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# Spherical functions on certain spherical homogeneous spaces over p-adic fields

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## §0 Introduction.

Throughout this paper, let $k$ be a $\mathfrak{p}$-adic field. Let $\mathbb{G}$ be an algebraic group defined over $k$, $G = \mathbb{G}(k)$, $K$ a special good maximal bounded subgroup of $G$, $\mathbb{X}$ a $\mathbb{G}$-homogeneous affine algebraic variety defined over $k$, and $X = \mathbb{X}(k)$. We write the action of $\mathbb{G}$ on $\mathbb{X}$ by $(g, x) \longmapsto g \star x$. Denote by $\mathcal{C}^\infty(K\backslash X)$ the set of left $K$-invariant $\mathbb{C}$-valued functions on $X$. The Hecke algebra $\mathcal{H}(G, K)$ acts on $\mathcal{C}^\infty(K\backslash X)$ from the left by the convolution product, which we write $(f, \Psi) \longmapsto f * \Psi$. A nonzero function $\Psi \in \mathcal{C}^\infty(K\backslash X)$ is called *a spherical function* if it is an $\mathcal{H}(G, K)$-common eigenfunction, which means, there exists a $\mathbb{C}$-algebra map $\lambda : \mathcal{H}(G, K) \longrightarrow \mathbb{C}$ satisfying

$$f * \Psi = \lambda(f)\Psi \quad \text{for } f \in \mathcal{H}(G, K).$$

Spherical functions are very interesting objects to investigate. The explicit expressions of spherical functions on $p$-adic groups have been given by I.G.Macdonald [Mac]. Later on, W.Casselman has reformulated them by representation theoretical method ([Cas]), for which there is an interpretative article written by P.Cartier([Car]). W. Casselman and J.Shalika carried forward this method to obtain explicit expressions of Whittaker functions associated to $p$-adic reductive group ([CasS]).

F.Sato and the author have investigated spherical functions on certain symmetric spaces; the space of alternating forms ([HS1]) and the spaces of hermitian and symmetric forms ([H1]-[H3]). In these cases, spherical functions can be regarded as generating functions of local densities of representations of forms by forms of the same kind. Hence, as an application, explicit formulas of local densities have been given(• [HS1], [HS2], [H3], [H4]).

In a similar method to [CasS], S. Kato has announced explicit expressions for spherical functions on certain spherical homogeneous spaces obtained by general linear groups([K2]), and S.Kato, A.Murase and T.Sugano have obtained explicit expressions for Whittaker-Shintani functions (spherical functions ) of certain spherical homogeneous spaces obtained by special orthogonal groups([KMS]). For the spaces which they investigated, the

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space of spherical functions attached to each Satake parameter, in other words, corresponding to each eigenvalue, is of dimension 1.

On the other hand, in a similar method to [Cas], the author has given an expression of spherical functions of certain spherical homogeneous spaces for which the dimension of the space of spherical functions is not necessarily one ([H3, Proposition 1.9] ), and applied it to the space of unramified hermitian forms and given the explicit expression of spherical functions (the dimension is $2^n$ according to the size $n$ of forms). This result has also used by K.Takano and S.Kato to give an explicit expression of spherical functions for the space $GL(n,k')/GL(n,k)$, where $k'$ is an unramified quadratic extension of $k$. In this case the space of spherical functions has dimension one([Tak]).

In the following, we investigate spherical functions on the following space:

$$\mathbf{G} = Sp_2 \times (Sp_1)^2, \qquad \mathbf{X} = Sp_2,$$

where $(Sp_1)^2$ is imbedded into $Sp_2$ and the action is given by

$$\tilde{g} \star x = g_1 x^t g_2, \quad \text{for } \tilde{g} = (g_1, g_2) \in Sp_2 \times (Sp_1)^2,\ x \in Sp_2,$$

(for the precise definition, see the beginning of Section 1). This $\mathbf{X}$ is a spherical homogeneous $\mathbf{G}$-space, which means $\mathbf{X}$ has a Zariski open orbit for a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$, and $\mathbf{X}$ is not a $\mathbf{G}$-symmetric space.

For this case, we will use the same result in [H3] in order to obtain a explicit formula of spherical functions. The space of spherical functions attached to each Satake parameter is of dimension 4. In [KMS], $SO(n) \times SO(n-1)$-space $SO(n)$ is considered, which is spherical and has an open Borel orbit over $k$ for every $n$, and the case when $n = 5$ is isogeneous to the present case. But there seems to have no direct correspondence between respective explicit formulas of spherical functions. Finally, $Sp_{2n} \times (Sp_n)^2$-space $Sp_{2n}$ is no longer spherical for $n \geq 2$.

We shall give a brief summary of our results. Taking a set $\{d_i \mid 1 \leq i \leq 4\}$ of basic relative $\mathbf{B}$-invariants (cf. (1.5)) and characters $\chi$ of $k^\times/(k^\times)^2$, we construct typical spherical functions (cf. (1.6))

$$\omega(x;\chi;s) = \int_K \chi(\prod_{i=1}^{4} d_i(k\star x)) \prod_{i=1}^{4} |d_i(k\star x)|^{s_i}\, dk, \qquad (x \in X,\ s \in \mathbf{C}^4),$$

where $|\ |$ is the absolute value on $k$ and $dk$ is the Haar measure on $K$, and the integral of the right hand side is absolutely convergent if $\mathrm{Re}(s_i) \geq 0$ $(1 \leq i \leq 4)$ and analytically continued to a rational function in $q^{s_1}, \ldots, q^{s_4}$, where $q$ is the residual number of $k$. We introduce a new variable $z$ related to $s$ by

$$\begin{aligned} z_1 &= s_1 + s_2 + s_3 + s_4 + 2, & z_2 &= s_3 + s_4 + 1, \\ z_3 &= s_1 + s_3 + 1, & z_4 &= s_2 + s_3 + 1, \end{aligned}$$

and write $\omega(x;\chi;z)$ in stead of $\omega(x;\chi;s)$.

These $\omega(x;\chi;z)$ are $\mathcal{H}(G,K)$-common eigenfunctions correspond to the same $\mathbb{C}$-algebra homomorphism $\lambda_z : \mathcal{H}(G,K) \longrightarrow \mathbb{C}$, which gives the Satake transform

$$\lambda_z : \mathcal{H}(G,K) \xrightarrow{\sim} \mathbb{C}[q^{\pm z_1}, q^{\pm z_2}, q^{\pm z_3}, q^{\pm z_4}]^W \quad \text{(Proposition 1.1)},$$

where $W$ is the Weyl group of $\mathbb{G}$.

Under the assumption that $k$ has odd residual characteristic, our main results are the following.

[1] To give a complete se of representatives of $K$-orbits in $X$ (Theorem 1).

[2] For each $\chi$, to give a rational function $F_\chi(z)$ for which$F_\chi(z)\cdot\omega(x;\chi;z)$ belongs to $\mathbb{C}[q^{\pm\frac{z_1}{2}}, q^{\pm\frac{z_2}{2}}, q^{\pm\frac{z_3}{2}}, q^{\pm\frac{z_4}{2}}]$ and $W$-invariant (Theorem 2).

[3] To give an explicit formula for $\omega(x;\chi;z)$ (Theorem 3).

[4] Employing spherical functions as kernel function, we give an $\mathcal{H}(G,K)$-module isomorphism (spherical transform)

$$\mathcal{S}(K\backslash X) \xrightarrow{\sim} \left(\mathbb{C}[q^{\pm z_1}, q^{\pm z_2}, q^{\pm z_3}, q^{\pm z_4}]^W \oplus \prod_{i=1}^{4}(q^{\frac{z_i}{2}} + q^{-\frac{z_i}{2}})\cdot\mathbb{C}[q^{\pm z_1}, q^{\pm z_2}, q^{\pm z_3}, q^{\pm z_4}]^W\right)^2.$$

Especially, $\mathcal{S}(K\backslash X)$ is a free $\mathcal{H}(G,K)$-module of rank 4, and we give a free basis (Theorem 4).

[5] Eigenvalues for spherical functions are parametrized by $z \in \left(\mathbb{C}/\frac{2\pi\sqrt{-1}}{\log q}\mathbb{Z}\right)^4/W$. The space of spherical functions on $X$ corresponding to $z \in \mathbb{C}^4$ has dimension 4 and a basis is given explicitly (Theorem 5).

Professor S. Böcherer has suggested to the author the significance of the investigation of this space $Sp_2$ from the view point of its relation to the global Gross-Prasad conjecture for $SO(5)$ (cf. [GR]). The explicit Hecke module structure of the Schwartz space of it would be helpful for the question whether the vanishing of the period integral on spherical vectors implies the vanishing of the period integral on the full modular representation space. The author would like to express her gratitude to him for these useful discussion.

Notation: Throughout this paper, we denote by $k$ a nonarchimedian local field of characteristic 0. Denote by $\mathcal{O}$ the ring of integers in $k$, $\mathfrak{p}$ the maximal ideal in $\mathcal{O}$, $\pi$ a fixed prime element of $k$, $q$ the cardinality of $\mathcal{O}/\mathfrak{p}$ and $|\ |$ the normalized absolute value on $k$. For convenience of notation, we understand $|0|^s = 0$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. For an algebraic set $\mathbb{Y}$ defined over $k$, we use the corresponding letter $Y$ for the set of $k$-rational points $\mathbb{Y}(k)$.

As usual, we denote by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$, respectively, the complex number field, the real number field, the rational number field, and the set of natural numbers.

# §1 The spherical homogeneous space $Sp_2$.

Set

$$Sp_n = \left\{ x \in GL_{2n} \,\middle|\, {}^t x J_n x = J_n \right\}, \qquad J_n = \left( \begin{array}{c|c} & 1_n \\ \hline -1_n & \end{array} \right), \tag{1.1}$$

and let $\mathbb{G} = Sp_2 \times (Sp_1)^2$ and we embed $(Sp_1)^2 = (SL_2)^2$ into $Sp_2$ by

$$\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right) \longmapsto \left( \begin{array}{cc|cc} a & & b & \\ & e & & f \\ \hline c & & d & \\ & g & & h \end{array} \right).$$

Hereafter, we understand empty places in matrices mean 0-entries.

Take $\mathbf{X} = Sp_2$, and consider the action of $\mathbb{G}$ on $\mathbf{X}$ defined by

$$\tilde{g} \star x = g_1 x^t g_2, \qquad \tilde{g} = (g_1, g_2) \in \mathbb{G},\ x \in \mathbf{X}.$$

We set the Borel subgroup $\mathbb{B} = \mathbb{B}_1 \times \mathbb{B}_2$ of $\mathbb{G}$ by

$$\mathbb{B}_1 = \begin{pmatrix} * & * & & \\ 0 & * & & * \\ & & * & 0 \\ \multicolumn{2}{c}{0} & * & * \end{pmatrix} \subset Sp_2, \quad \mathbb{B}_2 = \left( \begin{array}{cc|cc} * & & 0 & \\ & * & & 0 \\ \hline * & & * & \\ & * & & * \end{array} \right) \subset (Sp_1)^2. \tag{1.2}$$

Let us write an element $\mathbf{b} \in \mathbb{B}$ as

$$\mathbf{b} = \left( \left( \begin{array}{cc|cc} * & * & & \\ & * & & \\ \hline & & b_1 & 0 \\ & & c & b_2 \end{array} \right) \left( \begin{array}{cc|cc} 1 & & x_1 & x_2 \\ & 1 & x_2 & x_3 \\ \hline \multicolumn{2}{c|}{0} & 1 & \\ & & & 1 \end{array} \right), \left( \begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline y_1 & & 1 & \\ & y_2 & & 1 \end{array} \right) \left( \begin{array}{cc|cc} b_3 & & & \\ & b_4 & & \\ \hline & & * & \\ & & & * \end{array} \right) \right),$$

where the entries at marked $*$ are automatically determined. Then the left invariant Haar measure on $\mathbb{B}(k)$ is given by

$$d\mathbf{b} = \frac{|b_3|\,|b_4|}{|b_1|\,|b_2|^2} \cdot |db_1|\,|db_2|\,|dc|\,|dx_1|\,|dx_2|\,|dx_3|\,|db_3|\,|db_4|\,|dy_1|\,|dy_2| \tag{1.3}$$

and the modulus character $\delta$ ( $d(bb') = \delta^{-1}(b')db$) is $\delta(b) = |b_1|^{-4}\,|b_2|^{-2}\,|b_3|^{-2}\,|b_4|^{-2}$.

Let $W = W_1 \times W_2$ be the Weyl group of $\mathbb{G}$ with respect to the maximal torus consisting of diagonal matrices in $\mathbb{G}$, which is isomorphic to $(C_2 \rtimes (C_2)^2) \times (C_2)^2$, and we fix generators $\{w_i \mid 1 \le i \le 4\}$ of $W$ by their action on the maximal torus

$$w_i : (b_1, b_2, b_3, b_4) \longmapsto \begin{cases} (b_2, b_1, b_3, b_4) & \text{if } i = 1 \\ (b_1, b_2^{-1}, b_3, b_4) & \text{if } i = 2 \\ (b_1, b_2, b_3^{-1}, b_4) & \text{if } i = 3 \\ (b_1, b_2, b_3, b_4^{-1}) & \text{if } i = 4. \end{cases} \tag{1.4}$$

A set of basic relative $\mathbb{B}$-invariants and corresponding characters of $\mathbb{B}$ is given as follows. Let $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{X}$ with 2 by 2 matrices $A, B, C$ and $D$ and we write $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in M_2$ for simplicity. Set

$$\begin{aligned} d_1(x) &= C_1, & \phi_1(\mathbf{b}) &= b_1 b_3 \\ d_2(x) &= C_2, & \phi_2(\mathbf{b}) &= b_1 b_4 \\ d_3(x) &= \det C = C_1C_4 - C_2C_3, & \phi_3(\mathbf{b}) &= b_1 b_2 b_3 b_4 \\ d_4(x) &= (\det C\,(C^{-1}D))_3 = C_1D_3 - C_3D_1, & \phi_4(\mathbf{b}) &= b_1 b_2, \end{aligned} \tag{1.5}$$

then $\{d_i \mid 1 \leq i \leq 4\}$ forms a basis for relative $\mathbf{B}$-invariants and $\mathfrak{X}(\mathbf{B}) = < \phi_i \mid 1 \leq i \leq 4 >$ becomes the group of rational characters of $\mathbf{B}$ which corresponds to relative $\mathbf{B}$-invariants.

Let $K = \mathbb{G}(\mathcal{O})$ and $\mathcal{H}(G, K)$ be the Hecke algebra of $G = \mathbb{G}(k)$ with respect to $K$. We consider the following integral. For $x \in X$, $s \in \mathbb{C}^4$ and a character $\chi$ of $k^\times / (k^\times)^2$,

$$\omega(x; s; \chi) = \int_K \chi(\prod_{i=1}^4 d_i(k \star x)) \prod_{i=1}^4 |d_i(k \star x)|^{s_i}\, dk, \tag{1.6}$$

where $dk$ is the normalized Haar measure on $K$. The right hand of (1.6) is absolutely convergent for $\mathrm{Re}(s_i) \geq 0$ $(1 \leq i \leq 4)$ and analytically continued to rational functions in $q^{s_1}, \dots, q^{s_4}$, which is a $\mathcal{H}(G, K)$-common eigenfunction with respect to the convolution product (cf. [H3, Remark 1.1, Proposition 1.1 ] ).

It is convenient to introduce a new variable $z$ which is related to $s$ as follows

$$\begin{cases} z_1 = s_1 + s_2 + s_3 + s_4 + 2 \\ z_2 = s_3 + s_4 + 1 \\ z_3 = s_1 + s_3 + 1 \\ z_4 = s_2 + s_3 + 1, \end{cases} \qquad \begin{cases} s_1 = \frac{1}{2}(z_1 - z_2 + z_3 - z_4 - 1) \\ s_2 = \frac{1}{2}(z_1 - z_2 - z_3 + z_4 - 1) \\ s_3 = \frac{1}{2}(-z_1 + z_2 + z_3 + z_4 - 1) \\ s_4 = \frac{1}{2}(z_1 + z_2 - z_3 - z_4 - 1), \end{cases} \tag{1.7}$$

and we write also

$$\omega(x; \chi; s) = \omega(x; \chi; z),$$

if there is no danger of confusion. It is easy to see

$$\prod_{i=1}^4 |d_i(bg \star x)|^{s_i} = (\xi \delta^{\frac{1}{2}})(b) \cdot \prod_{i=1}^4 |d_i(g \star x)|^{s_i}, \qquad (b \in B, g \in G, x \in X),$$

where

$$\xi(b) = |b_1|^{s_1+s_2+s_3+s_4+2}\, |b_2|^{s_3+s_4+1}\, |b_3|^{s_1+s_3+1}\, |b_4|^{s_2+s_3+1} = |b_1|^{z_1}\, |b_2|^{z_2}\, |b_3|^{z_3}\, |b_4|^{z_4}$$

for $b = \left(\left(\begin{array}{cc|cc} * & & * & \\ \hline 0 & & b_1 & \\ & & & b_2 \end{array}\right), \left(\begin{array}{cc|c} b_3 & & 0 \\ & b_4 & \\ \hline & * & * \end{array}\right)\right) \in B$. The Weyl group $W$ acts on the set $\{z_1, z_2, z_3, z_4\}$ through its action on the character $\xi$ of $B$, and we have

$$w_i(z_1, z_2, z_3, z_4) = \begin{cases} (z_2, z_1, z_3, z_4) & \text{for } i = 1 \\ (z_1, -z_2, z_3, z_4) & \text{for } i = 2 \\ (z_1, z_2, -z_3, z_4) & \text{for } i = 3 \\ (z_1, z_2, z_3, -z_4) & \text{for } i - 4. \end{cases} \tag{1.8}$$

The following statements can be calculated directly, though they are a special case of Satake transform of algebraic groups [Si] and spherical functions on homogeneous spaces [H3, Proposition 1.1].

**Proposition 1.1** *For every $f \in \mathcal{H}(G, K)$, let*

$$\tilde{f}(z) = \int_G f(g)\xi^{-1}\delta^{\frac{1}{2}}(p(g))dg,$$

*where $dg$ is the Haar measure on $G$ normalized by $\int_K dg = 1$ and $g = p(g)k \in G = BK$. Then, by the map $f \longmapsto \tilde{f}(z)$, we have*

$$\mathcal{H}(G, K) \cong \mathbb{C}[q^{z_1} + q^{-z_1} + q^{z_2} + q^{-z_2},\ (q^{z_1} + q^{-z_1})(q^{z_2} + q^{-z_2}),\ q^{z_3} + q^{-z_3},\ q^{z_4} + q^{-z_4}],$$

*and for every $f \in \mathcal{H}(G, K)$*

$$(f * \omega(\ ; \chi; z))(x) = \tilde{f}(z) \cdot \omega(x; \chi; z) \qquad (x \in X).$$

We recall the Bruhat decomposition of $\mathbf{X} = Sp_2$

$$\mathbf{X} = \bigsqcup_{w \in W_1} \mathbf{B}_1 w \mathbf{B}_1, \tag{1.9}$$

where $W_1$ is the Weyl group of $Sp_2$ and the symbol $\sqcup$ means disjoint union. It is easy to see that

$$\mathbf{B}_1 = \bigsqcup_{s,t} E_{s,t}\mathbf{B}_S, \qquad \text{with } \mathbf{B}_S = {}^t\mathbf{B}_2,\ E_{s,t} = \left(\begin{array}{cc|cc} 1 & s & st & t \\ & 1 & t & \\ \hline & & 1 & \\ & & -s & 1 \end{array}\right),$$

where $s, t$ runs over the algebraic closure $\bar{k}$ of $k$, so we get for each $w \in W_1$ that

$$\mathbf{B}_1 w \mathbf{B}_1 = \bigcup_{s,t} \mathbf{B}_1 w E_{s,t} \mathbf{B}_S = \bigcup_{s,t} \mathbf{B} \star w E_{s,t}\ . \tag{1.10}$$

$$w_0 = \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & -1 & & \end{array}\right) \ ( = w_2 w_1 w_2 w_1 \in W).$$

The following Proposition tells us that our space is spherical homogeneous.

**Proposition 1.2** *The set*

$$\mathbb{Y} = \left\{ x \in \mathbb{X} \;\middle|\; \prod_{i=1}^{4} d_i(x) \neq 0 \right\}$$

*is an open* $\mathbb{B}$*-orbit over the algebraic closure of* $k$

$$\mathbb{Y} = \mathbb{B} \star x_0 \qquad \textit{with} \quad x_0 = \left(\begin{array}{cc|cc} & & 1 & 0 \\ & & 1 & 1 \\ \hline -1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right) (= w_0 E_{-1,-1}).$$

*Further, the $B$-orbit decomposition of the set of $k$-rational points in $\mathbb{Y}$ is given by*

$$\mathbb{Y}(k) = \bigsqcup_{u \in k^\times/(k^\times)^2} Y_u,$$

*where*

$$Y_u = \left\{ x \in X \;\middle|\; \prod_{i=1}^{4} d_i(x) \equiv u \bmod (k^\times)^2 \right\} \ni w_0 E_{-1,-u} = \left(\begin{array}{cc|cc} & 0 & 1 & 0 \\ & & 1 & 1 \\ \hline -1 & 1 & -u & u \\ 0 & -1 & u & 0 \end{array}\right).$$

**Remark.** By Proposition 1.2 and the injectivity of Poisson integral (cf. [K1]), we see that $\omega(x; \chi; z)$ is not identically zero for generic $z$ and linearly independent for characters $\chi$. Indeed, we will see that the space of spherical functions has dimension 4 and we give a basis by modifying $\omega(x; \chi; z)$ for various $\chi$ (cf. Theorem 5 in Section 5).

Before closing this section, we confirm the assumption (A2) of [H3]. Denote by $\mathbb{H}$ the stabilizer $\mathbb{G}_{x_0}$ of $x_0$ in $\mathbb{G}$ and consider the action of $\mathbb{B} \times \mathbb{H}$ on $\mathbb{G}$ by

$$(b, h) * g = bgh^{-1} \quad (b, h) \in \mathbb{B} \times \mathbb{H}, \ g \in \mathbb{G},$$

then $\mathbb{X} \cong \mathbb{G}/\mathbb{H}$ as $\mathbb{G}$-sets. Further, we see that $\mathbb{B}\mathbb{H} = (\mathbb{B} \times \mathbb{H}) * 1$ is an open orbit in $\mathbb{G}$ and $\mathbb{G}$ is decomposed into a finite number of $\mathbb{B} \times \mathbb{H}$-orbits.

For $g \in \mathbb{G}$, denote by $\mathbb{B}_{(g)}$ the image of the stabilizer $(\mathbb{B} \times \mathbb{H})_g$ by the projection $\mathbb{B} \times \mathbb{H} \longrightarrow \mathbb{B}$. Then we have

**Lemma 1.3** *For each $g \in \mathbb{G}$, $g \notin \mathbb{B}\mathbb{H}$, there exists a rational character in $\mathfrak{X}(\mathbb{B})$ which is nontrivial on $\mathbb{B}_{(g)}$.*

## §2 Cartan decomposition

Hereafter we assume that $k$ has odd residual characteristic. In this section we consider "Cartan decomposition" of $X$, that is we give a complete set of representatives of $K$-orbits in $X$.

To state the result, we introduce some notation: Let

$$\begin{aligned} \Lambda &= \left\{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\in \mathbb{Z}^4\cup(\tfrac{1}{2}+\mathbb{Z})^4 \,\middle|\, \lambda_1\ge\lambda_2\ge 0,\ \lambda_3\ge 0,\ \lambda_4\ge 0\right\}, \\ \Lambda_* &= \{\lambda\in\Lambda \mid \lambda_1>\lambda_2>0,\ \lambda_3>0,\ \lambda_4>0\}, \end{aligned} \tag{2.1}$$

and for $\lambda\in\Lambda$ and $\xi\in\mathcal{O}^\times$ set

$$\pi_{(\lambda;\xi)} = \left(\begin{array}{cc|cc} & & -\pi^{\lambda_1+\lambda_3} & \\ & & \xi\pi^{\lambda_2+\lambda_3} & -\pi^{\lambda_2+\lambda_4} \\ \hline \pi^{-\lambda_1-\lambda_3} & \xi\pi^{-\lambda_1-\lambda_4} & \xi\pi^{-\lambda_1+\lambda_3} & \pi^{-\lambda_1+\lambda_4} \\ & \pi^{-\lambda_2-\lambda_4} & \pi^{-\lambda_2+\lambda_3} & \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} \pi^{\lambda_1} & & & \\ & \pi^{\lambda_2} & & \\ \hline & & \pi^{-\lambda_1} & \\ & & & \pi^{-\lambda_2} \end{array}\right) \left(\begin{array}{cc|cc} & & -1 & \\ & & \xi & -1 \\ \hline 1 & \xi & \xi & 1 \\ & 1 & 1 & \end{array}\right) \left(\begin{array}{cc|cc} \pi^{-\lambda_3} & & & \\ & \pi^{-\lambda_4} & & \\ \hline & & \pi^{\lambda_3} & \\ & & & \pi^{\lambda_4} \end{array}\right)$$

Then our main result is the following.

**Theorem 1** *Let*

$$\widetilde{\mathcal{R}} = \left\{ \pi_{(\lambda;\xi)} \,\middle|\, \begin{array}{l} \lambda\in\Lambda,\ \xi\in\mathcal{O}^\times/(\mathcal{O}^\times)^2 \\ \xi = 1 \text{ unless } \lambda\in\Lambda_* \end{array} \right\},$$

*then $\widetilde{\mathcal{R}}$ makes a complete set of representatives of $K$-orbits in $X$.*

In order to prove Theorem 1, we first construct another complete set of representatives. We introduce some more notation. Set $K_1 = Sp_2(\mathcal{O})$ and $K_2 = (Sp_1(\mathcal{O}))^2 (\subset K_1)$, then it suffices to consider the representatives of double cosets in the space $K_1\backslash X/K_2$. Set

$$\begin{aligned} T_{(x,y,z,w)} &= \left(\begin{array}{cc|cc} x^{-1} & -x^{-1}y^{-1}z & & \\ & y^{-1} & & \\ \hline & & x & \\ & & z & y \end{array}\right) \left(\begin{array}{c|cc} 1_2 & & w \\ & w & \\ \hline & & 1_2 \end{array}\right) \\ &= \left(\begin{array}{cc|cc} x^{-1} & -x^{-1}y^{-1}z & -x^{-1}y^{-1}zw & x^{-1}w \\ & y^{-1} & y^{-1}w & \\ \hline & & x & \\ & & z & y \end{array}\right). \end{aligned}$$

and for $a,b,c,d\in\mathbb{Z}$ and $\varepsilon\in\mathcal{O}^\times$, set

$$\begin{aligned} A_{(a,b)} &= T_{(\pi^a,\pi^b,0,0)}, & B_{(a,b,c)} &= T_{(\pi^a,\pi^b,\pi^c,0)}, \\ C_{(a,b,d)} &= T_{(\pi^a,\pi^b,0,\pi^d)}, & D_{(a,b,c,d;\varepsilon)} &= T_{(\pi^a,\pi^b,\varepsilon\pi^c,\pi^d)}. \end{aligned}$$

**Proposition 2.1** *The set* $\mathcal{R} = \bigsqcup_{i=1}^{4} \mathcal{R}_i$ *is a complete set of representatives of* $K\backslash X$, *where*

$$\begin{aligned}
\mathcal{R}_1 &= \left\{ A_{(a,b)} \,\middle|\, a \geq 0,\ b \geq 0 \right\}, \quad \mathcal{R}_2 = \left\{ B_{(a,b,c)} \,\middle|\, a > c \geq 0,\ b \geq 0 \right\}, \\
\mathcal{R}_3 &= \left\{ C_{(a,b,d)} \,\middle|\, \begin{array}{l} a > 0,\ b > 0,\ a+b > d \geq 0 \\ a \geq b \ \text{if}\ d = 0 \end{array} \right\}, \\
\mathcal{R}_4 &= \left\{ D_{(a,b,c,d;\varepsilon)} \,\middle|\, \begin{array}{l} a > c,\ b+c > d,\ b+d > c,\ c+d > b \\ \varepsilon \in \mathcal{O}^\times/(\mathcal{O}^\times)^2 \end{array} \right\}.
\end{aligned}$$

**Remark 2.1.** (1) One proves that every $K$-orbit has a representative in the set $\mathcal{R}$ by Lemmas 2.2 and 2.3. It is possible but tedious to show directly that there occurs no $K$-equivalence within $\mathcal{R}$, so we take another way.

We will see (in Corollary 5.3) that spherical functions $\omega(x, \chi, z)$ take different values at each element of $\mathcal{R}$, by using their explicit formulas. Since spherical functions are $K$-invariant function, it means that each element in $\mathcal{R}$ belongs to the different $K$-orbit in $X$, and we see that $\mathcal{R}$ is a complete set of representatives of $K$-orbit of $X$. Thus we establish Proposition 2.1.

(2) The set $\mathcal{R}_4$ corresponds bijectively to the set

$$\widetilde{\mathcal{R}_*} = \left\{ \pi_{(\lambda;\xi)} \,\middle|\, \lambda \in \Lambda_*,\ \xi \in \mathcal{O}^\times/(\mathcal{O}^\times)^2 \right\}. \tag{2.2}$$

(3) In a direct calculation, the assumption on the residual characteristic is needed only for the proof that there occurs no $K$-equivalence within $\mathcal{R}_4$. For the even residual characteristic case, we have to choose a suitable subset within $\mathcal{R}_4$ (or within $\widetilde{\mathcal{R}_*}$).

**Lemma 2.2** *Set* $\mathcal{R}' = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}'_3 \cup \mathcal{R}'_4$ *with*

$$\begin{aligned}
\mathcal{R}'_3 &= \left\{ C_{(a,b,d)} \,\middle|\, a \geq 0,\ b \geq 0,\ d \geq 0 \right\}, \\
\mathcal{R}'_4 &= \left\{ D_{(a,b,c,d;\varepsilon)} \,\middle|\, a > c \geq 0,\ b \geq 0,\ d \geq 0,\ \varepsilon \in \mathcal{O}^\times/(\mathcal{O}^\times)^2 \right\}.
\end{aligned}$$

*Then every* $K$*-orbit in* $X$ *has a representative in* $\mathcal{R}'$.

**Lemma 2.3** *Because of the following relations, one can replace* $\mathcal{R}'_3$ *and* $\mathcal{R}'_4$ *by* $\mathcal{R}_3$ *and* $\mathcal{R}_4$, *respectively.*

$$C_{(a,b,d)} \sim_K A_{(a,b)} \quad \text{if}\ d \geq a+b. \tag{2.3}$$

$$C_{(a,0,d)} \sim_K B_{(a,0,d)}. \tag{2.4}$$

$$C_{(0,b,d)} \sim_K B_{(b-d,d,0)} \quad \text{if}\ b \geq d. \tag{2.5}$$

$$C_{(a,b,0)} \sim_K C_{(b,a,0)}. \tag{2.6}$$

$$D_{(a,b,c,d;\varepsilon)} \sim_K B_{(a,b,d)} \quad \text{if}\ d \geq b+c. \tag{2.7}$$

$$D_{(a,b,c,d;\varepsilon)} \sim_K C_{(c,a+b-c,d)} \quad \text{if}\ b \geq c+d. \tag{2.8}$$

$$D_{(a,b,c,d;\varepsilon)} \sim_K C_{(a,b,d)} \quad \text{if}\ c \geq b+d. \tag{2.9}$$

Now we make each element of $\mathcal{R}$ correspond systematically to an element in $\widetilde{\mathcal{R}}$. Set

$$\widetilde{D}_{(a,b,c,d;x)} = \left(\begin{array}{c|c} 0 & -1_2 \\ \hline 1_2 & 0 \end{array}\right) \cdot D_{(a,b,c,d;\varepsilon)} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & -\pi^{a} & 0 \\ & & -\varepsilon\pi^{c} & -\pi^{b} \\ \hline \pi^{-a} & -\varepsilon\pi^{-a-b+c} & -\varepsilon\pi^{-a-b+c+d} & \pi^{-a+d} \\ 0 & \pi^{-b} & \pi^{-b+d} & 0 \end{array}\right),$$

then

$$\pi_{(\lambda;\xi)} = \widetilde{D}_{(a,b,c,d;\varepsilon)}$$

for

$$a = \lambda_1 + \lambda_3,\ b = \lambda_2 + \lambda_4,\ c = \lambda_2 + \lambda_3,\ d = \lambda_3 + \lambda_4,$$
$$\lambda_1 = \frac{2a+b-c-d}{2},\ \lambda_2 = \frac{b+c-d}{2},\ \lambda_3 = \frac{-b+c+d}{2},\ \lambda_4 = \frac{b-c+d}{2},$$
$$\varepsilon = -\xi.$$

Then $\mathcal{R}$ corresponds bijectively to $\widetilde{\mathcal{R}}$, in particular $\mathcal{R}_4$ corresponds to $\widetilde{\mathcal{R}_*}$.

# §3 Functional equations and rationality of spherical functions

The functional equations for $\omega(x;z;\chi)$ and $\omega(x;z;w_i(\chi))$ for $w_i \in W$, $1 \le i \le 4$ can be obtained by taking suitable parabolic subgroup $\mathbf{P}_i$ containing $\mathbf{B}$ and prehomogeneous space $(\mathbf{P}_i \times GL_1, \mathbf{X} \times M_{2,1})$, for the details see [H5, §3]. Then we have the following theorem, which gives us some information on the location of poles and zeros of spherical functions.

**Theorem 2** *For each character $\chi$ of $k^\times/(k^\times)^2$, set*

$$F_\chi(z) = G_\chi(z)\big/G(z),$$

*where*

$$G(z) = (1 - q^{-z_1+z_2-1})(1 - q^{-z_1-z_2-1})\prod_{i=1}^{4}(1 - q^{-z_i-1}),$$

$$G_\chi(z) = \begin{cases} \{(+---)(-++-)(-+-+)(-+--)(--++)(--+-) & \\ \quad\times(---+)(----)\}_\varepsilon & \text{if } \chi(\mathcal{O}^\times) = 1 \text{ and } \chi(\pi) = \varepsilon \\ q^{-\frac{3z_1+z_2+z_3+z_4}{2}} & \text{if } \chi(\mathcal{O}^\times) \neq 1, \end{cases}$$

*and*

$$(\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4)_\varepsilon = 1 - \varepsilon q^{\frac{1}{2}(\varepsilon_1 z_1 + \varepsilon_2 z_2 + \varepsilon_3 z_3 + \varepsilon_4 z_4 - 1)} \quad (\varepsilon_i = +, -,\ \varepsilon = 1, -1).$$

*Then $F_\chi(z) \cdot \omega(x;z;\chi)$ belongs to $\mathbb{C}[q^{\pm\frac{z_1}{2}}, q^{\pm\frac{z_2}{2}}, q^{\pm\frac{z_3}{2}}, q^{\pm\frac{z_4}{2}}]$ and is invariant under the action of the Weyl group $W$ of $G$.*

## §4 Explicit expressions of spherical functions

In this section we give explicit expressions of spherical functions $\omega(x;\chi;z)$ for each element in $\widetilde{\mathcal{R}}$ following the method of [H3, §1]. Since spherical functions are $K$-invariant, it is enough to give such formulas for the representatives of $K\backslash X$. In Section 2, we have given a set $\widetilde{\mathcal{R}}$ of representatives of $K\backslash X$ and left the proof that there is no $K$-equivalence within $\widetilde{\mathcal{R}}$, which will be proved through the explicit formula $\omega(x;\chi;z)$ in Corollary 5.5.

Set

$$\mathcal{P}(x;\chi;z) = \int_U \chi(\prod_{i=1}^4 d_i(u\star x)) \prod_{i=1}^4 |d_i(u\star x)|^{s_i}\, du, \tag{4.1}$$

where the variable $z\in\mathbf{C}^4$ is related to $s\in\mathbf{C}^4$ by (1.7), $U$ is the Iwahori subgroup of $G$ compatible with $B$ and $du$ is the Haar measure on $U$ normalized by $\int_U du = 1$. The right hand side of (4.1) is absolutely convergent for $\mathrm{Re}(s_i)\geq 0$ $(1\leq i\leq 4)$ and analytically continued to a rational function in $q^{s_1},\ldots,q^{s_4}$.

Applying [H3, Proposition 1.9] to our case, we have the following.

**Proposition 4.1** *Let $G(z)$ and $G_\chi(z)$ be as in Theorem 2, and set*

$$H(z) = (1-q^{-z_1+z_2})(1-q^{-z_1-z_2})\cdot\prod_{i=1}^4(1-q^{-z_i}),$$

*where the variable $z\in\mathbf{C}^4$ is related to $s\in\mathbf{C}^4$ by (1.7). Then we have*

$$\omega(x;\chi;z) = \frac{1}{(1+q^{-1})^4(1+q^{-2})}\cdot\frac{G(z)}{G_\chi(z)}\cdot\sum_{\sigma\in W}\sigma\left(\frac{G_\chi(z)}{H(z)}\cdot\mathcal{P}(x;\chi;z)\right).$$

We set

$$\widetilde{\mathcal{R}_+} = \left\{\pi_{(\lambda;\xi)}\;\middle|\;\lambda\in\Lambda,\ \xi\in\mathcal{O}^\times/(\mathcal{O}^\times)^2\right\},$$

and calculate $\mathcal{P}(x;\chi;z)$ for $x\in\widetilde{\mathcal{R}_+}$.

**Proposition 4.2** *For $\pi_{(\lambda;\xi)}\in\widetilde{\mathcal{R}_+}$, we have*

$$\mathcal{P}(\pi_{(\lambda;\xi)};\chi;z) = \chi(\xi)\chi(\pi)^{2\lambda_1}q^{-\|\lambda\|-\lambda_1}\cdot q^{<\lambda,z>},$$

*where $\|\lambda\| = \sum_{i=1}^4\lambda_i$ and $<\lambda,z> = \sum_{i=1}^4\lambda_i z_i$.*

The following Proposition is an easy consequence of Propositions 4.1 and 4.2.

**Proposition 4.3** *Let $\chi$ be nontrivial on $\mathcal{O}^\times$ and $x\in X$ be $K$-equivalent to some element in $\widetilde{\mathcal{R}}\backslash\widetilde{\mathcal{R}_*}$. Then $\omega(x;\chi;z)=0$.*

For an element $\sigma$ of the Weyl group $W$, we set $\varepsilon(\sigma) = 1$ (resp. $-1$) if $\sigma$ is expressed by a product of even (resp. odd) numbers of $\{w_1, w_2, w_3, w_4\}$.

By Proposions 4.1, 4.2 and 4.3, we obtain our main results on explicit expressions of spherical functions.

**Theorem 3** *For each $\lambda \in \Lambda$, $\xi \in \mathcal{O}^\times$ and character $\chi$ of $k^\times/(k^\times)^2$, set*

$$c_{\lambda,\xi,\chi}(z) = \frac{\chi(\xi)\chi(\pi)^{2\lambda_1} q^{-\|\lambda\|-\lambda_1}}{(1+q^{-1})^4(1+q^{-2})} \cdot \frac{G(z)}{G_\chi(z)} \cdot \frac{1}{H_0(z)},$$

*where $G(z)/G_\chi(z) = F_\chi(z)^{-1}$ is given in Theorem 2 and*

$$H_0(z) = (q^{z_1} - q^{z_2})(1 - q^{-z_1-z_2}) \cdot \prod_{i=1}^{4}(q^{\frac{z_i}{2}} - q^{\frac{-z_i}{2}}) \left(= q^{\frac{3z_1+z_2+z_3+z_4}{2}} \cdot H(z)\right);$$

*so if $\chi$ is nontrivial on $\mathcal{O}^\times$, $G(z)/G_\chi(z)H_0(z)$ coincides with the c-function $G(z)/H(z)$ of $G$. Then the explicit formulas of spherical functions are given in the following.*

(i) *If $\chi$ is trivial on $\mathcal{O}^\times$, we have*

$$\omega(\pi_{(\lambda,\xi)}; \chi; z) = c_{\lambda,1,\chi}(z) \cdot \sum_{\sigma \in W} \varepsilon(\sigma) \cdot \sigma\left(G_\chi(z) \cdot q^{<\tilde{\lambda}, z>}\right),$$

*where $\tilde{\lambda} = (\lambda_1 + \frac{3}{2}, \lambda_2 + \frac{1}{2}, \lambda_3 + \frac{1}{2}, \lambda_4 + \frac{1}{2}) (\in \Lambda_*)$.*

(ii) *Let $\chi$ be nontrivial on $\mathcal{O}^\times$. Then $\omega(\pi_{(\lambda;\xi)}; \chi; z) = 0$ unless $\lambda \in \Lambda_*$, and if $\lambda \in \Lambda_*$, we have*

$$\begin{aligned}
&\omega(\pi_{(\lambda;\xi)}; \chi; z) \\
&= c_{\lambda,\xi,\chi}(z) \cdot \left(\left(q^{\lambda_1 z_1} - q^{-\lambda_1 z_1}\right)\left(q^{\lambda_2 z_2} - q^{-\lambda_2 z_2}\right) - \left(q^{\lambda_2 z_1} - q^{-\lambda_2 z_1}\right)\left(q^{\lambda_1 z_2} - q^{-\lambda_1 z_2}\right)\right) \\
&\quad \times \prod_{i=3,4} \left(q^{\lambda_i z_i} - q^{-\lambda_i z_i}\right).
\end{aligned}$$

# §5 Spherical Fourier transform

Let $\mathcal{S}(K\backslash X)$ be set of $K$-invariant Schwartz-Bruhat functions on $X$:

$$\mathcal{S}(K\backslash X) = \{\varphi \in \mathcal{C}^\infty(K\backslash X) \mid \textit{compactly supported}\},$$

and we introduce the spherical transform on $\mathcal{S}(K\backslash X)$ in the following. Set

$$\Psi_1(x; z) = F_1(z) \cdot \omega(x; 1; z), \qquad \Psi_2(x; z) = F_{\chi^*}(z) \cdot \omega(x; \chi^*; z),$$

where 1 is the trivial character and $\chi^*$ is the character for which $\chi^*(\pi) = 1$ and $\chi^*(\varepsilon) = \left(\frac{\varepsilon}{\mathfrak{p}}\right)$ for $\varepsilon \in \mathcal{O}^\times$, and $F_\chi(z)$ is the function defined in Theorem 2. By Theorem 2, we know that $\Psi_i(x; z)$, $i = 1, 2$ belong to

$$\mathbb{C}[q^{\pm\frac{z_1}{2}}, q^{\pm\frac{z_2}{2}}, q^{\pm\frac{z_3}{2}}, q^{\pm\frac{z_4}{2}}]^W (-\mathcal{C}_0, \text{say}).$$

On the other hand, as we saw in Proposition 1.1, $\mathcal{H}(G, K)$ is isomorphic to $\mathcal{C}_0$ by Satake isomorphism.

Now we define the spherical Fourier transform on $\mathcal{S}(K\backslash X)$ for $i = 1, 2$

$$\begin{array}{rccl} \mathcal{F}_i: & \mathcal{S}(K\backslash X) & \longrightarrow & \mathbb{C}[q^{\pm\frac{z_1}{2}}, q^{\pm\frac{z_2}{2}}, q^{\pm\frac{z_3}{2}}, q^{\pm\frac{z_4}{2}}]^W (= \mathcal{C}_0, \text{say}) \\ & \varphi & \longmapsto & \mathcal{F}_i(\varphi)(z) \end{array}$$

by

$$\mathcal{F}_i(\varphi)(z) = \int_X \varphi(x) \cdot \Psi_i(x; z) dx,$$

where $dx$ is the normalized $G$-invariant measure on $X$. Since $\mathcal{F}_i$ satisfies for every $f \in \mathcal{H}(G, K)$

$$\mathcal{F}_i(f * \varphi)(z) - \widetilde{\check{f}}(z) \cdot \mathcal{F}_i(\varphi)(z), \qquad \check{f}(g) - f(g^{-1}),$$

$\mathcal{F}_i$ is an $\mathcal{H}(G, K)$-module homomorphism, $i = 1, 2$.

Let us recall the sets $\Lambda$ and $\Lambda_*$ defined in the beginning of Section 2. Set $\Lambda_0 = \Lambda \setminus \Lambda_*$. For $\lambda \in \Lambda$, denote by $\varphi_\lambda$ the characteristic function of the $K$-orbit containing $\pi_{(\lambda;1)}$ and by $\varphi_{\lambda*}$ the characteristic function of the $K$-orbit containing $\pi_{(\lambda;\xi)}$ for $\xi \in \mathcal{O}^\times$, $\xi \notin (\mathcal{O}^\times)^2$. Then $\mathcal{S}(K\backslash X)$ is generated by $\{\varphi_\lambda \mid \lambda \in \Lambda_0\} \cup \{\varphi_\lambda, \varphi_{\lambda*} \mid \lambda \in \Lambda_*\}$.

For simplicity, we set

$$\eta(z) = \prod_{i=1}^{4} \left(q^{\frac{z_i}{2}} + q^{-\frac{z_i}{2}}\right), \qquad \mathcal{C} = \mathcal{C}_0 \oplus \eta(z) \cdot \mathcal{C}_0,$$

here we regard $\mathcal{C}_0$ and $\mathcal{C}$ as free $\mathcal{H}(G, K)$-modules through the Satake transform.

Our main theorem is the following.

**Theorem 4** *Set*

$$\begin{aligned} \mathcal{S}_1 &= <\varphi_\lambda \mid \lambda \in \Lambda_0>_{\mathcal{C}} + <\varphi_\lambda + \varphi_{\lambda*} \mid \lambda \in \Lambda_*>_{\mathcal{C}}, \\ \mathcal{S}_2 &= <\varphi_\lambda - \varphi_{\lambda*} \mid \lambda \in \Lambda_*>_{\mathcal{C}}. \end{aligned}$$

*Then $\mathcal{S}(K\backslash X) = \mathcal{S}_1 \oplus \mathcal{S}_2$ as an $\mathcal{H}(G, K)$-module, and $\mathcal{F}_j$ induces the $\mathcal{H}(G, K)$-module isomorphism $\mathcal{S}_j \cong \mathcal{C}$ for $j = 1, 2$.*

*In particular, $\mathcal{S}(K\backslash X)$ is a free $\mathcal{H}(G, K)$-module of rank 4 with basis*

$$\left\{\varphi_\lambda \,\middle|\, \lambda = (0,0,0,0),\ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\right\} \cup \left\{\varphi_\lambda - \varphi_{\lambda*} \,\middle|\, \lambda = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (2,1,1,1)\right\}.$$

It is clear that Ker$\mathcal{F}_1 \supset \mathcal{S}_2$, Ker$\mathcal{F}_2 \supset \mathcal{S}_1$ and $\mathcal{F}_2$ is injective on $\mathcal{S}_2$. Theorem 5 follows from Propositions 5.1 and 5.2 below.

**Proposition 5.1** *For $\lambda \in \Lambda_*$, set*

$$\widetilde{m_\lambda}(z) = \sum_{\sigma \in W} \sigma\left(\frac{q^{<\lambda,z>}}{H_0(z)}\right).$$

*Then*

$$\mathcal{F}_2(\varphi_\lambda - \varphi_{\lambda *}) \equiv \widetilde{m_\lambda}(z) \ (\mathrm{mod}\, \mathbb{C}^\times),$$

*$\widetilde{m_\lambda}(z) \in \mathcal{C}_0$ (resp. $\eta(z)\mathcal{C}_0$) if $\lambda_1 \in \frac{1}{2} + \mathbb{Z}$ (resp. $\lambda_1 \in \mathbb{Z}$), and*

$$\widetilde{m_\lambda}(z) = \begin{cases} 1 & \text{if } \lambda = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \\ \eta(z) & \text{if } \lambda = (2, 1, 1, 1). \end{cases}$$

*In Particular, $\mathcal{F}_2$ gives an $\mathcal{H}(G, K)$-module isomorphism $\mathcal{S}_2 \cong \mathcal{C}$.*

**Proposition 5.2** *For $\lambda \in \Lambda$, set*

$$K_\lambda(z) = \sum_{\sigma \in W} \sigma\left(\frac{G_1(z) \cdot q^{<\lambda,z>}}{H_0(z)}\right).$$

*Then,*

$$\mathcal{F}_1(\varphi_\lambda) = \mathcal{F}_1(\varphi_{\lambda *}) \equiv K_{\tilde{\lambda}}(z) \ (\mathrm{mod}\, \mathbb{C}^\times), \quad \tilde{\lambda} = (\lambda_1 + \frac{3}{2}, \lambda_2 + \frac{1}{2}, \lambda_3 + \frac{1}{2}, \lambda_4 + \frac{1}{2}),$$

*and $\lambda \in \Lambda_*$, $K_\lambda(z)$ can be expressed as*

$$K_\lambda(z) = c_\lambda \widetilde{m_\lambda}(z) + \sum_{\substack{\mu \in \Lambda_* \\ \lambda \succ \mu}} c_\mu \widetilde{m_\mu}(z), \ \text{with some } c_\lambda \in \mathbb{C}^\times, \ c_\mu \in \mathbb{C},$$

*where $\lambda \succ \mu$ means that $\|\lambda\| > \|\mu\|$ or $\|\lambda\| = \|\mu\|$, $\lambda_1 > \mu_1$. In Particular, $\mathcal{F}_1$ gives an $\mathcal{H}(G, K)$-module isomorphism $\mathcal{S}_1 \cong \mathcal{C}$. In particular*

Since $\omega(x; \chi^*; z)$ vanishes on $\widetilde{\mathcal{R}_0} = \widetilde{\mathcal{R}} \backslash \widetilde{\mathcal{R}_*}$ and takes a different value at each element of $\widetilde{\mathcal{R}_*}$ and $\omega(x; 1; z)$ takes a different value at each element of $\widetilde{\mathcal{R}_0}$, we conclude the proof of Cartan decomposition given in Section 2.

**Corollary 5.3** *The set $\widetilde{\mathcal{R}}$, as well as $\mathcal{R}$, is a complete set of representatives of $K$-orbit in $X$.*

Finally, we give a parametrization of spherical functions. The characters on $k^\times/(k^\times)^2$ are given by $\{1, \chi^*, \chi_\pi, \chi_\pi^*\}$, where $\chi_\pi(\pi) = -1$, $\chi_\pi(\mathcal{O}^\times) = 1$ and $\chi_\pi^* = \chi^* \chi_\pi$. We set for each $\chi$

$$\Psi_\chi(x; z) = F_\chi(z) \cdot \omega(x; \chi; z),$$

so $\Psi_{\chi^*}(x; z) = \Psi_2(x; z)$ in the previous notation.

**Theorem 5** *Eigenvalues for spherical functions are parametrized by* $z \in \left(\mathbb{C}/\frac{2\pi\sqrt{-1}}{\log q}\mathbb{Z}\right)^4/W$ *through the Satake transform* $\mathcal{H}(G,K) \longrightarrow \mathbb{C}$, $f \longmapsto \tilde{f}(z)$ *(cf. Proposition 1.1). The set*

$$\left\{ \Psi_1(x;z) + \Psi_{\chi_\pi}(x;z),\ \Psi_{\chi^*}(x;z) - \Psi_{\chi_\pi^*}(x;z),\ \frac{\Psi_1(x;z) - \Psi_{\chi_\pi}(x;z)}{\eta(z)},\ \frac{\Psi_{\chi^*}(x;z) + \Psi_{\chi_\pi^*}(x;z)}{\eta(z)} \right\}$$

*forms a basis of the space of spherical functions on* $X$ *corresponding to* $z \in \mathbb{C}^4$.

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