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Homeomorphism groups of finite topological spaces and Group actions

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概要

As being pointed out by several authors, finite topological spaces have more interesting topological properties than one might at first expect. In this short article, we investigate the homeomorphism groups of finite spaces with group action. In particular, we study the homeomorphism groups of fixed point set X^G and G-actions on homeomorphism groups induced by given G-action on X, where X is a finite topological space with a G-action.

1 Introduction

Let X be a finite set, and let X_n denote the n-point set $\{x_1, x_2, \cdots, x_n\}$. Let $\mathcal{T}$ be a topology on X, that is, $\mathcal{T}$ is a family of subsets of X which satisfies:

(1) $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$;
(2) $A, B \in \mathcal{T} \Rightarrow A \cup B \in \mathcal{T}$;
(3) $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$.

A finite set X with a topology is called a *finite topological space* or *finite space* briefly. A finite topological group is also defined canonically, but it is not assumed to satisfy any separation axioms. We say that a finite topological space $(X, \mathcal{T})$ is a finite T_0-space if it satisfies the T_0-separation axiom.

As several authors have pointed out, finite topological spaces have more interesting topological properties than one might at first expect. It is remarkable that for every finite topological space X, there exists a simplicial complex K such that X is weak homotopy

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equivalent to $|K|$ ([4]), and that the classification of finite topological spaces by homotopy type is reduced to a certain homeomorphism problem ([6]). Some relations with simple homotopy theory are revealed in [5]. Group actions on finite spaces have been also studied by several authors ([1], [3], [7]). In [7], Stong proved rather surprising results for the equivariant homotopy theory for finite T_0-spaces. The homeomorphism groups of finite topological spaces were studied in [3]. One can find a survey of the theory of the finite topological spaces from topological viewpoints in [2].

Let G be a finite topological group, $(X, \mathcal{T})$ a finite space with G-action. The purpose of the present article is to study the homeomorphism groups of the G-fixed point set X^G and the G-actions on Homeo(X). According to [4], for every finite space X, there exists a quotient space $\hat{X}$ of X such that $\hat{X}$ is homotopic to X and satisfies T_0-separation axiom. In [3], we proved the following splitting exact sequence

$$1 \longrightarrow \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x])) \stackrel{\iota}{\longrightarrow} \mathrm{Homeo}(X) \stackrel{\pi}{\longrightarrow} \mathrm{Homeo}(\hat{X})_X \longrightarrow 1,$$

where $\mathrm{Homeo}(\hat{X})_X$ is a subgroup of Homeo(X) which is denoted as $\mathrm{Homeo}_X(\hat{X})$ in [3].

The rest of this article is organized as follows. In section 2, we review the structure of Homeo(X) which was studied in [3]. In section 3, we study the homeomorphism groups of fixed point set X^G, where X is a finite topological space with a G-action. Section 4 is devoted to studying G-actions on Homeo(X), in particular,investigating the fixed point sets.

2 Homeomorphism groups of finite topological spaces

Let $(X_n, \mathcal{T})$ be a finite topological space. Let U_i denote the minimal open set which contains x_i, that is, U_i is the intersection of all open sets containing x_i. We see that $\{U_1, U_2, \cdots U_n\}$ is an open basis of $\mathcal{T}$.

Let X be a finite topological space. We define an equivalence relation $\sim$ on X by

$$x_i \sim x_j \quad \text{if} \quad U_i = U_j.$$

Let $\hat{X}$ be the quotient space $X/\sim$, and $\nu_X : X \to \hat{X}$ the quotient map. We note that

$$\nu_X(x_i) = U_i \cap C_i,$$

where C_i is the smallest closed set containing x_i. ¿From now on, we denote $\nu_X(x) \in \hat{X}$ by $[x]_X$ or briefly $[x]$. The following proposition bridges the gap between general finite topological spaces and finite T_0-spaces.

Proposition 2.1 ([4] : Theorem 4). *Let X and Y be finite topological spaces. Then the following hold.*

(1) The quotient map $\nu_X : X \to \hat{X}$ is a homotopy equivalence.

(2) The quotient space $\hat{X}$ is a finite T_0-space.

(3) For each continuous map $\varphi : X \to Y$, there exists a unique continuous map $\hat{\varphi} : \hat{X} \to \hat{Y}$ such that $\nu_Y \circ \varphi = \hat{\varphi} \circ \nu_X$.

Let θ_X : Homeo$(X) \times X \to X$ be the natural action. According to [3], there exists unique continuous homomorphism π : Homeo$(X) \to$ Homeo$(\hat{X})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Homeo}(X) \times X & \xrightarrow{\theta_X} & X \\ {\scriptstyle \pi \times \nu_X}\downarrow & & \downarrow{\scriptstyle \nu_X} \\ \mathrm{Homeo}(\hat{X}) \times \hat{X} & \xrightarrow[\theta_{\hat{X}}]{} & \hat{X} \end{array}$$

The product $\prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))$ is identified with the set of maps $F : \hat{X} \to \coprod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))$ with $F([x]) \in \mathrm{Homeo}(\nu_X^{-1}([x]))$ for every $[x] \in \hat{X}$. Let F be an element of $\prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))$. Then, F defines a map $\iota(F) : X \to X$ by $\iota(F)(x) = F([x])(x)$, under above identification. It is easy to see that ι is a group homomorphism. Set a subset Homeo$(\hat{X})_X$ of Homeo$(\hat{X})$ by

$$\mathrm{Homeo}(\hat{X})_X = \left\{ f \in \mathrm{Homeo}(\hat{X}) \;\middle|\; \begin{array}{l} \#f([x]) = \#[x] \text{ for every } [x] \in \hat{X}, \\ \text{where the numbers are counted as subsets of } X \end{array} \right\}.$$

We see that Homeo$(\hat{X})_X$ is a subgroup of Homeo$(\hat{X})$. In [3], we proved the following theorem.

Theorem 2.2 ([3] : Theorem 4.7). *Let X be a finite topological space. Then, the following hold.*

(1) Homeo$(\hat{X})_X = \mathrm{Im}(\pi)$.

(2) The sequence

$$1 \longrightarrow \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x])) \xrightarrow{\iota} \mathrm{Homeo}(X) \xrightarrow{\pi} \mathrm{Homeo}(\hat{X})_X \longrightarrow 1$$

is a splitting exact sequence.

3 Homeomorphism groups of fixed point sets

Let G be a finite topological group, $(X, \mathcal{T})$ a finite space with G-action. In this section we study the homeomorphism groups of the G-fixed point set X^G. In this section, we will prove the following theorem.

Theorem 3.1. *Let G be a finite topological group, $(X,\mathcal{T})$ a finite space with G-action. Then, there exists a splitting exact sequence*

$$1 \longrightarrow \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x])^G) \xrightarrow{\iota^G} \mathrm{Homeo}(X^G) \xrightarrow{\pi^G} \mathrm{Homeo}(\hat{X^G})_{X^G} \longrightarrow 1.$$

Lemma 3.2. *Let G be a finite topological group, $(X,\mathcal{T})$ a finite space with G-action. Then, it holds that*

$$\prod_{[x]\in\hat{X^G}} \mathrm{Homeo}(\nu_X^{-1}([x])^G) = \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x])^G).$$

Proof. Since $\hat{X^G}$ is a subset of $\hat{X}$, we see that $\prod_{[x]\in\hat{X^G}} \mathrm{Homeo}(\nu_X^{-1}([x])^G) \subset \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x])^G)$. Since $\nu_X^{-1}([x])^G = \emptyset$ for $[x] \in \hat{X} - \hat{X^G}$, the equality holds. □

Lemma 3.3. *Let G be a finite topological group, $(X,\mathcal{T})$ a finite space with G-action. For every $x \in X^G$, it holds that*

$$\nu_{X^G}{}^{-1}([x]_{X^G}) = (\nu_X^{-1}([x]_X))^G.$$

Proof. Under the condition $x \in X^G$, it holds the following equivalences.

$$\begin{aligned} y \in \nu_{X^G}{}^{-1}([x]_{X^G}) &\Longleftrightarrow \nu_{X^G}(y) = [x]_{X^G} \\ &\Longleftrightarrow x \sim y \text{ in } X^G \\ &\Longleftrightarrow x \sim y \text{ in } X \text{ and } x, y \in X^G \\ &\Longleftrightarrow y \in X^G \text{ and } \nu_X(y) = [x]_X \\ &\Longleftrightarrow y \in X^G \text{ and } y \in \nu_X^{-1}([x]_X) \\ &\Longleftrightarrow y \in (\nu_X^{-1}([x]_X))^G \end{aligned}$$

□

Proof of Theorem 3.1 By Lemma 3.2 and Lemma 3.3, we have

$$\prod_{[x]\in\hat{X^G}} \mathrm{Homeo}(\nu_{X^G}{}^{-1}([x]_{X^G})) = \prod_{[x]\in\hat{X^G}} \mathrm{Homeo}(\nu_X^{-1}([x])^G) = \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x])^G).$$

If we apply the exact sequence in Theorem 2.2 for X^G, then the second term is nothing but $\prod_{[x]\in\hat{X^G}} \mathrm{Homeo}(\nu_{X^G}{}^{-1}([x]_{X^G}))$. This completes the proof of Theorem 3.1. □

Corollary 3.4. *Let $(X,\mathcal{T})$ be a finite space. Put $G = \mathrm{Homeo}(X)$ with the compact open topology. Then the G-fixed point set X^G of the natural continuous G-action on X is a T_0-space.*

Proof If $\nu_X^{-1}([x])$ consists of more than two points for $[x] \in \hat{x}$, it holds that $\nu_X^{-1}([x])^G = \emptyset$, thereby the group $\prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x])^G)$ is trivial. Hence we have $\mathrm{Homeo}(X^G) \cong \mathrm{Homeo}(\hat{X^G})_{X^G}$, which has discrete topology. According to [3], we see that X^G is a T_0-space. □

4 Group actions on homeomorphism groups

Throughout this section, let G be a finite topological group, $(X, \mathcal{T})$ a finite space with continuous G-action $\varphi : G \times X \to X$. Let $\Phi : G \to \mathrm{Homeo}(X)$ be the continuous homomorphism satisfying $\varphi = \theta \circ (\Phi \times id_X)$, where θ is the natural action of $\mathrm{Homeo}(X)$ on X. In this section, we consider group actions on $\mathrm{Homeo}(X)$.

Lemma 4.1. *Let G be a finite topological group. For any $g \in G$, let U_g denote the minimal open neighbourhood of g. For given elements g and h of G ,if an open set O contains gh, O also contains $U_gU_h = \{g'h' | g' \in U_g \text{ and } h' \in U_h\}$.*

Proof. Since G is a finite topological group, U_g and U_h are connected components of G. By the continuity of group operations, U_gU_h is a connected neighbourhood of gh. Hence, we have $U_gU_h \subset U_{gh} \subset O$. □

Lemma 4.2. *The map $\psi : G \times \mathrm{Homeo}(X) \to \mathrm{Homeo}(X)$ defined by $\psi(g, f) = \Phi(g) \circ f \circ \Phi(g^{-1})$ is a continuous action of G on* $\mathrm{Homeo}(X)$.

Proof. It is easy to see that ψ is a G-action. We prove the continuity of ψ. For any non-empty open set O of $\mathrm{Homeo}(X)$, fix an arbitrary point $(g,\ f) \in \psi^{-1}(O)$. Since $\psi(g,\ f) = \Phi(g) \circ f \circ \Phi(g^{-1}) \in O$ and both $\Phi(g)$ and f are homeomorphisms on X, by Lemma 4.1,

$$\psi(g,\ f) \in U_{\Phi(g)}\ U_f\ U_{\Phi(g^{-1})} \subset O$$

holds. This implies that

$$(g,\ f) \in \Phi^{-1}(U_{\Phi(g)}) \times U_f \subset \psi^{-1}(O).$$

Since its middle term is open in $G \times \mathrm{Homeo}(X)$, it completes the proof. □

In this section, we study the structure of the G-fixed point set $\mathrm{Homeo}(X)^G$, which is denoted by $\mathrm{Homeo}(X)^G$. For using the exact sequence of Theorem 2.2, we will define similar G-actions on $\prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))$ and $\mathrm{Homeo}(\hat{X})_X$.

Lemma 4.3. *Let F be an element of $\prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))$. Given $g \in G$, there exists a unique element $F' \in \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))$ such that $\iota(F') = \Phi(g) \circ \iota(F) \circ \Phi(g^{-1})$.*

Proof. For any $x \in X$, it holds that

$$\Phi(g) \circ \iota(F) \circ \Phi(g^{-1})(x) = \Phi(g)(\iota(F)(\Phi(g^{-1})(x))) = \Phi(g)(F([\Phi(g^{-1})(x)])(\Phi(g^{-1})(x))).$$

We see that $F([\Phi(g^{-1})(x)])(\Phi(g^{-1})(x)) \in \nu_X^{-1}([\Phi(g^{-1})(x)])$. Since $\Phi(g)$ maps $\nu_X^{-1}([\Phi(g^{-1})(x$ onto $\nu_X^{-1}([\Phi(g)\circ\Phi(g^{-1})(x)]) = \nu_X^{-1}([x]_X)$, it holds that $\Phi(g)\circ\iota(F)\circ\Phi(g^{-1})(x) \in \nu_X^{-1}([x]_X)$. Hence, there exists an element $F' \in \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))$ such that $\iota(F') = \Phi(g)\circ\iota(F)\circ \Phi(g^{-1})$. Since ι is injective, F' is determined uniquely. □

Lemma 4.4. *The map* $\psi_0 : G \times \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x])) \to \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))$ *defined by* $\psi_0(g, F) = \iota^{-1}(\Phi(g) \circ \iota(F) \circ \Phi(g^{-1}))$ *is a continuous action of* G *on* $\prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))$.

Proof. It is easy to see that ψ_0 is a G-action. Since $\prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))$ has the trivial topology, ψ_0 is continuous. □

Lemma 4.5. *Let* G *be a finite topological group,* $(X, \mathcal{T})$ *a finite space with continuous* G*-action* $\varphi : G \times X \to X$. *Then, the map* $\hat{\varphi} : G \times \hat{X} \to \hat{X}$ *defined by* $\hat{\varphi}(g, [x]) = [\Phi(g)(x)]$ *is a continuous action of* G *on* $\hat{X}$.

Proof. Since a homeomorphism preserves the equivalence relation, $\hat{\varphi}$ is well defined. It is easy to see that $\hat{\varphi}$ is a G-action on $\hat{X}$. Since in the following commutative diagram, $id \times \nu_X$ is able to be regarded as a quotient map, the continuity of $\nu_X \circ \varphi$ implies the continuity of $\hat{\varphi}$.

$$\begin{array}{ccc} G \times X & \xrightarrow{\varphi} & X \\ {\scriptstyle id\times\nu_X}\downarrow & & \downarrow{\scriptstyle \nu_X} \\ G \times \hat{X} & \xrightarrow[\hat{\varphi}]{} & \hat{X} \end{array}$$

Lemma 4.6. *Let* $\hat{\varphi}_g : \hat{X} \to \hat{X}$ *be a homeomorphism on* $\hat{X}$ *defined by* $\hat{\varphi}_g([x]) = \hat{\varphi}(g, [x])$. *The map* $\psi_1 : G \times \mathrm{Homeo}(\hat{X}) \to \mathrm{Homeo}(\hat{X})$ *defined by* $\psi_1(g, f)([x]) = \hat{\varphi}_g \circ f \circ \hat{\varphi}_{g^{-1}}([x])$ *is a continuous action of* G *on* $\mathrm{Homeo}(\hat{X})$ *and it holds that*

$$\psi_1(g, f) = \pi(\Phi(g)) \circ f \circ \pi(\Phi(g^{-1})).$$

Proof. It is easy to see that ψ_1 is a G-action on $\mathrm{Homeo}(\hat{X})$. The continuity of ψ_1 is obtained by an analogous discussion to Lemma 4.2. For any $g \in G$ and $[x] \in \hat{X}$, we have $\hat{\varphi}_g([x]) = \hat{\varphi}(g, [x]) = [\Phi(g)(x)] = \pi(\Phi(g))([x])$, which proves the required formula. □

Lemma 4.7. *Under the* G*-action* $\psi_1 : G \times \mathrm{Homeo}(\hat{X}) \to \mathrm{Homeo}(\hat{X})$, $\mathrm{Homeo}(\hat{X})_X$ *is a* G*-invariant subgroup of* $\mathrm{Homeo}(\hat{X})$.

Proof. Since $\Phi(g)$ is a homeomorphism on $\mathrm{Homeo}(X)$, we have $\#(\nu_X^{-1}([\Phi(g)(x)])) = \#(\nu_X^{-1}([x]))$. Hence, we obtain $\pi(\Phi(g)) \in \mathrm{Homeo}(\hat{X})_X$, thereby $\psi_1(g, f) = \pi(\Phi(g)) \circ f \circ \pi(\Phi(g^{-1})) \in \mathrm{Homeo}(\hat{X})_X$, which implies that $\mathrm{Homeo}(\hat{X})_X$ is a G-invariant subgroup of $\mathrm{Homeo}(X)$. □

Theorem 4.8. *A sequence of the* G*-fixed point sets*

$$1 \longrightarrow \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))^G \xrightarrow{\iota^G} \mathrm{Homeo}(X)^G \xrightarrow{\pi^G} \mathrm{Homeo}(\hat{X})_X^G$$

is an exact sequence of finite topological groups, where ι^G *and* π^G *are the restrictions of* ι *and* π *to the fixed point sets respectively.*

Proof. The continuity of ι^G and π^G are follows from the continuity of ι and π. Since ι is injective, ι^G is injective.

Since for any element $F \in \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))^G$ it holds that $\iota^{-1}(\Phi(g) \circ \iota^G(F) \circ \Phi(g^{-1})) = F$, $\iota^G(F)$ is in $\mathrm{Homeo}(X)^G$. For any $f \in \mathrm{Homeo}(X)^G$, we observe that

$$\psi_1(g, \pi^G(f)) = \pi(\Phi(g)) \circ \pi^G(f) \circ \pi(\Phi(g^{-1})) = \pi(\Phi(g) \circ f \circ \Phi(g^{-1})) = \pi(f) = \pi^G(f).$$

According to definition, we obtain

$$((\pi^G \circ \iota^G)(F))([x]) = [\iota^G(F)(x)] = [F([x])(x)] = [x] = id_{\hat{X}}([x])$$

for every $F \in \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))^G$ and every $[x] \in \hat{X}$. Hence, it holds that $\pi^G \circ \iota^G(F) = id_{\hat{X}}$ for every $F \in \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))$. Let f be an element of $\ker \pi^G$. Then, $f(x) \in [x]$ for every $x \in X$, thereby f defines an element $F \in \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))$ by $F([x])(x) = f(x)$ for every $x \in X$. Then $\iota(F) = f$. Moreover, F satisfies that

$$\iota^{-1}(\Phi(g) \circ \iota(F) \circ \Phi(g^{-1})) = \iota^{-1}(\Phi(g) \circ f \circ \Phi(g^{-1})) = \iota^{-1}(f) = F,$$

which shows $\ker(\pi^G) \subset \mathrm{Im}(\iota^G)$. □

Remark 4.9. By Theorem 4.8, we have an exact sequence of finite topological groups

$$1 \longrightarrow \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))^G \xrightarrow{\iota^G} \mathrm{Homeo}(X)^G \xrightarrow{\pi^G} \mathrm{Im}(\pi^G) \longrightarrow 1.$$

In general, however, this sequence does not split and $\mathrm{Im}(\pi^G) \neq \mathrm{Homeo}(\hat{X})^G_X$.

Example 4.10. Let $(X_8, \mathcal{T})$ be a finite topological space which was treated in [3], that is, let $(X_8, \mathcal{T})$ be a finite topological space with the topology which has the following open basis.

$$\{\ \{x_1, x_2\}, \{x_1, x_2, x_3\}, \{x_4, x_5\}, \{x_4, x_5, x_6\}, \{x_7\}, \{x_7, x_8\}\ \}.$$

Then, the quotient space $\hat{X}$ is the set of six points

$$\{[x_1] = [x_2],\ [x_3],\ [x_4] = [x_5],\ [x_6],\ [x_7],\ [x_8]\}$$

with the topology generated by a open basis

$$\{\{[x_1]\}, \{[x_1], [x_3]\}, \{[x_4]\}, \{[x_4], [x_6]\}, \{[x_7]\}, \{[x_7], [x_8]\}\}.$$

We see that

$$\mathrm{Homeo}(\hat{X}) \cong \mathfrak{S}_3,\ \mathrm{Homeo}_X(\hat{X}) \cong \mathbb{Z}_2,\ \prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x])) \cong \mathbb{Z}_2 \times \mathbb{Z}_2,$$

and consequently,

$$\mathrm{Homeo}(X) \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \cong D_4,$$

where D_4 is a dihedral group of order 8.

Let a and b be two homeomorphisms on X_8 defined by

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 2 & 1 & 3 & 7 & 8 \end{pmatrix},$$

that is, $a(x_i) = x_{a(x_i)}$ and $b(x_j) = x_{b(x_j)}$. We observe that

$$\mathrm{Homeo}(X) \cong D_4 = < a,\ b > \quad \text{and} \quad a^2 = b^4 = abab = 1.$$

For various group actions on X we investigate the group actions on the homeomorphism groups induced by them. We note that $\mathrm{Homeo}(\hat{X})^G_{\hat{X}} \cong \mathbb{Z}_2$ for any finite topological group G. We classify the fixed point sets of G-actions by observing $\mathrm{Im}\,\Phi : G \to \mathrm{Homeo}(X)$.

$\mathrm{Im}(\Phi)$	$\prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x]))^G$	$\mathrm{Homeo}(X)^G$	$\mathrm{Im}(\pi^G)$
$\mathrm{Homeo}(X)$	$< \iota^{-1}(b^2) > \cong \mathbb{Z}_2$	$< b^2 > \cong \mathbb{Z}_2$	1
$< b >$	$< \iota^{-1}(b^2) > \cong \mathbb{Z}_2$	$< b > \cong \mathbb{Z}_4$	$< \pi(b) > = \mathrm{Homeo}(\hat{X})^G_{\hat{X}} \cong \mathbb{Z}_2$
$< a >$ or $< ab^2 >$	$< \iota^{-1}(a), \iota^{-1}(ab^2) > \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$< a, ab^2 > \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	1
$< b^2 >$	$\prod_{[x]\in\hat{X}} \mathrm{Homeo}(\nu_X^{-1}([x])) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathrm{Homeo}(X) \cong D_4$	$< \pi(b) > = \mathrm{Homeo}(\hat{X})^G_{\hat{X}} \cong \mathbb{Z}_2$
$< ab >$ or $< ab^3 >$	$< \iota^{-1}(b^2) > \cong \mathbb{Z}_2$	$< ab, ab^3 > \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$< \pi(b) > = \mathrm{Homeo}(\hat{X})^G_{\hat{X}} \cong \mathbb{Z}_2$

We note that if $\mathrm{Im}(\Phi)$ is $< b^2 >, < ab >$ or $< ab^3 >$, the exact sequences of Remark 4.9 split.

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