数理解析研究所講究録 1338 巻 2003 年 41-52

THETA CORRESPONDENCE AND REPRESENTATION THEORY

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ABSTRACT. After reviewing the relation between the theta integral (= theta lifting) and Howe correspondence, we give an example of the preservation of the associated cycles by the theta lifting (joint work with C.-B. Zhu).

Namely, let (G, G') be a type I dual pair strictly in the stable range (we assume that G' is the smaller member), and π' a unitary highest weight module of $\widetilde{G'}$. Then the associated cycle of the theta lift $\pi = \theta(\pi')$ of π' can be given as $\mathcal{AC}(\theta(\pi')) = \theta(\mathcal{AC}(\pi'))$, where the theta lifting of associated cycle is naturally defined using the lifting of nilpotent orbits. We also give a naive introduction to the basic property of associated cycles and the lifting of nilpotent orbits.

1. Theta integral

The content of this section is mainly quoted from [10, 11] and [4].

Let F be a number field and $\mathbb{A}$ a ring of adeles of F. For simplicity, we consider *one of type I dual pairs* defined over F in the following. It is constructed as follows. Take a vector space

$$V\ /F \text{ with non-degenerate symmetric bilinear form } (,) = (,)_V$$
$$V'\ /F \text{ with non-degenerate skew-symmetric bilinear form } (,)' = (,)_{V'}$$

Then $W = V \otimes_F V'$ inherits a skew-symmetric form defined by $(,)_W = (,)_V \otimes_F (,)_{V'}$. We put

$$\begin{cases} G = O(V) & \text{orthogonal group} \\ G' = Sp(V') & \text{symplectic group} \end{cases}$$

They are naturally subgroups of $Sp(W)$ commuting with each other, which form a type I dual pair (G, G') in $Sp(W)$. We denote by $G(\mathbb{A}), G'(\mathbb{A})$ or $Sp(W)_{\mathbb{A}}$, the global adelic groups. For each place v of F, let F_v be the completion of F at v, and G_v or G'_v denotes the corresponding groups over the local field F_v.

$Sp(W)_{\mathbb{A}}$ has a non-trivial double cover $Mp(W)_{\mathbb{A}}$ called the metaplectic group. This group has a distinguished representation called the Weil representation. We do not give an exact construction of the representation but use an explicit realization. For this, we refer the readers to [20], [4], [3], [17], et al.

Let $W = X \oplus Y$ be a complete polarization, and take a character χ of $\mathbb{A}$ which is trivial on F. Then the Weil representation $\omega = \omega_\chi$ of $Mp(W)_{\mathbb{A}}$ is realized on the Hilbert space $L^2(X(\mathbb{A}))$. It is unitary, and the space of smooth vectors coincides with the space of Schwartz-Bruhat functions $\mathcal{S} = \mathcal{S}(X(\mathbb{A}))$ on $X(\mathbb{A})$.

Proceedings of RIMS Workshop on "Automorphic Forms and Representations of Algebraic Groups over Local Fields", January 20 – January 24, 2003.

Let θ be a tempered distribution on $\mathcal{S}$ defined by

$$\theta(\varphi) = \sum_{\xi \in X(F)} \varphi(\xi) \qquad (\varphi \in \mathcal{S}), \tag{1.1}$$

which converges absolutely. Then θ is $Sp(W)_F$-invariant distribution, i.e.,

$$\theta(\omega(\gamma)\varphi) = \theta(\varphi) \qquad (\gamma \in Sp(W)_F,\ \varphi \in \mathcal{S}).$$

Note that $Sp(W)_F$ is embedded into $Mp(W)_{\mathbb{A}}$ as a discrete subgroup. This property characterizes θ up to constant multiple ([4]). Let $\widetilde{G}(\mathbb{A})$ denote the inverse image of $G(\mathbb{A})$ of the covering map $Mp(W)_{\mathbb{A}} \to Sp(W)_{\mathbb{A}}$. The same notation applies to arbitrary subgroup of $Sp(W)_{\mathbb{A}}$. For $(g, h) \in \widetilde{G}(\mathbb{A}) \times \widetilde{G'}(\mathbb{A})$, we put

$$\theta_\varphi(g, h) = \theta(\omega(g \cdot h)\varphi) \qquad (\varphi \in \mathcal{S}). \tag{1.2}$$

Then, appropriate choice of φ and G, G' will give various types of classical theta functions.

Assume that π' is an automorphic representation realized on a Hilbert space

$$\mathcal{H}_{\pi'} \subset L^2(G'(F)\backslash\widetilde{G'}(\mathbb{A})).$$

For $f \in \mathcal{H}_{\pi'}^\infty =$ (smooth vectors), we define the *theta integral* by

$$\theta_\varphi^f(g) = \int_{G'(F)\backslash\widetilde{G'}(\mathbb{A})} \theta_\varphi(g, h) f(h) dh. \tag{1.3}$$

If π' is a cuspidal representation, then the integral converges and defines a slowly increasing function on $G(F)\backslash\widetilde{G}(\mathbb{A})$. In the following (in this section), *we assume π' to be cuspidal.*

Formally θ_φ^f gives an automorphic form on $G(F)\backslash\widetilde{G}(\mathbb{A})$ and one may expect that

$$\{\theta_\varphi^f \mid f \in \mathcal{H}_{\pi'}^\infty, \varphi \in \mathcal{S}(X(\mathbb{A}))\}$$

gives an automorphic representation π of $\widetilde{G}(\mathbb{A})$ after some *completion.* Thus we want to see when the integral

$$\langle \theta_{\varphi_1}^{f_1}, \theta_{\varphi_2}^{f_2} \rangle = \int_{G(F)\backslash\widetilde{G}(\mathbb{A})} \theta_{\varphi_1}^{f_1}(g)\, \overline{\theta_{\varphi_2}^{f_2}(g)}\, dg \tag{1.4}$$

converges (under some assumption), and gives a non-zero value for some choice of $\{f_i\}$ and $\{\varphi_i\}$.

Theorem 1.1 (Rallis' inner product formula). *Assume* $\dim V > 2\dim V' + 2$. *Then the above inner product* (1.4) *converges absolutely. Moreover, we have*

$$\begin{aligned}\langle \theta_{\varphi_1}^{f_1}, \theta_{\varphi_2}^{f_2} \rangle &= \int_{\widetilde{G'}(\mathbb{A})} \langle \omega(h)\varphi_1, \varphi_2 \rangle \langle \pi'(h) f_1, f_2 \rangle dh \\ &= \prod_{v \in P(F)} \int_{\widetilde{G'}(F_v)} \langle \omega_v(h)\varphi_{1v}, \varphi_{2v} \rangle \langle \pi'_v(h) f_{1v}, f_{2v} \rangle dh\end{aligned}$$

Here, $P(F)$ denotes the set of all places of F.

Proof. For proof, see [10, Theorem 2.1]. Essentially, the following two ingredients prove the theorem; (i) Howe's technique of doubling variables; (ii) Siegel-Weil formula, which claims that θ-integral coincides with an Eisenstein series. □

Thus we should consider the integral

$$\int_{\widetilde{G'}(F_v)} \langle \omega_v(h)\varphi_{1\,v}, \varphi_{2\,v}\rangle \langle \pi'_v(h) f_{1\,v}, f_{2\,v}\rangle dh$$

at various places $v \in P(F)$. For finite places at which F is not ramified, it is given by special values of L-functions. In the following sections, we will concentrate on real places.

2. Theta correspondence over reals

From now on, we assume the ground field is $\mathbb{R}$, thus V, V' are now considered as vector spaces over $\mathbb{R}$ with symmetric (respectively skew-symmetric) non-degenerate bilinear form. We also write $G = O(V)$ and $G' = Sp(V')$, which are *real Lie groups*. The Weil representation $\omega = \omega_X$ is realized on the space of L^2-functions $L^2(X)$ on a maximal totally isotropic space X of W.

Let us consider the integral

$$\int_{\widetilde{G'}} \langle \omega(h)\varphi_1, \varphi_2\rangle \langle \pi'(h) f_1, f_2\rangle dh \qquad (\varphi_i \in \mathcal{S}(X),\ f_i \in \mathcal{H}_{\pi'}^{\infty}) \tag{2.1}$$

for a genuine irreducible unitary representation π' of $\widetilde{G'}$ on a Hilbert space $\mathcal{H}_{\pi'}$. A representation of $\widetilde{G'}$ is called *genuine* if it is not factor through to the representation of G', i.e., if it is non-trivial on the kernel of the covering map. Note that π' is not necessarily automorphic nor cuspidal now, and everything is considered over $\mathbb{R}$. If we put $\Phi_i = \varphi_i \otimes f_i \in \mathcal{S} \otimes \mathcal{H}_{\pi'}^{\infty}$, the above formula becomes

$$(\Phi_1, \Phi_2)_{\pi'} = \int_{G'} \langle (\omega \otimes \pi')(h)\Phi_1, \Phi_2\rangle dh. \tag{2.2}$$

Since ω and π' are both genuine, $\omega \otimes \pi'$ factors through to a representation of G', and we do not need a cover anymore.

It may be useful to consider this integral for a compact group (or a compact dual pair) as a toy model. Thus, only in this short paragraph, let us pretend as if G' was a compact group and ω was a representation of G'. Then ω decomposes discretely as

$$\omega \simeq \sum\nolimits_{\xi \in \widehat{G'}}^{\oplus} \mathrm{Hom}_{G'}(\xi, \omega) \otimes \xi, \qquad \text{hence}$$

$$\omega \otimes \pi' \simeq \sum\nolimits_{\xi \in \widehat{G'}}^{\oplus} \mathrm{Hom}_{G'}(\xi, \omega) \otimes (\xi \otimes \pi').$$

Then an integral $\int_{G'} \langle (\omega \otimes \pi')(h) v_1, v_2\rangle dh$ survives only if $\xi \simeq (\pi')^*$ for some $\xi \in \widehat{G'}$ and the collection of $(\Phi_1, \Phi_2)_{\pi'}$ will give an inner product on the space of multiplicities $\mathrm{Hom}_{G'}((\pi')^*, \omega)$. In some sense, this is carried over to our present situation.

Now let us return to our original settings in this section. For the convergence of the integral, the following theorem holds.

Theorem 2.1 (Li [8], [11, Theorem 2.1]). *Suppose we are in one of the following two situations.*

(1) *The pair (G, G') is in the* stable range, *i.e.,*

$$\dim(\textit{maximal totally isotropic space in } V) \geq \dim V'. \tag{2.3}$$

(2) *π' is in the discrete series and* $\dim V \geq \dim V'$.

Then the above integral (2.1) *converges absolutely for any choice of* $\{\varphi_i\} \subset \mathcal{S}$ *and* $\{f_i\} \subset \mathcal{H}_{\pi'}^{\infty}$.

Now assume the above (1) or (2) from now on.

Put $R =$ (kernel of $(\ ,\ )_{\pi'}$), and make a completion of $(\mathcal{S}\otimes\mathcal{H}_{\pi'}^{\infty})/R$ by the inner product $(\ ,\ )_{\pi'}$.

$$\mathfrak{H} = (\text{completion of } (\mathcal{S} \otimes \mathcal{H}_{\pi'}^{\infty})/R) \tag{2.4}$$

Since $\widetilde{G}$ acts on $\mathcal{S} \otimes \mathcal{H}_{\pi'}^{\infty}$ which leaves R stable, the resulting Hilbert space $\mathfrak{H}$ carries a unitary representation π of $\widetilde{G}$ (but still it may be zero). The following theorem is proved by Li for general type I dual pairs ([9, Proposition 2.4]), and independently by Moeglin for the pair $(O(2p, 2q), Sp(2n, \mathbb{R}))$.

Theorem 2.2 (Moeglin, Li). (1) *If* $\mathfrak{H}$ *is not zero, then it carries a genuine unitary representation* $(\pi, \mathfrak{H})$ *of* $\widetilde{G}$, *which is the theta lift of* $(\pi')^*$ *in the sense of Howe* ([6]; *see* §3) ; $\pi = \theta((\pi')^*)$.

(2) *If* (G, G') *is in the stable range,* $\mathfrak{H}$ *is non-zero for any unitary irreducible representation* π' *of* $\widetilde{G}$, *which is genuine.*

(3) *If* π' *is in the discrete series which is "sufficiently regular", then* $\mathfrak{H}$ *is non-zero and* $\pi = A_{\mathfrak{q}}(\lambda)$; *a representation with non-zero cohomology defined by Vogan and Zuckerman* ([19]).

This theorem tells us that the representation $(\pi, \mathfrak{H})$ of $\widetilde{G}$ so-obtained is in correspondence with the dual of $(\pi', \mathcal{H})$ in the sense of Howe (one may call it Howe correspondence). In this sense, the notion of theta lifting and Howe correspondence are almost the same.

In the next section, we briefly review the definition and basic properties of Howe correspondence.

3. Howe correspondence

Let ω be the Weil representation of $Mp(W)$, and we choose a complete polarization $W = X \oplus Y$. Let $\Omega = \mathfrak{H}_{\boldsymbol{K}}$ be the space of $\boldsymbol{K}$-finite vectors of ω, where $\boldsymbol{K}$ is a maximal compact subgroup of $Mp(W)$. Then Ω is a $(\mathfrak{G}, \boldsymbol{K})$-module, where $\mathfrak{G}$ is the complexified Lie algebra of $Mp(W)$, and Ω is called the Harish-Chandra module of ω.

In this section, we only consider the Harish-Chandra module Ω, and by abuse of notation, we often denote the action of $(\mathfrak{G}, \boldsymbol{K})$ by the same letter ω, or simply write it by module notation. It is well known that Ω can be identified with the space of polynomials on $X_{\mathbb{C}} = X \otimes_{\mathbb{R}} \mathbb{C}$. In this realization, $\boldsymbol{K}$ is identified with the determinantal double cover of the unitary group $U(X_{\mathbb{C}})$, and the action of $\boldsymbol{K}$ is given by the left translation of polynomials times its determinant, i.e.,

$$\omega(k)f(x) = \sqrt{\det(k)}f(k^{-1}x) \qquad (k \in \boldsymbol{K}, f(x) \in \mathbb{C}[X_{\mathbb{C}}], x \in X_{\mathbb{C}}) \tag{3.1}$$

Let $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$ be the complexified Cartan decomposition along the Lie algebra of the maximal compact subgroup. Then, the action of $\mathfrak{K}$ is given by the differential of $\omega(\boldsymbol{K})$, and the action of $\mathfrak{P}$ is given by the multiplication of polynomials of degree two (if it is a root vector of a positive root), or the differentiation by a constant coefficient differential operator of degree two (if it is a root vector of a negative root). For more detailed realization, we refer to [5] (or [12], for example).

Take an irreducible Harish-Chandra module π' of $\widetilde{G'}$, i.e., π' is a $(\mathfrak{g}', \widetilde{K'})$-module, where $\mathfrak{g}'$ denotes the complexified Lie algebra of $\widetilde{G'}$ and K' is a maximal compact subgroup of G' (similarly we will denote by $\mathfrak{g}$ the complexified Lie algebra of $\widetilde{G}$ and by K a maximal compact subgroup of G). Put

$$\mathbb{H} = \mathrm{Hom}_{(\mathfrak{g}', \widetilde{K'})}(\Omega, \pi') \qquad \text{(morphisms of Harish-Chandra } (\mathfrak{g}', \widetilde{K'})\text{-modules)}, \tag{3.2}$$

and consider

$$\Omega/N; \quad N = \bigcap_{\varphi \in \mathbb{H}} \mathrm{Ker}\, \varphi.$$

Then there exists a quasi-simple $(\mathfrak{g}, \widetilde{K})$-module $\Omega(\pi')$ of finite length such that

$$\Omega/N \simeq \Omega(\pi') \otimes \pi'$$

as $(\mathfrak{g} \oplus \mathfrak{g}', \widetilde{K} \times \widetilde{K'})$-modules. $\Omega(\pi')$ is called *Howe's maximal quotient* for π'.

Theorem 3.1 (Howe). *If $\Omega(\pi')$ is not zero, it has a unique irreducible quotient, which is denoted by $\theta(\pi')$ and called the* theta lift *of π'.*

In fact, the correspondence $\pi' \leftrightarrow \pi = \theta(\pi')$ is bijective between the genuine irreducible representations which appear in ω as quotients. Note that π and π' are in correspondence if and only if there exists a non-trivial $(\mathfrak{g} \oplus \mathfrak{g}', \widetilde{K} \times \widetilde{K'})$-module morphism $\Omega \to \pi \otimes \pi'$. As a formal convention, we put $\theta(\pi') = 0$ if $\Omega(\pi') = 0$, i.e., π' does not appear as a quotient of ω.

Lemma 3.2. *Let us denote by $\Omega(\pi')^*$ the $\widetilde{K}$-finite dual of $\Omega(\pi')$. Then we have*

$$\Omega(\pi')^* \simeq \mathrm{Hom}_{(\mathfrak{g}', \widetilde{K'})}(\Omega, \pi')_{\widetilde{K}\text{-finite}} = \mathbb{H}_{\widetilde{K}}.$$

Proof. Take $v^* \in \Omega(\pi')^*$. Then

$$\Omega \xrightarrow{\text{proj.}} \Omega/N \simeq \Omega(\pi') \otimes \pi' \xrightarrow{v^* \otimes 1} \pi'$$

gives an element of $\mathbb{H}_{\widetilde{K}}$.

Conversely, any $f \in \mathbb{H}_{\widetilde{K}}$ factors through Ω/N by the definition of N. Thus we get $\Omega(\pi') \otimes \pi' \simeq \Omega/N \xrightarrow{\overline{f}} \pi'$, which is $(\mathfrak{g}', \widetilde{K'})$-equivariant. Since π' is irreducible, $\overline{f}$: $v \otimes \pi' \xrightarrow{\sim} \pi'$ gives a scalar $v_f^*(v)$. This gives the inverse map. □

Theorem 3.3 (N-Zhu [15]). *Assume that (G, G') is an irreducible type I dual pair strictly in the stable range. If π' is a unitary highest weight module for $\widetilde{G'}$ (so that G'/K' must be a Hermitian symmetric space), then $\Omega(\pi')$ is irreducible, hence $\Omega(\pi') = \theta(\pi')$ gives the theta lift.*

Remark 3.4. We say that the pair (G, G') is *strictly in the stable range* if (G, G') is in the following list.

This condition is a little bit stronger than the stable range condition (due to J.-S. Li) given above. Note that it is ambiguously called "stable range" in [15]. Though the theorem itself is valid for all the above three pairs, we are only treating Case $\mathbf{R}$ in this

TABLE 1. The dual pairs strictly in the stable range

	the pair (G, G')	strictly stable range condition
Case $\mathbb{R}$:	$(O(p,q), Sp(2n,\mathbb{R}))$	$2n < \min(p,q)$
Case $\mathbb{C}$:	$(U(p,q), U(m,n))$	$m+n \leq \min(p,q)$
Case $\mathbb{H}$:	$(Sp(p,q), O^*(2n))$	$n \leq \min(p,q)$

To give an idea of the proof of this theorem, let us briefly indicate how to compute K-types of $\Omega(\pi')$ (which is proved to be $\theta(\pi')$ afterwards).

Let us remind that V is an indefinite quadratic space over $\mathbb{R}$, and $G = O(V)$. Let $V = V^+ \oplus V^-$ be a decomposition for which

$$\begin{cases} V^+ \text{ is positive definite} & p = \dim V^+, \\ V^- \text{ is negative definite} & q = \dim V^-. \end{cases} \tag{3.3}$$

We denote $K^\pm = O(V^\pm)$, so that $K = K^+ \times K^-$ gives a maximal compact subgroup of G. Recall the complete polarization $V' = X' \oplus Y'$ of the symplectic space V'. Then according to the decomposition, we can take a maximal totally isotropic space X as

$$X = V \otimes Y' = (V^+ \otimes Y') \oplus (V^- \otimes Y'). \tag{3.4}$$

Therefore, the Weil representation ω is realized on the L^2-space

$$L^2(X) = L^2(V^+ \otimes Y') \otimes L^2(V^- \otimes Y').$$

We note that $L^2(V^\pm \otimes Y')$ carries the Weil representation for compact dual pairs $(K^\pm, G')$ $= (O(V^\pm), Sp(V'))$, whose decomposition is well known by the work of Kashiwara and Vergne [7]. Up to twisting by a genuine character of the double cover of $O(V^\pm)$, we have

$$\begin{cases} L^2(V^+ \otimes Y') \simeq \sum_{\sigma_1 \in O(V^+)^\wedge} \sigma_1^{\chi_1} \otimes L^+(\sigma_1), \\ L^2(V^- \otimes Y') \simeq \sum_{\sigma_2 \in O(V^-)^\wedge} \sigma_2^{\chi_2} \otimes L^-(\sigma_2), \end{cases} \tag{3.5}$$

where $L^+(\sigma_1)$ (respectively $L^-(\sigma_2)$) is a unitary highest (respectively lowest) weight module of $\widetilde{G'} = \widetilde{Sp}(V')$, which is genuine; and $\sigma_i^{\chi_i} = \chi_i \otimes \sigma_i$ is a genuine irreducible finite dimensional representation of the double cover $\widetilde{O}(V^\pm)$ obtained from the irreducible representation $\sigma_i \in O(V^\pm)^\wedge$ twisted by a certain genuine character χ_i. Note, however, the double covers $\widetilde{G'}$ differ according to $V^\pm$ if the parities of p and q are different. The reason is that the cover is taken in the different metaplectic groups $Mp(V^\pm \otimes V')$. Similarly, $\widetilde{G'} \subset Mp(W)$ may be different from $\widetilde{G'} \subset Mp(V^\pm \otimes V')$. But it is too subtle to denote the dependence, so we will omit it.

Under the condition that the pair is strictly in the stable range, $L^+(\sigma_1)$ is a holomorphic discrete series, and $L^-(\sigma_2)$ is an anti-holomorphic one. This will make our arguments particularly simple.

Using (3.5), we get

$$\begin{aligned} \operatorname{Hom}_{\widetilde{G'}}(\omega, \pi')_{\widetilde{K}} &= \operatorname{Hom}_{\widetilde{G'}}(L^2(V^+ \otimes Y') \otimes L^2(V^- \otimes Y'), \pi')_{\widetilde{K}} \\ &= \sum_{\sigma_1, \sigma_2} \operatorname{Hom}_{\widetilde{G'}}(L^+(\sigma_1) \otimes L^-(\sigma_2), \pi') \otimes (\sigma_1^{\chi_1} \otimes \sigma_2^{\chi_2})^*. \end{aligned}$$

Since π' is a unitary highest weight module, the multiplicity

$$\mathrm{Hom}_{\widetilde{G'}}(L^+(\sigma_1) \otimes L^-(\sigma_2), \pi') \simeq \mathrm{Hom}_{\widetilde{G'}}(L^+(\sigma_1), L^-(\sigma_2)^* \otimes \pi')$$

is of finite dimension. Moreover, it can be described in terms of finite dimensional representations. Namely, if

$$\begin{cases} \tau_1 \text{ is the minimal } \widetilde{K'}\text{-type of } L^+(\sigma_1), \text{ and} \\ \tau_2 \text{ is the minimal } \widetilde{K'}\text{-type of } L^-(\sigma_2)^*, \end{cases}$$

then the above multiplicity equals to

$$\mathrm{Hom}_{\widetilde{K'}}(\tau_1, \tau_2 \otimes (\pi'|_{\widetilde{K'}})).$$

Thus, by Lemma 3.2, finally we obtain

$$\Omega(\pi')|_{\widetilde{K}} \simeq \sum_{\sigma_1 \in O(V^+)^\wedge, \sigma_2 \in O(V^-)^\wedge} \mathrm{Hom}_{\widetilde{K'}}(\tau_1, \tau_2 \otimes (\pi'|_{\widetilde{K'}}))^* \otimes (\sigma_1^{\chi_1} \otimes \sigma_2^{\chi_2}). \tag{3.6}$$

Since $\theta(\pi')$ is the unique irreducible quotient of $\Omega(\pi')$, the multiplicity of $\widetilde{K}$-types in $\theta(\pi')$ cannot exceed $\dim \mathrm{Hom}_{\widetilde{K'}}(\tau_1, \tau_2 \otimes (\pi'|_{\widetilde{K'}}))^*$. However, if we choose appropriate $\widetilde{K}$-types in $\mathcal{S} \otimes \mathcal{H}^\infty_{(\pi')^*}$ and its $\widetilde{K'}$-finite vectors, the theta integral (2.2) converges for such vectors and gives a non-degenerate inner product. This means that the above multiplicity should survive after taking the quotient by $R = \mathrm{Ker}(\ ,\ )_{(\pi')^*}$ (cf. (2.4)). This means that the multiplicity of $\widetilde{K}$-types in $\theta(\pi')$ and $\Omega(\pi')$ is the same, which proves $\theta(\pi') = \Omega(\pi')$.

We summarize the above arguments into

Theorem 3.5 (N-Zhu). *Let π' be a genuine unitary highest weight module of $\widetilde{G'}$ and $\pi = \theta(\pi')$ its theta lift. Then the $\widetilde{K}$-type decomposition of π is given by*

$$\pi|_{\widetilde{K}} \simeq \sum_{\sigma_1 \in O(V^+)^\wedge, \sigma_2 \in O(V^-)^\wedge} \mathrm{Hom}_{\widetilde{K'}}(\tau_1, \tau_2 \otimes (\pi'|_{\widetilde{K'}}))^* \otimes (\sigma_1^{\chi_1} \otimes \sigma_2^{\chi_2}).$$

The multiplicities in the above decomposition formula is efficiently computable. See [12] for example.

4. Associated Cycle

Let $(\pi, \mathfrak{X})$ be a Harish-Chandra $(\mathfrak{g}, K)$ module, where $\mathfrak{g}$ is the complexified Lie algebra of G and K is a maximal compact subgroup. For simplicity, we assume that $\mathfrak{X}$ is quasi-simple, i.e., the center $\mathfrak{Z}(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ acts on $\mathfrak{X}$ as scalars.

Take a finite dimensional generating space $\mathfrak{X}_0 \subset \mathfrak{X}$, which is K-stable. Let $\{U_n(\mathfrak{g})\}_{n=0}^\infty$ be the standard filtration of the enveloping algebra $U(\mathfrak{g})$. We define a filtration of $\mathfrak{X}$ as $\mathfrak{X}_n = U_n(\mathfrak{g})\mathfrak{X}_0$, which is K-stable. Moreover, it satisfies $U_m(\mathfrak{g})\mathfrak{X}_n = \mathfrak{X}_{n+m}$. If a filtration satisfies this condition for sufficiently large n and arbitrary $m \geq 0$, it is called *good*. Thus $\{\mathfrak{X}_n\}_{n=0}^\infty$ is a K-stable good filtration of $\mathfrak{X}$. Let

$$\mathrm{gr}\,\mathfrak{X} = \sum_{n \geq 0}^{\oplus} \mathfrak{X}_n / \mathfrak{X}_{n-1} \qquad (\mathfrak{X}_{-1} = 0) \tag{4.1}$$

be the associated graded module of $\mathrm{gr}\, U(\mathfrak{g}) = S(\mathfrak{g})$ (the symmetric algebra of $\mathfrak{g}$).

In general, let M be a finitely generated module over a Noetherian ring A. (In our present case, we take $A = S(\mathfrak{g})$ and $M = \operatorname{gr}\mathfrak{X}$.) Let $\{P_i\}_{i=1}^{l}$ be the set of all the minimal prime ideals containing the annihilator ideal $\operatorname{Ann} M = \{a \in A \mid aM = 0\}$. Clearly,

$$\operatorname{Supp} M := \operatorname{Spec}(\operatorname{Ann} M) = \bigcup\nolimits_{i=1}^{l} \operatorname{Spec}(A/P_i) \tag{4.2}$$

gives an irreducible decomposition of $\operatorname{Supp} M$. In addition to this, we associate a multiplicity $m_i = m(M, P_i)$ with each irreducible component $\operatorname{Spec}(A/P_i)$, where m_i is defined as the length of A_{P_i}-module M_{P_i}. Here A_{P_i} or M_{P_i} denotes the localization at P_i. Note that M_{P_i} is an Artinian A_{P_i}-module, so that the multiplicity is a positive integer. The *associated cycle* $\mathcal{AC}(M)$ of M is defined to be a formal sum

$$\mathcal{AC}(M) = \sum\nolimits_{i=1}^{l} m_i \cdot \operatorname{Spec}(A/P_i). \tag{4.3}$$

If $\mathcal{M}$ is a coherent sheaf over $\operatorname{Spec} A$ corresponding to M, then the support of $\mathcal{M}$ is $\operatorname{Supp} M$ above, and the usual notion of characteristic cycle $\operatorname{Ch}(\mathcal{M})$ coincides with $\mathcal{AC}(M)$.

Let us return to the $S(\mathfrak{g})$-module $\operatorname{gr}\mathfrak{X}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition, and we identify $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}]$ via Killing form, so that $\text{m-Spec}\, S(\mathfrak{g}) \simeq \mathfrak{g}$, where $\text{m-Spec}\, A$ denotes the set of maximum spectrum of A.

Theorem 4.1 (Vogan [18]). *The associated cycle $\mathcal{AC}(\operatorname{gr}\mathfrak{X})$ does not depend on the choice of the generating space $\mathfrak{X}_0$. We denote it by $\mathcal{AC}(\mathfrak{X})$. Then $\mathcal{AC}(\mathfrak{X})$ is a finite union of the closure of $K_\mathbb{C}$-nilpotent orbits in $\mathfrak{p}$ with multiplicity.*

$$\mathcal{AC}(\mathfrak{X}) = \sum_{i=1}^{l} m_i \cdot [\overline{\mathcal{O}_i}] \qquad (\mathcal{O}_i : \ K_\mathbb{C}\text{-nilpotent orbit in } \mathfrak{p}) \tag{4.4}$$

We call $\operatorname{Supp}(\operatorname{gr}\mathfrak{X}) = \cup_i \overline{\mathcal{O}_i}$ *the* associated variety *of $\mathfrak{X}$, and denote it by $\mathcal{AV}(\mathfrak{X})$.*

Proof. We skip the proof of independency of $\mathcal{AC}(\mathfrak{X})$ from the choice of the $K_\mathbb{C}$-stable generating space.

Since the filtration $\{\mathfrak{X}_n\}_{n=0}^{\infty}$ is K-stable, the action of $\mathfrak{k} = \operatorname{Lie}(K)_\mathbb{C}$ kills $\operatorname{gr}\mathfrak{X}$. Thus, in fact, $\operatorname{gr}\mathfrak{X}$ is an $S(\mathfrak{g}/\mathfrak{k})$-module. This means the support of $\operatorname{gr}\mathfrak{X}$ is contained in $\mathfrak{p} \simeq (\mathfrak{g}/\mathfrak{k})^*$. Moreover, there is an action of $K_\mathbb{C}$ on $\operatorname{Supp}(\operatorname{gr}\mathfrak{X})$ induced by $K_\mathbb{C}$-module structure of $\operatorname{gr}\mathfrak{X}$, hence $\mathcal{AV}(\mathfrak{X})$ is a union of $K_\mathbb{C}$-orbits.

Since $\mathfrak{X}$ is assumed to be quasi-simple, $\operatorname{gr}\mathfrak{Z}(\mathfrak{g}) = S(\mathfrak{g})^G$ acts on $\operatorname{gr}\mathfrak{X}$ trivially. Thus the invariants of positive degree $S(\mathfrak{g})_+^G$ kills $\operatorname{gr}\mathfrak{X}$. By the result of Kostant, it is known that $S(\mathfrak{g})_+^G$ generates a prime ideal, which is an annihilator ideal of the nilpotent variety. Thus $\operatorname{Supp}(\operatorname{gr}\mathfrak{X}) = \mathcal{AV}(\mathfrak{X})$ is contained in the nilpotent variety. □

We give some examples of associated cycles here.

Example 4.2. If τ is a finite dimensional representation of G, its associated cycle is supported on the point $\{0\}$. The multiplicity is given by the dimension $\dim\tau$.

Example 4.3 (Yamashita, N-Ochiai-Taniguchi). We will give the associated cycles of unitary highest/lowest weight modules. For details, we refer the readers to [13].

First we describe certain nilpotent orbits. Let $(G, G') = (O(p,q), Sp(2n, \mathbb{R}))$ be our type I dual pair. A choice of a maximal compact subgroup $K' \subset G'$ determines a complexified Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$. Since $G' = Sp(V')$ is a Hermitian symmetric type, there is a $K'_\mathbb{C}$-stable decomposition $\mathfrak{p}' = \mathfrak{p}'_+ \oplus \mathfrak{p}'_-$. One can identify $\mathfrak{p}'_+ = \operatorname{Sym}_n(\mathbb{C})$ (the space of symmetric matrices of order n), where $n = \frac{1}{2}\dim V'$ is the real rank of G'. The action of $K'_{\mathbb{C}} \simeq GL_n(\mathbb{C})$ is the usual one; $gX\,{}^tg$ ($g \in GL_n(\mathbb{C}), X \in \mathrm{Sym}_n(\mathbb{C})$). Then $\mathfrak{p}'_{\pm}$ is contained in the nilpotent variety, and the nilpotent $K'_{\mathbb{C}}$-orbits in $\mathfrak{p}'_{\pm}$ is classified by the rank of symmetric matrices. We denote the nilpotent orbit in $\mathfrak{p}'_{+}$ of rank k by $\mathcal{O}_k$. In particular, $\mathcal{O}_0 = \{0\}$ is the trivial orbit, and $\mathcal{O}_n = \{A \in \mathrm{Sym}_n(\mathbb{C}) \mid \det A \neq 0\}$ is dense open in $\mathfrak{p}'_{+}$.

Let us recall the decomposition (3.5). Thus unitary highest weight modules $L^+(\sigma)$ are parametrized by irreducible finite dimensional representations $\sigma \in O(V^+)^\wedge$ for various positive definite quadratic space V^+. Here (only in this example), we do *not* assume any condition between the dimensions of $V^{\pm}$ and V'. Therefore, $L^+(\sigma)$ need not be in holomorphic discrete series, but it can be an arbitrary unitary highest weight representation including singular unitary highest weight modules.

The associated cycle of $L^+(\sigma)$ ($\sigma \in O(V^+)^\wedge$) is given by

$$\mathcal{AC}(L^+(\sigma)) = \begin{cases} \dim\sigma \cdot [\overline{\mathcal{O}_p}] & \text{if } p = \dim V^+ \leq n \\ \dim\sigma^{O(p-n)} \cdot [\overline{\mathcal{O}_n}] & \text{if } p = \dim V^+ > n \end{cases} \tag{4.5}$$

Here $O(p-n)$ is embedded into $O(p)$ diagonally, and $\sigma^{O(p-n)}$ denotes $O(p-n)$-invariants in σ. Note that $\overline{\mathcal{O}_n} = \mathfrak{p}'_{+}$.

By a result of Yamashita ([21]), the multiplicity of $\mathcal{AC}(L^+(\sigma))$ is also interpreted as the dimension of the space of generalized Whittaker vectors. This is one of the motivation to calculate associated cycles.

5. Theta lift of associated cycles

First we recall the notion of the lifting of nilpotent orbits for symmetric pairs [14]:

$$\begin{aligned} G' = Sp(2n,\mathbb{R}) &\longrightarrow G = O(p,q), \\ \mathcal{N}(\mathfrak{p}') \supset \mathcal{O}' &\longrightarrow \mathcal{O} \subset \mathcal{N}(\mathfrak{p}), \end{aligned}$$

where for a subset $\mathfrak{s} \subset \mathfrak{g}$ we denote the set of nilpotent elements in $\mathfrak{s}$ by $\mathcal{N}(\mathfrak{s})$. We always assume that the pair (G, G') is strictly in the stable range, which amounts to assume that $2n < \min(p,q)$.

Now $W = \mathbb{R}^{p,q} \otimes \mathbb{R}^{2n}$ has a complex structure such that the imaginary part of the standard Hermitian form gives our symplectic form. With this complex structure, we consider W as a complex vector space:

$$W = M_{p+q,n}(\mathbb{C}) = \left\{ \begin{pmatrix} A \\ B \end{pmatrix} \mid A \in M_{p,n}(\mathbb{C}), B \in M_{q,n}(\mathbb{C}) \right\} = M_{p,n}(\mathbb{C}) \oplus M_{q,n}(\mathbb{C}).$$

Then the action of $K_{\mathbb{C}} = O(p,\mathbb{C}) \times O(q,\mathbb{C})$ and $K'_{\mathbb{C}} = GL_n(\mathbb{C})$ on W can be given as

$$\begin{pmatrix} kA\,{}^tg \\ hBg^{-1} \end{pmatrix}; \qquad \begin{pmatrix} A \\ B \end{pmatrix} \in W,\ (k,h) \in O(p,\mathbb{C}) \times O(q,\mathbb{C}),\ g \in GL_n(\mathbb{C}).$$

We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (resp. $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$) as

$$\mathfrak{g} = \mathfrak{o}(p+q,\mathbb{C}) = \left(\begin{array}{c|c} \mathrm{Alt}_p(\mathbb{C}) & 0 \\ \hline 0 & \mathrm{Alt}_q(\mathbb{C}) \end{array}\right) \oplus \left(\begin{array}{c|c} 0 & M_{p,q}(\mathbb{C}) \\ \hline {}^tM_{p,q}(\mathbb{C}) & 0 \end{array}\right) = \mathfrak{k} \oplus \mathfrak{p},$$

$$\mathfrak{g}' = \mathfrak{sp}(2n,\mathbb{C}) = \left(\begin{array}{c|c} M_n(\mathbb{C}) & 0 \\ \hline 0 & -{}^tM_n(\mathbb{C}) \end{array}\right) \oplus \left(\begin{array}{c|c} 0 & \mathrm{Sym}_n(\mathbb{C}) \\ \hline \mathrm{Sym}_n(\mathbb{C}) & 0 \end{array}\right) = \mathfrak{k}' \oplus \mathfrak{p}'.$$

Thus, we can identify $\mathfrak{p} = M_{p,q}(\mathbb{C})$ and $\mathfrak{p}' = \mathfrak{p}'_+ \oplus \mathfrak{p}'_- = \mathrm{Sym}_n(\mathbb{C}) \oplus \mathrm{Sym}_n(\mathbb{C})$. To define the lifting, we consider the following double fibration map

$$\begin{tikzcd} & W = M_{p,n} \oplus M_{q,n} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ \mathfrak{p} = M_{p,q} & & \mathrm{Sym}_n \oplus \mathrm{Sym}_n = \mathfrak{p}' \end{tikzcd}$$

where the moment maps φ and ψ are explicitly given by

$$(A,B) \in M_{p,n} \oplus M_{q,n} = W,$$
$$\begin{cases} \varphi(A,B) = A\,{}^tB \in M_{p,q} = \mathfrak{p}, \\ \psi(A,B) = ({}^tAA, {}^tBB) \in \mathrm{Sym}_n \oplus \mathrm{Sym}_n = \mathfrak{p}'. \end{cases}$$

These maps are equivariant quotient maps onto their images. For example, φ is a quotient map by $GL_n(\mathbb{C})$ onto its image (rank $\leq n$ matrices in $M_{p,q}(\mathbb{C})$), and it is $K_{\mathbb{C}}$-equivariant. Note that ψ is surjective by our assumption that the pair is strictly in the stable range.

The following theorem is established in [14]. It is also obtained by Ohta [16] and Daszkiewicz-Kraśkiewicz-Przebinda [1] independently.

Theorem 5.1. *Take a nilpotent $K'_{\mathbb{C}}$-orbit $\mathcal{O}'$ in $\mathfrak{p}'$. The push-down of the inverse image $\varphi(\psi^{-1}(\overline{\mathcal{O}'}))$ of the closure of $\mathcal{O}'$ is equal to the closure of a nilpotent $K_{\mathbb{C}}$-orbit $\overline{\mathcal{O}}$. This gives a one-to-one correspondence from the set of nilpotent $K'_{\mathbb{C}}$-orbits in $\mathfrak{p}'$ to the set of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}$.*

We write this correspondence as $\mathcal{O} = \theta(\mathcal{O}')$, and call it the *theta lift* of $\mathcal{O}'$. For associated cycles we can extend the theta lifting by

$$\theta(\textstyle\sum_i m_i[\overline{\mathcal{O}'_i}]) = \sum_i m_i[\overline{\theta(\mathcal{O}'_i)}]. \tag{5.1}$$

We can now state our main theorem.

Theorem 5.2 (N-Zhu). *Let (G,G') be a reductive dual pair of type I. We assume that the pair is strictly in the stable range with G' the smaller member, and that G' is of Hermitian type (see Table 1 in §3). Let π' be a genuine unitary highest weight representation of $\widetilde{G'}$ which appears in the Howe correspondence of a compact dual pair. Then the associated cycle is preserved by the theta lifting.*

$$\theta(\mathcal{AC}(\pi')) = \mathcal{AC}(\theta(\pi')) \tag{5.2}$$

More precisely, if the associated cycle of π' *is given by* $\mathcal{AC}(\pi') = m_{\pi'}[\overline{\mathcal{O}'}]$, *then* $\mathcal{AC}(\theta(\pi')) = m_{\pi'}[\overline{\theta(\mathcal{O}')}]$ *with the same multiplicity.*

Some remarks are in order.

First, for the pairs $(O(p,q), Sp(2n,\mathbb{R}))$ and $(U(p,q), U(m,n))$, all the unitary highest weight module of $\widetilde{G'}$ appears in the Howe correspondence for some compact dual pairs. This is proved in [7]. However, for the pair $(Sp(p,q), O^*(2n))$, there are small exception. See [2].

Second, $\mathcal{AC}(\pi')$ is well-understood; $\mathcal{AV}(\pi')$ is irreducible, and the multiplicity $m_{\pi'}$ can be given by the dimension of certain subspace of representations of compact groups. See Example 4.3.

Third, if π' is a singular unitary highest weight representation, the formula of $\mathcal{AC}(\pi')$ is also interpreted as the preservation of the associated cycle under the theta lifting. In fact, π' is the theta lift of a finite dimensional representation in the stable range. However, if π' is not singular, we can see that the associated cycle is no longer preserved by the theta lifting. Thus the assumption of the stable range condition is necessary.

For the proof of this theorem, we refer to [15].

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