the local decomposition theorems of distribution solutions for those systems. In particular, the solutions of our systems on Kähler manifolds are decomposed into 4 solutions with different propagation speeds.

Definition 1. Let $\bigwedge^{(p)} T^*M$ be a vector bundle of p-differential forms on M. Let $\mathcal{E}_M^{(p)}$ be a sheaf of p-forms on M with C^∞ coefficients, and $\mathcal{D}b_M^{(p)}$ a sheaf of p-currents on M; that is, p-forms with distribution coefficients. In this article, we do not mean distributions the dual space of $C_0^\infty(M)$. Our distributions behave as "functions" for coordinate transformations.

Definition 2. We put $\widetilde{M} := \mathbb{R}_t \times M$. We denote by $\widetilde{\mathcal{E}}_M^{(p)}$, $\widetilde{\mathcal{D}b}_M^{(p)}$ the sheaves of sections of $\mathcal{E}_{\widetilde{M}}^{(p)}$, $\mathcal{D}b_{\widetilde{M}}^{(p)}$ which do not include the covariant vector dt . That is, setting the projection $\pi : \widetilde{M} \to M$, we define

$$\widetilde{\mathcal{E}}_M^{(p)} := \mathcal{E}_{\widetilde{M}}^{(0)} \underset{\pi^{-1}\mathcal{E}_M^{(0)}}{\otimes} \pi^{-1}\mathcal{E}_M^{(p)} , \qquad \widetilde{\mathcal{D}b}_M^{(p)} := \mathcal{D}b_{\widetilde{M}}^{(0)} \underset{\pi^{-1}\mathcal{E}_M^{(0)}}{\otimes} \pi^{-1}\mathcal{E}_M^{(p)} .$$

Definition 3. The inner products $\langle \cdot, \cdot \rangle : \bigwedge^{(1)} T_x^* M \times \bigwedge^{(1)} T_x M \to \mathbb{R}$, $\langle \cdot, \cdot \rangle^* : \bigwedge^{(p)} T_x^* M \times \bigwedge^{(p)} T_x^* M \to \mathbb{R}$, are defined as follows. We choose a positive orthonormal system $(\omega^1, \cdots, \omega^n)$ of C^∞ sections of T^*M concerning the Riemannian metric; that is, there is a positive number α such that $\omega^1 \wedge \cdots \wedge \omega^n = \alpha \Omega_x > 0$. Then for

$$\sigma = \sum_{1 \le i \le n} \sigma_i dx^i, \qquad \tau = \sum_{1 \le i \le n} \tau^i \partial_i \ ,$$

we define

$$\langle \sigma, \tau \rangle := \sum_{1 \le i \le n} \sigma_i \tau^i,$$

and for

$$\phi = \sum_{1 \le i_1 < \cdots < i_p \le n} \phi_{i_1 \cdots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p},$$
$$\psi = \sum_{1 \le i_1 < \cdots < i_p \le n} \psi_{i_1 \cdots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p},$$

we define

$$\langle \phi, \psi \rangle^* := \sum_{1 \le i_1 < \cdots < i_p \le n} \phi_{i_1 \cdots i_p} \psi^{i_1 \cdots i_p}$$
$$:= \sum_{\substack{1 \le i_1 < \cdots < i_p \le n \\ 1 \le j_1 < \cdots < j_p \le n}} \phi_{i_1 \cdots i_p} g^{i_1 j_1} \cdots g^{i_p j_p} \psi_{j_1 \cdots j_p}.$$

Definition 4. We denote by $d : \mathcal{D}b_M^{(p)} \to \mathcal{D}b_M^{(p+1)}$ the exterior differential operator which acts on $\mathcal{D}b_M^{(p)}$ as a sheaf morphism. Then the following formulas are well-known:

$$\begin{cases} 1.\ d(\phi \pm \psi) = d\phi \pm d\psi & (\phi, \psi \in \mathcal{D}b_M^{(p)}), \\ 2.\ d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^p \phi \wedge d\psi & (\phi \in \mathcal{D}_M^{(p)}, \psi \in \mathcal{D}b_M^{(q)}), \\ 3.\ d(d\phi) = 0 & (\phi \in \mathcal{D}b_M^{(p)}), \\ 4.\ \text{For } f \in \mathcal{D}b_M^{(0)},\ df := \sum \dfrac{\partial f}{\partial x_j} dx^j \in \mathcal{D}b_M^{(1)}. & \end{cases}$$

Here $0 \le p \le n$. If $p = n$, $d\phi = 0$ holds.

Definition 5. The isomorphism $* : \bigwedge T^*M \to \bigwedge T^*M$ of vector bundle is defined as follows:

$$\begin{cases} 1.\ * : \bigwedge^{(p)} T_x^*M \mapsto \bigwedge^{(n-p)} T_x^*M \quad \text{is a linear map,} \\ 2.\ * (\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}) = (-1)^{(i_1-1)+\cdots+(i_p-p)} \omega^{j_1} \wedge \cdots \wedge \omega^{j_{n-p}}, \\ \qquad \text{for any permutation } (i_1, \cdots, i_p, j_1, \cdots, j_{n-p}) \text{ of } (1, \cdots, n). \end{cases}$$

Here $(i_1 \cdots i_p)$ and $(j_1 \cdots j_{n-p})$ are indices satisfying

$$\begin{cases} 1.\ (i_1 \cdots i_p j_1 \cdots j_{n-p}) \text{ is a permitation of } (1 \cdots n), \\ 2.\ 1 \le i_1 < \cdots < i_p \le n,\ 1 \le j_1 < \cdots < j_{n-p} \le n. \end{cases}$$

Remark 6. The definition above does not depend on the choice of the positive orthonormal system $\{\omega^1, \cdots, \omega^n\}$.

Proposition 7. *We set* $\phi, \psi \in \bigwedge^{(p)} T_x^*M$. *Then we obtain*

$$\begin{cases} 1.\ \phi \wedge *\psi = (*\phi) \wedge \psi = \langle \phi, \psi \rangle^* \, \omega^1 \wedge \cdots \wedge \omega^n, \\ 2.\ * 1 = \omega^1 \wedge \cdots \wedge \omega^n = \sqrt{g}\, dx^1 \wedge \cdots \wedge dx^n, \\ 3.\ * \phi = (-1)^{(i_1-1)+\cdots+(i_p-p)} \sqrt{g}\, g^{i_1 j_1} \cdots g^{i_p j_p} \phi_{i_1 \cdots i_p}\, dx^{j_1} \wedge \cdots \wedge dx^{j_{n-p}} \\ \qquad \in \bigwedge^{(n-p)} T_x^*M. \end{cases}$$

Here $g = \det(g_{\lambda\kappa})$.

Let $U \subset M$ be an open subset. Let $\alpha^{(p)} \in \mathcal{D}b_M^{(p)}(U)$, $\beta^{(p)} \in \mathcal{E}_M^{(p)}(U)$ be sections. We suppose that $\beta^{(p)}$ has a compact support in U. Then the following integral is well-defined.

$$(\alpha^{(p)}, \beta^{(p)}) := \int_M \langle \alpha^{(p)}, \beta^{(p)} \rangle^* \, \omega^1 \wedge \cdots \wedge \omega^n.$$

Definition 8. Let $\alpha^{(p)} \in \mathcal{D}b_M^{(p)}$, $\beta^{(p-1)} \in \mathcal{E}_M^{(p-1)}$ be sections. We suppose $\beta^{(p-1)}$ has a compact support. Then the sheaf morphism $\delta : \mathcal{D}b_M^{(p)} \to \mathcal{D}b_M^{(p-1)}$ is defined as

$$(\delta\alpha^{(p)}, \beta^{(p-1)}) = (\alpha^{(p)}, d\beta^{(p-1)}).$$

Hence we have

$$\delta = (-1)^{n(p-1)+1} * d * .$$

Definition 9. Let $\mathfrak{X}_s^r$ be the sheaf of $\bigotimes^r T_x M \otimes \bigotimes^s T_x^* M$-valued C^∞ functions, and $\mathcal{D}b_s^r$ the sheaf of $\bigotimes^r T_x M \otimes \bigotimes^s T_x^* M$-valued distributions. Then, the sheaf morphisms $\nabla : \mathfrak{X}_s^r \to \mathfrak{X}_{s+1}^r$, $\mathcal{D}b_s^r \to \mathcal{D}b_{s+1}^r$ are defined as follows:

$$\begin{cases} 1.\ \text{For } a(x) \in \mathfrak{X}_0^0, & \text{we have} \quad \nabla a(x) = \dfrac{\partial a}{\partial x^j} dx^j. \\ 2.\ \text{For } \dfrac{\partial}{\partial x^j} \in \mathfrak{X}_0^1, & \text{we have} \quad \nabla \left(\dfrac{\partial}{\partial x^j}\right) = \Gamma_j{}^i{}_k \dfrac{\partial}{\partial x^i} \otimes dx^k. \\ 3.\ \text{For } dx^j \in \mathfrak{X}_1^0, & \text{we have} \quad \nabla \left(dx^j\right) = -\Gamma_i{}^j{}_k dx^i \otimes dx^k. \\ 4.\ \text{For } e \in \mathfrak{X}_s^r, f \in \mathfrak{X}_{s'}^{r'}, & \text{we have} \quad \nabla(e \otimes f) = (\nabla e) \otimes f + e \otimes \nabla f. \end{cases}$$

Here,

$$\left\{ \Gamma_i^j{}_k = g^{jl}\Gamma_{ilk} = g^{jl} \cdot \frac{1}{2} \left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^l} \right) \right\}$$

are the Riemann-Christoffel symbols.

Proposition 10. *We set*

$$e = e^r_{i_1 \cdots i_s} dx^{i_1} \otimes \cdots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}} \in \mathfrak{X}_s^r.$$

Then we have

$$\nabla e = \left(\partial_k e^r_{i_1 \cdots i_s} + e^q_{i_1 \cdots i_s} \Gamma_q{}^r{}_k + e^r_{i_1 \cdots i_{p-1} q i_{p+1} \cdots i_s} \Gamma_{i_p}{}^q{}_k \right) \\ \times dx^k \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}}.$$

Hence we call the following *the covariant differentiation* :

$$\nabla_k e = \left(\partial_k e^r_{i_1\cdots i_s} + e^q_{i_1\cdots i_s}\Gamma_q{}^r{}_k + e^r_{i_1\cdots i_{p-1}qi_{p+1}\cdots i_s}\Gamma_{i_p}{}^q{}_k\right)$$
$$\times\, dx^{i_1}\otimes\cdots\otimes dx^{i_s}\otimes\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_r}}.$$

For

$$u = \sum_{1\le i_1<\cdots<i_p\le n} u_{i_1\cdots i_p} dx^{i_1}\wedge\cdots\wedge dx^{i_p} \in \widetilde{\mathcal{D}b}^{(p)}_M,$$

we define an operator P_{R} for $\widetilde{\mathcal{D}b}^{(p)}_M$ on M $(1\le p\le n-1)$, where the coefficients $\{u_{i_1\cdots i_p}\}$ are supposed to be alternating with respect to $(i_1\cdots i_p)$.

Definition 11. We define sheaf-morphisms $P_{\mathrm{R}} : \widetilde{\mathcal{D}b}^{(p)}_M \longrightarrow \widetilde{\mathcal{D}b}^{(p)}_M$ by

$$P_{\mathrm{R}}\, u := \rho\frac{\partial^2}{\partial t^2}u + (\lambda+2\mu)d\delta u + \mu\delta d u,$$

where the density constant ρ and the Làme constants λ, μ are positive.

For $p=1$, this equation is the covariant form of $P_{\mathrm{R}}\, u^i$.

When $p=0$ or n, $P_{\mathrm{R}}\, u = 0$ reduces to a wave equation. Therefore we suppose $1\le p\le n-1$.

For $u\in\widetilde{\mathcal{D}b}^{(p)}_M$, we define equations $\mathfrak{M}^{\mathrm{R}}$, $\mathfrak{M}^{\mathrm{R}}_1$, $\mathfrak{M}^{\mathrm{R}}_2$, $\mathfrak{M}^{\mathrm{R}}_0$ as follows:

$$\mathfrak{M}^{\mathrm{R}} :\quad P_{\mathrm{R}}\, u = 0,$$

$$\mathfrak{M}^{\mathrm{R}}_1 : \begin{cases} P_{\mathrm{R}}\, u = 0, \\ du = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \alpha\Delta)u = 0, \\ du = 0, \end{cases}$$

$$\mathfrak{M}^{\mathrm{R}}_2 : \begin{cases} P_{\mathrm{R}}\, u = 0, \\ \delta u = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \beta\Delta)u = 0, \\ \delta u = 0, \end{cases}$$

$$\mathfrak{M}^{\mathrm{R}}_0 : \begin{cases} P_{\mathrm{R}}\, u = 0, \\ du = 0, \\ \delta u = 0, \end{cases} \iff \begin{cases} \partial_t^2 u = 0, \\ du = 0, \\ \delta u = 0. \end{cases}$$

Here, $\alpha = (\lambda + 2\mu)/\rho$, $\beta = \mu/\rho$ and $\Delta = d\delta + \delta d : \widetilde{\mathcal{D}b}_M^{(p)} \to \widetilde{\mathcal{D}b}_M^{(p)}$ is the Laplacian on M.

Further we define subsheaves $\mathcal{S}ol(\mathfrak{M}^{\mathrm{R}}; p)$, $\mathcal{S}ol(\mathfrak{M}_j^{\mathrm{R}}; p)$, $(j = 0, 1, 2)$ of $\widetilde{\mathcal{D}b}_M^{(p)}$ as follows: For $\mathfrak{N}^{\mathrm{R}} = \mathfrak{M}^{\mathrm{R}}, \mathfrak{M}_j^{\mathrm{R}}$,

$$\mathcal{S}ol(\mathfrak{N}^{\mathrm{R}}; p) := \left\{ u \in \widetilde{\mathcal{D}b}_M^{(p)} \;\middle|\; u \text{ satisfies } \mathfrak{N}^{\mathrm{R}} \right\}.$$

Then, we have the following theorem.

Theorem 12. *For any germ $u \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{R}}; p)\Big|_{(\mathring{t},\mathring{x})}$, there exist some germs $u_j \in \mathcal{S}ol(\mathfrak{M}_j^{\mathrm{R}}; p)\Big|_{(\mathring{t},\mathring{x})}$ $(j = 1, 2)$ such that $u = u_1 + u_2$.*

Further, the equation $u = u_1 + u_2 = 0$ implies $u_1, u_2 \in \mathcal{S}ol(\mathfrak{M}_0^{\mathrm{R}}; p)\Big|_{(\mathring{t},\mathring{x})}$. Equivalently, we have the following exact sequence:

$$0 \longrightarrow \mathcal{S}ol(\mathfrak{M}_0^{\mathrm{R}}; p) \longrightarrow \mathcal{S}ol(\mathfrak{M}_1^{\mathrm{R}}; p) \oplus \mathcal{S}ol(\mathfrak{M}_2^{\mathrm{R}}; p) \longrightarrow \mathcal{S}ol(\mathfrak{M}^{\mathrm{R}}; p) \longrightarrow 0,$$

where $F(U) = U \oplus (-U)$, $G(U_1 \oplus U_2) = U_1 + U_2$.

Remark 13. For the case $p = 1$, the contravariant form of this decomposition means the decomposition $u^i = u_1^i + u_2^i \in \widetilde{\mathcal{D}b}_0^1$ satisfying the next conditions:

$$\nabla_i \, u_1{}^i = 0, \quad \nabla^i u_2{}^j - \nabla^j u_2{}^i = 0.$$

Let X be an n-dimensional complex manifold with a Hermitian metric, and $\bigwedge^{(q,r)} T^*X$ a vector bundle of (q,r)-type differential forms on X. Let $\mathcal{E}_X^{(q,r)}$ be a sheaf of (q,r)-forms on X with C^∞ coefficients, and $\mathcal{D}b_X^{(q,r)}$ a sheaf of (q,r)-currents on X. Setting $\widetilde{X} = \mathbb{R}_t \times X$, we also define $\widetilde{\mathcal{E}}_X^{(q,r)}$, $\widetilde{\mathcal{D}b}_X^{(q,r)}$ similarly to $\widetilde{\mathcal{E}}_M^{(p)}$, $\widetilde{\mathcal{D}b}_M^{(p)}$.

Definition 14. We denote by $\partial : \mathcal{D}b_X^{(q,r)} \to \mathcal{D}b_X^{(q+1,r)}$ the exterior differential operator which acts on $\mathcal{D}b_X^{(q,r)}$ as a sheaf morphism and $\overline{\partial} : \mathcal{D}b_X^{(q,r)} \to \mathcal{D}b_X^{(q,r+1)}$ the conjugate exterior differential operator. For a section

$$\phi = \phi_{i_1 \cdots i_q \bar{j}_1 \cdots \bar{j}_r} \, dz^{i_1} \wedge \cdots \wedge dz^{i_q} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_r} \quad \text{of } \mathcal{D}b_X^{(q,r)},$$

the following formulas are well-known:

$$\begin{cases} d\phi &= (\partial + \overline{\partial})\phi, \\ \partial\phi &= \dfrac{\partial \phi}{\partial z^k} \, dz^k \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_q} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_r} \in \mathcal{D}b_X^{(q+1,r)}, \\ \overline{\partial}\phi &= \dfrac{\partial \phi}{\partial \bar{z}^k} \, d\bar{z}^k \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_q} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_r} \in \mathcal{D}b_X^{(q,r+1)}. \end{cases}$$

Definition 15. The linear operator $*$ on X induces isomorphisms $\bigwedge^{(q,r)} T^*X \longrightarrow \bigwedge^{(n-r,n-q)} T^*X$ of vector bundle. Hence we have sheaf-morphisms $* : \mathcal{D}b_X^{(q,r)} \longrightarrow \mathcal{D}b_X^{(n-r,n-q)}$ on X as follows: For

$$\psi = \psi_{I\bar{J}} \, \omega^I \wedge \overline{\omega}^J \in \mathcal{D}b_X^{(q,r)},$$

we have

$$*\psi = \delta \begin{pmatrix} 1 \cdots n \bar{1} \cdots \bar{n} \\ I \ \ \bar{J} \ \ \bar{J}^C \ \ I^C \end{pmatrix} \psi_{I\bar{J}} \, \omega^{J^C} \wedge \overline{\omega}^{I^C} \in \mathcal{D}b_X^{(n-r,n-q)},$$

where $\{\omega^1, \cdots, \omega^n\}$ is a local orthonormal system of C^∞ sections of T^*X concerning the Hermitian metric and $I^C := \{1, \cdots, n\} \backslash I$. Here $\delta(\cdot) = \pm 1$ is the signature of the permutation $(I\bar{J}\bar{J}^C I^C)$ of $(1 \cdots n \bar{1} \cdots \bar{n})$.

Let $U \subset X$ be an open subset. Let $\alpha^{(q,r)} = \alpha_{I\bar{J}} \, \omega^I \wedge \overline{\omega}^J \in \mathcal{D}b_X^{(q,r)}(U)$, $\beta^{(q,r)} = \beta_{I\bar{J}} \, \omega^I \wedge \overline{\omega}^J \in \mathcal{E}_X^{(q,r)}(U)$ be sections. We suppose that $\beta^{(q,r)}$ has a compact support in U. Then the following integral is well-defined.

$$(\alpha^{(q,r)}, \beta^{(q,r)}) := \int_X \langle \alpha^{(q,r)}, \beta^{(q,r)} \rangle^* \, \omega^1 \wedge \cdots \wedge \omega^n \wedge \overline{\omega}^1 \wedge \cdots \wedge \overline{\omega}^n,$$

where, $\langle \alpha^{(q,r)}, \beta^{(q,r)} \rangle^* = \sum_{I,J} \alpha_{I\bar{J}} \overline{\beta^{I\bar{J}}}$.

Definition 16. Let $\alpha^{(q,r)} \in \mathcal{D}b_X^{(q,r)}$, $\beta^{(q-1,r)} \in \mathcal{E}_X^{(q-1,r)}$, and $\gamma^{(q,r-1)} \in \mathcal{E}_X^{(q,r-1)}$ be sections. We suppose $\beta^{(q-1,r)}$ and $\gamma^{(q,r-1)}$ have compact supports. Then sheaf morphisms $\overline{\vartheta} : \mathcal{D}b_X^{(q,r)} \to \mathcal{D}b_X^{(q-1,r)}$ and $\vartheta : \mathcal{D}b_X^{(q,r)} \to \mathcal{D}b_X^{(q,r-1)}$ are defined as

$$(\overline{\vartheta}\alpha^{(q,r)}, \beta^{(q-1,r)}) = (\alpha^{(q,r)}, \partial\beta^{(q-1,r)}),$$
$$(\vartheta\alpha^{(q,r)}, \gamma^{(q,r-1)}) = (\alpha^{(q,r)}, \overline{\partial}\gamma^{(q,r-1)}).$$

Further they satisfy the following equations:

$$\begin{cases} \delta &= \overline{\vartheta} + \vartheta, \\ \overline{\vartheta} &= - * \overline{\partial} *, \\ \vartheta &= - * \partial * . \end{cases}$$

Now we assume that X is a Kähler manifold; that is, for the Hermitian metric h ,we have the equation $d\left(\sum h_{j\overline{k}}(z) dz^j \wedge d\overline{z}^k\right) = 0$, and we know that $h_{j\overline{k}}$ can be described as $h_{j\overline{k}} = \partial_j \overline{\partial}_k \phi$ with a smooth real function ϕ locally. Then the following equations for operators on $\widetilde{\mathcal{D}b}_X^{(q,r)}$ are well-known:

$$\begin{cases} \Box = \overline{\Box} = \frac{1}{2}\Delta, \\ \partial\vartheta + \vartheta\partial = 0, \quad \overline{\partial}\,\overline{\vartheta} + \overline{\vartheta}\,\overline{\partial} = 0, \\ \partial\overline{\partial} + \overline{\partial}\partial = 0, \quad \vartheta\overline{\vartheta} + \overline{\vartheta}\vartheta = 0. \end{cases}$$

Definition 17. We define sheaf-morphisms $P_{\mathrm{K}} : \widetilde{\mathcal{D}b}_X^{(q,r)} \longrightarrow \widetilde{\mathcal{D}b}_X^{(q,r)}$ on $\widetilde{X}$ by

$$P_{\mathrm{K}} = \frac{\partial^2}{\partial t^2} + \alpha_1 \partial\overline{\vartheta} + \alpha_2 \overline{\vartheta}\partial + \alpha_3 \overline{\partial}\vartheta + \alpha_4 \vartheta\overline{\partial}.$$

Here, $\alpha_1, \alpha_2, \alpha_3$ and α_4 are positive coefficients.

When $q, r = 0$ or n, $P_{\mathrm{K}}\, u = 0$ reduces to a wave equation. Therefore, we suppose $1 \leq q, r \leq n-1$.

For $u \in \widetilde{\mathcal{D}b}_X^{(q,r)}$, we define equations $\mathfrak{M}^{\mathrm{K}}$, $\mathfrak{M}_i^{\mathrm{K}}$ $(i = 1, 2, 3, 4)$, $\mathfrak{M}_{jk}^{\mathrm{K}}$, $\mathfrak{M}_{jk0}^{\mathrm{K}}$ $\left((jk) = (13), (14), (23), (24)\right)$ as follows:

$$
\begin{aligned}
&\mathfrak{M}^{\mathrm{K}} \;:\quad P_{\mathrm{K}}\, u = 0,\\[2mm]
&\mathfrak{M}_1^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\, u = 0,\\ \partial u = 0,\\ \overline{\partial} u = 0, \end{cases} \quad\Longleftrightarrow\quad \begin{cases} \left(\partial_t^2 + \dfrac{\alpha_1+\alpha_3}{2}\Delta\right) u = 0,\\ \partial u = 0,\\ \overline{\partial} u = 0, \end{cases}\\[2mm]
&\mathfrak{M}_2^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\, u = 0,\\ \overline{\vartheta} u = 0,\\ \overline{\partial} u = 0, \end{cases} \quad\Longleftrightarrow\quad \begin{cases} \left(\partial_t^2 + \dfrac{\alpha_2+\alpha_3}{2}\Delta\right) u = 0,\\ \overline{\vartheta} u = 0,\\ \overline{\partial} u = 0, \end{cases}\\[2mm]
&\mathfrak{M}_3^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\, u = 0,\\ \partial u = 0,\\ \vartheta u = 0, \end{cases} \quad\Longleftrightarrow\quad \begin{cases} \left(\partial_t^2 + \dfrac{\alpha_1+\alpha_4}{2}\Delta\right) u = 0,\\ \partial u = 0,\\ \vartheta u = 0, \end{cases}\\[2mm]
&\mathfrak{M}_4^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\, u = 0,\\ \overline{\vartheta} u = 0,\\ \vartheta u = 0, \end{cases} \quad\Longleftrightarrow\quad \begin{cases} \left(\partial_t^2 + \dfrac{\alpha_2+\alpha_4}{2}\Delta\right) u = 0,\\ \overline{\vartheta} u = 0,\\ \vartheta u = 0, \end{cases}\\[2mm]
&\mathfrak{M}_{13}^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\, u = 0,\\ \partial u = 0,\\ \overline{\partial} u = 0,\\ \vartheta u = 0, \end{cases} \quad\Longleftrightarrow\quad \begin{cases} \partial_t^2 u = 0,\\ \partial u = 0,\\ \overline{\partial} u = 0,\\ \vartheta u = 0, \end{cases}\\[2mm]
&\mathfrak{M}_{24}^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\, u = 0,\\ \overline{\vartheta} u = 0,\\ \overline{\partial} u = 0,\\ \vartheta u = 0, \end{cases} \quad\Longleftrightarrow\quad \begin{cases} \partial_t^2 u = 0,\\ \overline{\vartheta} u = 0,\\ \overline{\partial} u = 0,\\ \vartheta u = 0, \end{cases}\\[2mm]
&\mathfrak{M}_{12}^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\, u = 0,\\ \overline{\partial} u = 0,\\ \overline{\vartheta} u = 0,\\ \partial u = 0, \end{cases} \quad\Longleftrightarrow\quad \begin{cases} \partial_t^2 u = 0,\\ \overline{\partial} u = 0,\\ \overline{\vartheta} u = 0,\\ \partial u = 0, \end{cases}
\end{aligned}
$$

$$\mathfrak{M}_{34}^{\mathrm{K}}: \begin{cases} P_{\mathrm{K}}\, u = 0, \\ \vartheta u = 0, \\ \overline{\vartheta} u = 0, \\ \partial u = 0, \end{cases} \iff \begin{cases} \partial_t^2 u = 0, \\ \vartheta u = 0, \\ \overline{\vartheta} u = 0, \\ \partial u = 0, \end{cases}$$

$$\mathfrak{M}_{0}^{\mathrm{K}}: \begin{cases} P_{\mathrm{K}}\, u = 0, \\ \partial u = 0, \\ \overline{\partial} u = 0, \\ \vartheta u = 0, \\ \overline{\vartheta} u = 0, \end{cases} \iff \begin{cases} \partial_t^2 u = 0, \\ \partial u = 0, \\ \overline{\partial} u = 0, \\ \vartheta u = 0, \\ \overline{\vartheta} u = 0. \end{cases}$$

Further we define subsheaves $\mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}; q, r)$, $\mathcal{S}ol(\mathfrak{M}_i^{\mathrm{K}}; q, r)$ $\big(i = 1, 2, 3, 4\big)$, $\mathcal{S}ol(\mathfrak{M}_{jk}^{\mathrm{K}}; q, r)$, $\mathcal{S}ol(\mathfrak{M}_{jk0}^{\mathrm{K}}; q, r)$ $\big((jk) = (13), (23), (14), (24)\big)$ of $\widetilde{\mathcal{D}b}_X^{(q,r)}$ as the sheaves of $\widetilde{\mathcal{D}b}_X^{(q,r)}$-solutions, respectively.

Then, we have the following theorem.

Theorem 18. *For any germ $u \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}; q, r)\Big|_{(\mathring{t}, \mathring{z})}$, there exist some germs $u_{ij} \in \mathcal{S}ol(\mathfrak{M}_{ij}^{\mathrm{K}}; q, r)\Big|_{(\mathring{t}, \mathring{z})}$ $((ij) = (13), (23), (14), (24))$ such that $u = u_{13} + u_{23} + u_{14} + u_{24}$.*

Further, we find that $u = u_{13} + u_{23} + u_{14} + u_{24} = 0$ implies

$$u_{jk} \in \mathcal{S}ol(\mathfrak{M}_{jk0}^{\mathrm{K}}; q, r) \quad \big((jk) = (13), (23), (14), (24)\big).$$

Equivalently, we have the following exact sequence:

$$0 \longrightarrow \bigoplus_{(ij)}{}' \mathcal{S}ol(\mathfrak{M}_{ij0}^{\mathrm{K}}; q, r) \xrightarrow{G} \bigoplus_{(ij)} \mathcal{S}ol(\mathfrak{M}_{ij}^{\mathrm{K}}; q, r) \xrightarrow{H} \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}; q, r) \longrightarrow 0.$$

Here,

$$\bigoplus_{(ij)}{}' \mathcal{S}ol(\mathfrak{M}_{ij0}^{\mathrm{K}}; q, r) := \Big\{ (u_{ij}) \in \bigoplus_{(ij)} \mathcal{S}ol(\mathfrak{M}_{ij0}^{\mathrm{K}}; q, r) \,\Big|\, \sum_{(ij)} u_{ij} = 0 \Big\},$$

$G(U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}) = U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}$, $H(U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}) = U_{13} + U_{23} + U_{14} + U_{24}$.

Example 19. We assume $X = \mathbb{C}^2$. Then, X is a Kähler manifold with the complex Euclidean metric. We find a solution $u \in \widetilde{\mathcal{D}b}_X^{(1,1)}$ of the form with $\zeta \equiv \zeta_1 dz^1 + \zeta_2 dz^2$ where $(\zeta_1, \zeta_2) \in \mathbb{C}^2 \backslash \{0\}$;

$$u(t, z) = U(t) e^{i(z \cdot \zeta + \overline{z} \cdot \overline{\zeta})}.$$

Then,

$$\begin{aligned} P_{\mathrm{K}}\, u = U'' + (\alpha_1 - \alpha_2)\ \zeta \wedge \Big(* \left(\overline{\zeta} \wedge *U \right) \Big) + \alpha_2 |\zeta|^2 U \\ + (\alpha_3 - \alpha_4)\ \overline{\zeta} \wedge \Big(* \left(\zeta \wedge *U \right) \Big) + \alpha_4 |\zeta|^2 U = 0. \end{aligned}$$

We put

$$U(t) = c_1(t)\ \zeta \wedge \overline{\zeta} + c_2(t)\ \zeta \wedge \overline{\zeta}^{\perp} + c_3(t)\ \zeta^{\perp} \wedge \overline{\zeta} + c_4(t)\ \zeta^{\perp} \wedge \overline{\zeta}^{\perp},$$

where $\zeta^{\perp} = \overline{\zeta}_2 dz^1 - \overline{\zeta}_1 dz^2$, $|\zeta| = |\zeta^{\perp}|$ hold. Then, we get

$$\begin{aligned} \Big(c_1'' + (\alpha_1 + \alpha_3)|\zeta|^2 c_1\Big)\zeta \wedge \overline{\zeta} + \Big(c_2'' + (\alpha_1 + \alpha_4)|\zeta|^2 c_2\Big)\zeta \wedge \overline{\zeta}^{\perp} \\ \Big(c_3'' + (\alpha_2 + \alpha_3)|\zeta|^2 c_3\Big)\zeta^{\perp} \wedge \overline{\zeta} + \Big(c_4'' + (\alpha_2 + \alpha_4)|\zeta|^2 c_4\Big)\zeta^{\perp} \wedge \overline{\zeta}^{\perp} = 0. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} c_1(t) &= A_{13}^{+} \exp\left(i\sqrt{\alpha_1 + \alpha_3}|\zeta|t\right) + A_{13}^{-} \exp\left(-i\sqrt{\alpha_1 + \alpha_3}|\zeta|t\right), \\ c_2(t) &= A_{14}^{+} \exp\left(i\sqrt{\alpha_1 + \alpha_4}|\zeta|t\right) + A_{14}^{-} \exp\left(-i\sqrt{\alpha_1 + \alpha_4}|\zeta|t\right), \\ c_3(t) &= A_{23}^{+} \exp\left(i\sqrt{\alpha_2 + \alpha_3}|\zeta|t\right) + A_{23}^{-} \exp\left(-i\sqrt{\alpha_2 + \alpha_3}|\zeta|t\right), \\ c_4(t) &= A_{24}^{+} \exp\left(i\sqrt{\alpha_2 + \alpha_4}|\zeta|t\right) + A_{24}^{-} \exp\left(-i\sqrt{\alpha_2 + \alpha_4}|\zeta|t\right). \end{aligned}$$

Since

$$
\begin{aligned}
U(0) =&(A_{13}^{+} + A_{13}^{-})\ \zeta \wedge \overline{\zeta} + (A_{14}^{+} + A_{14}^{-})\ \zeta \wedge \overline{\zeta}^{\perp} \\
&+ (A_{23}^{+} + A_{23}^{-})\ \zeta^{\perp} \wedge \overline{\zeta} + (A_{24}^{+} + A_{24}^{-})\ \zeta^{\perp} \wedge \overline{\zeta}^{\perp}, \\
\frac{\partial}{\partial t} U(0) =& i\sqrt{\alpha_1 + \alpha_3}|\zeta|(A_{13}^{+} - A_{13}^{-})\ \zeta \wedge \overline{\zeta} \\
&+ i\sqrt{\alpha_1 + \alpha_4}|\zeta|(A_{14}^{+} - A_{14}^{-})\ \zeta \wedge \overline{\zeta}^{\perp} \\
&+ i\sqrt{\alpha_2 + \alpha_3}|\zeta|(A_{23}^{+} - A_{23}^{-})\ \zeta^{\perp} \wedge \overline{\zeta} \\
&+ i\sqrt{\alpha_2 + \alpha_4}|\zeta|(A_{24}^{+} - A_{24}^{-})\ \zeta^{\perp} \wedge \overline{\zeta}^{\perp},
\end{aligned}
$$

we get

$$
\begin{aligned}
A_{13}^{+} &= \frac{\langle U(0), \zeta \wedge \overline{\zeta} \rangle^*}{2|\zeta|^4} - i\ \frac{\langle \frac{\partial}{\partial t} U(0), \zeta \wedge \overline{\zeta} \rangle^*}{2\sqrt{\alpha_1 + \alpha_3}|\zeta|^5}\ , \\
A_{13}^{-} &= \frac{\langle U(0), \zeta \wedge \overline{\zeta} \rangle^*}{2|\zeta|^4} + i\ \frac{\langle \frac{\partial}{\partial t} U(0), \zeta \wedge \overline{\zeta} \rangle^*}{2\sqrt{\alpha_1 + \alpha_3}|\zeta|^5}\ , \\
A_{14}^{+} &= \frac{\langle U(0), \zeta \wedge \overline{\zeta}^{\perp} \rangle^*}{2|\zeta|^4} - i\ \frac{\langle \frac{\partial}{\partial t} U(0), \zeta \wedge \overline{\zeta}^{\perp} \rangle^*}{2\sqrt{\alpha_1 + \alpha_4}|\zeta|^5}\ , \\
A_{14}^{-} &= \frac{\langle U(0), \zeta \wedge \overline{\zeta}^{\perp} \rangle^*}{2|\zeta|^4} + i\ \frac{\langle \frac{\partial}{\partial t} U(0), \zeta \wedge \overline{\zeta}^{\perp} \rangle^*}{2\sqrt{\alpha_1 + \alpha_4}|\zeta|^5}\ , \\
A_{23}^{+} &= \frac{\langle U(0), \zeta^{\perp} \wedge \overline{\zeta} \rangle^*}{2|\zeta|^4} - i\ \frac{\langle \frac{\partial}{\partial t} U(0), \zeta^{\perp} \wedge \overline{\zeta} \rangle^*}{2\sqrt{\alpha_2 + \alpha_3}|\zeta|^5}\ , \\
A_{23}^{-} &= \frac{\langle U(0), \zeta^{\perp} \wedge \overline{\zeta} \rangle^*}{2|\zeta|^4} + i\ \frac{\langle \frac{\partial}{\partial t} U(0), \zeta^{\perp} \wedge \overline{\zeta} \rangle^*}{2\sqrt{\alpha_2 + \alpha_3}|\zeta|^5}\ , \\
A_{24}^{+} &= \frac{\langle U(0), \zeta^{\perp} \wedge \overline{\zeta}^{\perp} \rangle^*}{2|\zeta|^4} - i\ \frac{\langle \frac{\partial}{\partial t} U(0), \zeta^{\perp} \wedge \overline{\zeta}^{\perp} \rangle^*}{2\sqrt{\alpha_2 + \alpha_4}|\zeta|^5}\ , \\
A_{24}^{-} &= \frac{\langle U(0), \zeta^{\perp} \wedge \overline{\zeta}^{\perp} \rangle^*}{2|\zeta|^4} + i\ \frac{\langle \frac{\partial}{\partial t} U(0), \zeta^{\perp} \wedge \overline{\zeta}^{\perp} \rangle^*}{2\sqrt{\alpha_2 + \alpha_4}|\zeta|^5}\ .
\end{aligned}
$$