数理解析研究所講究録 1347 巻 2003 年 156-166

Existence Results for Some Quasilinear Elliptic Equations in an Unbounded Domain

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1 Introduction

This paper is concerned with the following quasilinear elliptic equations involving critical Sobolev exponents:

$$\text{(QE)} \qquad \begin{cases} -\Delta_p u - p\nabla\theta(x)\cdot\nabla u|\nabla u|^{p-2} = \lambda a(x)|u|^{p-2}u + K(x)|u|^{q-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ with $1 < p < N$, $p < q \le p^* := Np/(N-p)$ and $\lambda \in \mathbf{R}$. Here $\Omega \subset \mathbf{R}^N$ is an unbounded domain such that $\Omega := \mathbf{R}^N \setminus \cup_{i=1}^k \overline{\omega}_i$, where ω_i is an open and connected set with smooth boundary. If $\cup_{i=1}^k \omega_i \neq \emptyset$, we impose zero Dirichlet boundary condition on the boundary $\partial\Omega$ of Ω. If $\cup_{i=1}^k \omega_i = \emptyset$, then the homogenous boundary condition is not required. In (QE) $\theta(x)$, $a(x)$ and $K(x)$ are positive (or non-negative) functions. Our aim is to look for solutions tending to zero as $|x| \to \infty$.

When $p = 2$, $\theta(x) = \frac{1}{8}|x|^2$, $a(x) = K(x) \equiv 1$ and $\Omega = \mathbf{R}^N$, (QE) is written as the following semilinear elliptic equation

$$-\Delta u - \frac{1}{2}x\cdot\nabla u = \lambda u + |u|^{q-2}u \quad \text{in } \mathbf{R}^N \tag{1.1}$$

with $q > 2$. Escobedo-Kavian [11] have shown that

(i) if $q < 2^* = 2N/(N-2)$, (1.1) admits a solution if and only if $\lambda < N/2$,

(ii) if $q = 2^*$, (1.1) admits a solution if and only if $\lambda \in (N/4, N/2)$ for $N \ge 4$,

where $N/2$ is the first eigenvalue of $-\Delta - \frac{1}{2}x\cdot\nabla$ on $\mathbf{R}^N$ (see [11, Theorem 4.10]). As a more general case than (1.1), we will study the solvability of (QE). We also study the range of λ for which (QE) admits a solution.

Equation (QE) is also written in the following divergence form:

$$-\mathrm{div}(e^{p\theta(x)}|\nabla u|^{p-2}\nabla u) = \lambda e^{p\theta(x)}a(x)|u|^{p-2}u + e^{p\theta(x)}K(x)|u|^{q-2}u \quad \text{in } \Omega; \tag{1.2}$$

so that the associated weak formulation is given by

$$\int_\Omega e^{p\theta(x)}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx = \int_\Omega e^{p\theta(x)}(\lambda a(x)|u|^{p-2}u + K(x)|u|^{q-2}u)\varphi\,dx \tag{1.3}$$

for all $\varphi \in C_0^\infty(\Omega)$. From (1.3), it is natural to introduce some Sobolev spaces with weight function $e^{p\theta(x)}$.

We first consider (QE) with $q < p^*$. We assume that $\theta \in C^2(\Omega)$ is a non-negative function which satisfies

$$\text{(A1)} \qquad \begin{cases} (\theta.1) \text{ there exists a constant } c_\theta > 0 \text{ such that } \Delta\theta \ge c_\theta \text{ for all } x \in \Omega, \\ (\theta.2) \text{ there exists a point } x_0 \in \Omega \text{ such that } (x - x_0)\cdot\nabla\theta \ge 0 \text{ for all } x \in \Omega, \\ (\theta.3) \ \big[(p-1)\Delta\theta + |\nabla\theta|^2\big]|\nabla\theta|^{p-2} \to +\infty \ \text{ as } \ |x| \to \infty. \end{cases}$$

One can easily check that $\theta(x) = |x|^2$ fulfills (A1). For such function θ, we introduce weighted Sobolev spaces $L^p(\theta, \Omega)$ and $W^{1,p}(\theta, \Omega)$ as follows:

$$L^p(\theta, \Omega) := \left\{ u \in L^p(\Omega) \middle| \int_\Omega e^{p\theta(x)} |u|^p dx < +\infty \right\}, \tag{1.4}$$

$$W^{1,p}(\theta, \Omega) := \left\{ u \in W_0^{1,p}(\Omega) \middle| \int_\Omega e^{p\theta(x)} (|u|^p + |\nabla u|^p) dx < +\infty \right\}. \tag{1.5}$$

Let $a(x)$ be a non-negative function satisfying

(B1) $\qquad a(x) \in L^r(\Omega) \quad \text{for some } r \in (N/p, \infty].$

Correspondingly to $W^{1,p}(\theta, \Omega)$, define

$$\lambda_1 := \inf_{u \in W^{1,p}(\theta,\Omega)\setminus\{0\}} \left\{ \int_\Omega e^{p\theta(x)} |\nabla u|^p dx \Big/ \int_\Omega e^{p\theta(x)} a(x) |u|^p dx \right\}.$$

From (A1) and (B1), one can show that λ_1 is positive (see Lemma 2.2). It is easy to see that λ_1 is the first eigenvalue for the following eigenvalue problem;

$$\begin{cases} -\Delta_p u - p\nabla\theta(x) \cdot \nabla u |\nabla u|^{p-2} = \lambda a(x) |u|^{p-2} u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Let $K(x)$ be a positive function such that

(C1) $\qquad V(x) := e^{(p-q)\theta(x)} K(x) \in L^r(\Omega) \quad \text{with } r \in (p^*/(p^* - q), \infty].$

We next define a weak solution of (QE): u is called a *weak solution* of (QE) if it satisfies (1.3) for every $\varphi \in W^{1,p}(\theta, \Omega)$.

Theorem 1.1 (case $q < p^*$ [18]). *Assume* (A1), (B1) and (C1). *Then* (QE) *admits a non-trivial weak solution* $u^* \in W^{1,p}(\theta, \Omega)$ *for every* $\lambda < \lambda_1$.

Next we will study (QE) in case $q = p^*$ by assuming, in addition to (A1) and (B1), that

(A2) $\qquad$ there exists $\alpha_\theta > 0$ such that $|\nabla\theta(x)| = \alpha_\theta |x - x_0| + o(|x - x_0|)$

and

(B2) $\qquad$ there exists $s \in [p-2, p)$ such that $a(x) = |x - x_0|^{-s} + o(|x - x_0|^{-s})$

as $|x - x_0| \to 0$. It follows from (B1) and (B2) that r, s must satisfy $rs < N$. We also put the following condition on positive function $K(x) \in C(\Omega)$:

(C2) $\qquad V(x) := e^{(p-p^*)\theta(x)} K(x)$ satisfies $V(x_0) = \|V\|_{L^\infty(\Omega)}$ and $\overline{\lim_{|x|\to\infty}} V(x) = 0$.

Theorem 1.2 (case $q = p^*$ [19]). *Let* $N \geq p^2 - s(p-1)$. *Assume* (A1), (B1), (B2) and (C2). *Then* (QE) *admits at least one non-trivial weak solution* $u^* \in W^{1,p}(\theta, \Omega)$ *for every*

i) $\lambda \in (0, \lambda_1)$ *if* $s \in (p-2, p)$,

ii) $\lambda \in (\lambda_0, \lambda_1)$ *if* $s = p-2$, *where*

$$\lambda_0 = \begin{cases} \alpha_\theta p\left(\dfrac{N-p}{p-1}\right)^{p-1} \overline{A} & \text{if } N > 3p-2, \\ \alpha_\theta p\left(\dfrac{N-p}{p-1}\right)^{p-1} & \text{if } N = 3p-2 \end{cases}$$

with

$$\overline{A} = \int_{\mathbf{R}^N} \frac{|y|^2}{[1+|y|^{p/(p-1)}]^{N-1}} dy \Big/ \int_{\mathbf{R}^N} \frac{|y|^{-(p-2)}}{[1+|y|^{p/(p-1)}]^{N-p}} dy.$$

Remark 1.1. It is not obvious whether λ_1 is greater than λ_0 or not. If $\|a(x)\|_{L^r(\Omega)}$ ($r \neq \infty$) is sufficiently small, then λ_1 is greater than λ_0. In this situation, (QE) with $q = p^*$ has a non-trivial solution $u^* \in W^{1,p}(\theta, \Omega)$ for any $\lambda \in (\lambda_0, \lambda_1)$.

Remark 1.2. Let $p = 2$ and $N \geq 4 - s$. If $r = \infty$ and $\|a(x)\|_{L^\infty(\Omega)} < 2c_\theta/\alpha_\theta N$, then we can show that (QE) with $q = 2^*$ admits at least one non-trivial solution $u^* \in W^{1,2}(\theta, \Omega)$ for every $\lambda \in (\alpha_\theta N, \lambda_1)$.

Remark 1.3. By using the technique of Egnell [10] that (QE) has no positive solution in $W^{1,p}(\theta, \Omega)$ for every $\lambda \geq \lambda_1$.

It is easily shown that weak solutions of (QE) are critical points of the following functional

$$I_\theta(u) := \frac{1}{p} \int_\Omega e^{p\theta(x)}(|\nabla u|^p - \lambda a(x)|u|^p)dx - \frac{1}{q} \int_\Omega e^{p\theta(x)} K(x)|u|^q dx. \tag{1.6}$$

To seek for critical points of I_θ, we first prepare some properties of weighted Sobolev spaces $L^p(\theta, \Omega)$ and $W^{1,p}(\theta, \Omega)$ in Section 2. These spaces are, in a sense, generalization of function spaces introduced by Escobedo-Kavian [11]. They have discussed $L^p(\theta)$ and $W^{1,p}(\theta)$ for $p = 2$ under slightly weaker conditions on θ (see Proposition 1.12 of [11]).We also refer to Kawashima [14] and Muramoto-Naito-Yoshida [17] in the special case $\theta = \alpha|x|^2$. Similarly to [11], we will prove the compactness of some Sobolev's embedding $W^{1,p}(\theta, \Omega)$ into $L^p(\theta, \Omega)$ under (A1).

Our analysis for Theorem 1.1 and 1.2 is based on the Moutain Pass Theorem. In case $q < p^*$, the idea of the proof of Theorem 1.1 is standard (see author's paper [18, Section 3]). We will mainly exhibit the strategy of the proof of Theorem 1.2 in this page.

In general, the embedding $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ is not compact for general $\Omega \subset \mathbf{R}^N$. In order to resolve this point, Lions [15, 16] has studied some behavior of sequences $\mu_m := |\nabla u_m|^p dx$ and $\nu_m := |u_m|^{p^*} dx$, where $\{u_m\}$ is a weakly convergent sequence in $D^{1,p}(\mathbf{R}^N)$. From his method, one can find useful information on these sequences at some local points. On the other hand, some author have introduced the idea for these behaviors at infinity in the affirmative sense. They are, for example, Ben-Naoum, Troestler and Willem [4], Bianch, Chabrowski and Szulkin [5] and Chabrowski [7]. In Section 3, we will study the behavior of sequences $\overline{\mu}_m := e^{p\theta(x)}|\nabla u_m|^p dx$ and $\overline{\nu}_m := e^{p\theta(x)} K(x)|u_m|^{p^*} dx$ corresponding to the functional I_θ. We also clarify the relationship between (μ_m, ν_m) and $(\overline{\mu}_m, \overline{\nu}_m)$.

Being based on these preparations, we will apply a standard variational argument to I_θ with $q = p^*$ in Section 4. To assure Palais-Smale condition, it is sufficient to show that the energy level associated to I_θ must be below a certain critical level. We estimate this energy level by using a special function defined by Talenti [23] in Section 5.

2 Preliminary and some notations corresponding to weighted Sobolev spaces

To solve (QE), we introduce some notations corresponding to weighted Sobolev spaces.

For $1 < p \leq \infty$, let $L^p(\Omega)$ and $L^p(\mathbf{R}^N)$ denote the Lebesgue spaces with norms $\|\cdot\|_{p,\Omega}$ and $\|\cdot\|_p$, respectively. Let $W_0^{1,p}(\Omega)$ and $W^{1,p}(\mathbf{R}^N)$ be the usual Sobolev spaces whose norms are defined by $\|u\|_{1,p,\Omega}^p := \|u\|_{p,\Omega}^p + \|\nabla u\|_{p,\Omega}^p$ and $\|u\|_{1,p}^p := \|u\|_p^p + \|\nabla u\|_p^p$, respectively. Denote by $D^{1,p}(\mathbf{R}^N)$ is the completion of $C_0^\infty(\mathbf{R}^N)$ with respect to norm $\|u\|_{D^{1,p}(\mathbf{R}^N)} := \|\nabla u\|_p$.

Let $\theta \in C^2(\Omega)$ be a non-negative function satisfying $(A1)$. For such θ, we define $L^p(\theta, \Omega)$ and $W^{1,p}(\theta, \Omega)$ by (1.4) and (1.5), respectively. The norm of $L^p(\theta, \Omega)$ is defined by

$$\|u\|_{p,\theta,\Omega} = \left\{ \int_\Omega e^{p\theta(x)} |u|^p dx \right\}^{\frac{1}{p}}.$$

We also define $\|u\|_{1,p,\theta,\Omega}^p = \|u\|_{p,\theta,\Omega}^p + \|\nabla u\|_{p,\theta,\Omega}^p$. It is easy to see that $W^{1,p}(\theta, \Omega)$ is a Banach space with norm $\|\cdot\|_{1,p,\theta,\Omega}$. If $\Omega = \mathbf{R}^N$, we simply write $L^p(\theta)$ and $W^{1,p}(\theta)$ in place of $L^p(\theta, \mathbf{R}^N)$ and $W^{1,p}(\theta, \mathbf{R}^N)$, respectively. The norms corresponding to $L^p(\theta)$ and $W^{1,p}(\theta)$ are written as $\|\cdot\|_{p,\theta} := \|\cdot\|_{p,\theta,\mathbf{R}^N}$ and $\|\cdot\|_{1,p,\theta} := \|\cdot\|_{1,p,\theta,\mathbf{R}^N}$, respectively.

Lemma 2.1. *Assume $(A1)$. Then there exists a positive constant C which depends on p, N, θ and Ω, such that*

$$C \int_\Omega e^{p\theta}(1 + |\nabla\theta|^p)|u|^p dx \leq \int_\Omega e^{p\theta} |\nabla u|^p dx \tag{2.1}$$

for all $u \in W^{1,p}(\theta, \Omega)$.

Lemma 2.2. *Assume $(A1)$ and $q \in [p, p^*]$. Then there exists $c > 0$ such that $\|u\|_{q,\theta,\Omega} \leq c\|u\|_{1,p,\theta,\Omega}$ for all $u \in W^{1,p}(\theta, \Omega)$. Moreover the embedding $W^{1,p}(\theta, \Omega) \subset L^q(\theta, \Omega)$ is compact for $q \in [p, p^*)$.*

Lemma 2.3. *Assume $(A1)$. Then $W^{1,p}(\theta, \Omega)$ is a reflexive Banach space for every $m \geq 0$.*

Esspecially for the case $q = p^*$, we introduce the following quotient:

$$Q_{\lambda,K,\theta}(u) := \frac{\displaystyle\int_\Omega e^{p\theta(x)} |\nabla u(x)|^p dx - \lambda \int_\Omega e^{p\theta(x)} a(x)|u(x)|^p dx}{\displaystyle\left(\int_\Omega e^{p\theta(x)} K(x)|u(x)|^{p^*} dx \right)^{p/p^*}}. \tag{2.2}$$

From the definition of $V(x)$, one can express $\int_\Omega e^{p\theta(x)} K(x)|u|^{p^*} dx = \int_\Omega V(x) \cdot e^{p^*\theta(x)} |u|^{p^*} dx$. So due to (B1) and (C2), $Q_{\lambda,K,\Omega} : W^{1,p}(\theta, \Omega) \to \mathbf{R}$ is well defined for every $\lambda \in \mathbf{R}$. Define

$$S_{\lambda,K,\theta}(\Omega) := \inf_{u \in W^{1,p}(\theta,\Omega) \setminus \{0\}} Q_{\lambda,K,\theta}(u).$$

In this case, there is a close relationship between seeking critical points of I_θ and seeking a minimizer of $S_{\lambda,K,\theta}(\Omega)$ in case $q = p^*$ (see Section 5). Furthermore

$$S_{0,\theta}^*(\Omega) := \inf_{u \in W^{1,p}(\theta,\Omega) \setminus \{0\}} \left\{ \int_\Omega e^{p\theta} |\nabla u|^p dx \Big/ \left(\int_\Omega e^{p^*\theta} |u|^{p^*} dx \right)^{p/p^*} \right\}.$$

This is the special case $\lambda = 0$, and $V(x) \equiv 1$ for $S_{\lambda,K,\theta}(\Omega)$. Note that $S^*_{0,\theta}(\Omega)$ is the best constant corresponding to the embedding $W^{1,p}(\theta,\Omega) \subset L^{p^*}(\theta,\Omega)$. In case $\Omega = \mathbf{R}^N$ we denote $S_{\lambda,K,\theta} := S_{\lambda,K,\theta}(\mathbf{R}^N)$ and $S^*_{0,\theta} := S^*_{0,\theta}(\mathbf{R}^N)$, respectively.

Finally, define $S_0(\Omega)$ as follows;

$$S_0(\Omega) := \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \left\{ \int_\Omega |\nabla u|^p dx \Big/ \left(\int_\Omega |u|^{p^*} dx \right)^{p/p^*} \right\}.$$

We simply write S_0 instead of $S_0(\mathbf{R}^N)$. Talenti [23] has shown that S_0 is attained by

$$v_\varepsilon(x) := \frac{1}{[\varepsilon + |x - x_0|^{p/(p-1)}]^{(N-p)/p}} \tag{2.3}$$

for any $\varepsilon > 0$ and $x_0 \in \mathbf{R}^N$.

3 Concentration-Compactness Principle corresponding to weighted Sobolev spaces

Proposition 3.1. *Let $\{u_m\}$ be a sequence which converges to u^* weakly in $W^{1,p}(\theta)$ (also converges weakly in $D^{1,p}(\mathbf{R}^N)$). Then there exist at most countable index sets $J, \overline{J}, J^*$, families $A = \{x_j, j \in J\}$, $B = \{\overline{x}_j, j \in \overline{J}\}$, $C = \{x^*_j, j \in J^*\}$ of distinct points in $\mathbf{R}^N$, and sets $\{\nu_j, \mu_j; j \in J\}$, $\{\overline{\nu}_j, \overline{\mu}_j; j \in \overline{J}\}$, $\{\nu^*_j, \mu^*_j; j \in J^*\}$ of positive numbers such that*

(i)
$$\nu_m := |u_m|^{p^*} dx \rightharpoonup \nu = |u^*|^{p^*} dx + \sum_{j \in J} \nu_j \delta_{x_j},$$
$$\mu_m := |\nabla u_m|^p dx \rightharpoonup \mu \geq |\nabla u^*|^p dx + \sum_{j \in J} \mu_j \delta_{x_j},$$

(ii)
$$\overline{\nu}_m := e^{p\theta(x)} K(x) |u_m|^{p^*} dx \rightharpoonup \overline{\nu} = e^{p\theta(x)} K(x) |u^*|^{p^*} dx + \sum_{j \in \overline{J}} \overline{\nu}_j \delta_{\overline{x}_j},$$
$$\overline{\mu}_m := e^{p\theta(x)} |\nabla u_m|^p dx \rightharpoonup \overline{\mu} \geq e^{p\theta(x)} |\nabla u^*|^p dx + \sum_{j \in \overline{J}} \overline{\mu}_j \delta_{\overline{x}_j},$$

(iii)
$$\nu^*_m := e^{p^*\theta(x)} |u_m|^{p^*} dx \rightharpoonup \nu^* = e^{p^*\theta(x)} |u^*|^{p^*} dx + \sum_{j \in J^*} \nu^*_j \delta_{x^*_j},$$
$$\mu^*_m := e^{p\theta(x)} |\nabla u_m|^p dx \rightharpoonup \mu^* \geq e^{p\theta(x)} |\nabla u^*|^p dx + \sum_{j \in J^*} \mu^*_j \delta_{x^*_j}.$$

Here $S_0(\nu_j)^{p/p^*} \leq \mu_j$ for all $j \in J$, $S_{0,K,\theta}(\overline{\nu}_j)^{p/p^*} \leq \overline{\mu}_j$ for all $j \in \overline{J}$ and $S^*_{0,\theta}(\nu^*_j)^{p/p^*} \leq \mu^*_j$ for all $j \in J^*$. In particular, $\sum_{j\in J}(\nu_j)^{p/p^*}, \sum_{j \in \overline{J}}(\overline{\nu}_j)^{p/p^*}$ and $\sum_{j \in J^*}(\nu^*_j)^{p/p^*}$ are bounded.

P. L. Lions [15, 16] has first established an effective method in the study of variational problems involving critical Sobolev exponents. It is so called as Concentration-Compactness Principle. It is well known that his proposition plays an important role when one encounters the lack of compactness due to the presence of critical Sobolev exponents. The proofs of them are also found in the monograph of Struwe [22, pp. 44-46]. Similar to their explanation, one can easily prove Proposition 3.1.

From our proposition, one can understand the behavior of weak convergent sequences at bounded points in detail, which converges weakly in some Sobolev spaces. Roughly speaking, these are only concerned with concentrations of a weakly convergent sequence at local points and do not provide any information about the loss of mass at infinity.

Proposition 3.2. *Let $\Omega \subset \mathbf{R}^N$ be a general unbounded domain. Let $\{u_m\} \subset W^{1,p}(\theta)$ satisfy the conditions of Proposition 3.1. Define*

$$\overline{\nu}_{\infty,\Omega} := \lim_{R\to\infty} \overline{\lim_{m\to\infty}} \int_{\Omega(R)} e^{p\theta} K|u_m|^{p^*}dx, \ \overline{\mu}_{\infty,\Omega} := \lim_{R\to\infty} \overline{\lim_{m\to\infty}} \int_{\Omega(R)} e^{p\theta}|\nabla u_m|^p dx,$$

$$\nu^*_{\infty,\Omega} := \lim_{R\to\infty} \overline{\lim_{m\to\infty}} \int_{\Omega(R)} e^{p^*\theta}|u_m|^{p^*}dx.$$

with $\Omega(R) := \Omega \cap \{|x| > R\}$. Then

$$\overline{\lim_{m\to\infty}} \int_\Omega e^{p\theta} K|u_m|^{p^*}dx = \int_\Omega d\overline{\nu} + \overline{\nu}_{\infty,\Omega}, \quad \overline{\lim_{m\to\infty}} \int_\Omega e^{p\theta}|\nabla u_m|^p dx = \int_\Omega d\overline{\mu} + \overline{\mu}_{\infty,\Omega},$$

$$\overline{\lim_{m\to\infty}} \int_\Omega e^{p^*\theta}|u_m|^{p^*}dx = \int_\Omega d\nu^* + \nu^*_{\infty,\Omega}.$$

In particular,

$$S_{0,K,\theta}(\Omega)(\overline{\nu}_{\infty,\Omega})^{p/p^*} \le \overline{\mu}_{\infty,\Omega} \text{ and } S^*_{0,\theta}(\Omega)(\nu^*_{\infty,\Omega})^{p/p^*} \le \overline{\mu}_{\infty,\Omega}.$$

Lemma 3.1. *Let $V(x) = e^{(p-p^*)\theta(x)}K(x) \in L^\infty(\Omega)$ and define $V(\infty) := \overline{\lim_{|x|\to\infty}} V(x)$. Then $\overline{\nu}_{\infty,\Omega} \le V(\infty)\nu^*_{\infty,\Omega}$.*

4 Proof of Theorem 1.2.

Theorem 4.1. *Assume* (A1), (B1), (B2) and (C2). *For every $\lambda < \lambda_1$, any sequence $\{u_m\}$ satisfying*

$$I_\theta(u_m) \to b_\theta, \tag{4.1}$$

$$I'_\theta(u_m) \to 0 \quad \text{in} \quad (W^{1,p}(\theta,\Omega))^* \tag{4.2}$$

contains a convergent subsequence in $W^{1,p}(\theta,\Omega)$, provided that

$$b_\theta < \frac{1}{N} S_0^{N/p} \|V\|_{\infty,\Omega}^{-(N-p)/p} \ (:= b^*_\theta).$$

Proof. Define the functional I_θ by (1.6) in Section 1. Set $X = W^{1,p}(\theta,\Omega)$ and define

$$\|u\|_X^p := \|\nabla u\|_{p,\theta,\Omega}^p = \int_\Omega e^{p\theta}|\nabla u|^p\, dx.$$

By Lemma 2.1, $\|\cdot\|_X$ gives an equivalent norm with $\|\cdot\|_{1,p,\theta,\Omega}$ in X.

From (4.1) and (4.2), one can easily check that $\|u_m\|_X$ is a bounded sequence; so there exists a subsequence (still denoted by $\{u_m\}$) such that

$$u_m \rightharpoonup u^* \qquad \text{weakly in } X.$$

We consider the natural extension of u_m and u^* by setting $u_m = u^* \equiv 0$ in $\mathbf{R}^N \backslash \Omega$. Without loss of generality, we may also assume

$$u_m \rightharpoonup u^* \qquad \text{weakly in } W^{1,p}(\theta).$$

So we can apply the concentration-compactness principle defined in Section 3.

First, by using (4.1) and (4.2), we can estimate the values of coefficients defined in the statements of Propositions 3.1-3.2 as follows.

Lemma 4.1. $\overline{\mu}_{\infty,\Omega} = \overline{\nu}_{\infty,\Omega} = 0$.

Lemma 4.2. *Let $\{u_m\}$ be a sequence satisfying* (4.1) *and* (4.2). *Then $A = \{x_j, j \in J\}$, $B = \{\overline{x}_j, j \in \overline{J}\}$ and $C = \{x_j^*, j \in J^*\}$ in Proposition 3.1 are finite sets. In particular, $\overline{\mu}_j = \overline{\nu}_j \geq S_0^{N/p} \|V\|_{\infty,\Omega}^{-(N-p)/p}$ for every $j \in J$.*

Secondly, from the fact that $\{u_m\}$ is a weakly convergent sequence in X, we can derive following two Lemmas.

Lemma 4.3. *Let $\{u_m\}$ be a sequence which converges to u^* weakly in $W^{1,p}(\theta, \Omega)$. If $V(x) = e^{(p-p^*)\theta(x)} K(x) \in L^\infty(\Omega)$, then there exists a subsequence $\{u_{m_k}\} \subset \{u_m\}$, still denoted by $\{u_m\}$, such that*

(i) $$e^{p\theta(x)} K(x) |u_m|^{p^*-2} u_m \rightharpoonup e^{p\theta(x)} K(x) |u^*|^{p^*-2} u^* \quad \text{weakly in} \quad L^{N/p}(\theta, \Omega)$$

and

(i) $$e^{p\theta(x)} a(x) |u_m|^{p-2} u_m \rightharpoonup e^{p\theta(x)} a(x) |u^*|^{p-2} u^* \quad \text{weakly in} \quad L^{p'}(\theta, \Omega)$$

with $1/p + 1/p' = 1$.

Proof. The proof is similar to Drábek-Huang [9, Proposition 2.3]. □

Lemma 4.4. *Assume* (A1), (B1) *and* (C1). *Suppose $u_m \rightharpoonup u^*$ weakly in $W^{1,p}(\theta, \Omega)$ and $e^{p\theta(x)} K(x) |u_m|^{p^*} dx \rightharpoonup e^{p\theta(x)} K(x) |u^*|^{p^*} dx + \sum_{j \in \overline{J}} \overline{\nu}_j \delta_{\overline{x}_j}$ in the weak*-sense of measures. If $\overline{J}$ is a finite set, then there exists a subsequence $\{u_{m_k}\} \subset \{u_m\}$, still denoted by $\{u_m\}$, such that for each $1 \leq i \leq N$;*

$$\begin{cases} e^{\theta(x)} \dfrac{\partial u_m}{\partial x_i} \to e^{\theta(x)} \dfrac{\partial u^*}{\partial x_i} & \text{a.e. on } \Omega, \\ e^{p\theta(x)} |\nabla u_m|^{p-2} \dfrac{\partial u_m}{\partial x_i} \rightharpoonup e^{p\theta(x)} |\nabla u^*|^{p-2} \dfrac{\partial u^*}{\partial x_i} & \text{weakly in } L^{p'}(\theta, \Omega) \end{cases} \tag{4.3}$$

with $1/p + 1/p' = 1$.

Proof. We follow the idea of Xiping [24, Theorem 3.1] to show (4.3). □

We continue the proof of Theorem 5.1. Here (4.2) implies $\langle I'_\theta(u_m), \varphi\rangle_X \to 0$ as $m \to +\infty$. That is,

$$\int_\Omega e^{p\theta(x)}|\nabla u_m|^{p-2}\nabla u_m \cdot \nabla\phi dx - \int_\Omega e^{p\theta(x)}(\lambda a(x)|u_m|^{p-2}u_m + K(x)|u_m|^{p^*-2}u_m)\phi dx \to 0$$

for all $\phi \in W^{1,p}(\theta,\Omega)$. Hence it follows from Lemmas 4.3-4.5 that

$$-\Delta_p u^* - p\nabla\theta(x)\cdot\nabla u^*|u^*|^{p-2} = \lambda a(x)|u^*|^{p-2}u^* + K(x)|u^*|^{p^*-2}u^*$$

in X^*; so that $I'_\theta(u^*) = 0$.

For any $\sigma > 0$, there exists $m > 0$ enough large so that

$$b_\theta + \sigma > I_\theta(u_m) = \frac{1}{p}\int_\Omega e^{p\theta(x)}(|\nabla u_m|^p - \lambda a(x)|u_m|^p)dx - \frac{1}{p^*}\int_\Omega e^{p\theta(x)}K(x)|u_m|^{p^*}dx.$$

Letting $\overline{\lim_{m\to\infty}}$ in above inequality, it follows from Propositions 3.1-3.2 and Lemmas 4.2-4.3 that

$$\begin{aligned} b_\theta + \sigma &> \overline{\lim_{m\to\infty}} I_\theta(u_m) = \overline{\lim_{m\to\infty}} I_\theta(u_m) - \frac{1}{p}\langle I'_\theta(u^*), u^*\rangle_X \\ &\geq \frac{1}{N}\int_\Omega e^{p\theta(x)}K(x)|u^*|^{p^*}dx + \frac{1}{p}\sum_{j\in\overline{J}}\overline{\mu}_j - \frac{1}{p^*}\sum_{j\in\overline{J}}\overline{\nu}_j \\ &\geq \frac{1}{N}\int_\Omega e^{p\theta(x)}K(x)|u^*|^{p^*}dx + \sum_{j\in\overline{J}} b^*_\theta. \end{aligned} \tag{4.4}$$

Since $b_\theta < b^*_\theta$, then we have $\overline{\nu}_j = 0$ for all $j \in \overline{J}$ from (4.4). This implies

$$\int_\Omega e^{p\theta(x)}K(x)|u_m|^{p^*}dx \to \int_\Omega e^{p\theta(x)}K(x)|u^*|^{p^*}dx$$

as $m \to \infty$.

Finally from the idea of Dinca-Jebelean-Mawhin [8], we can conclude that $\|\nabla u_m\|_X \to \|\nabla u^*\|_X$. So one can see $u_m \to u^*$ in X. Thus I_θ satisfies Palais-Smale condition. □

5 Estimate of mini-max level b_θ

By Theorem 5.1, the proof of Theorem 1.2 will be complete if we can show

$$b_\theta = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\theta(\gamma(t)) < \frac{1}{N}S_0^{N/p}\|V\|_{\infty,\Omega}^{-(N-p)/p} \tag{5.1}$$

where $\Gamma : [0,1] \to W^{1,p}(\theta,\Omega)$ is a set of continuous paths which connect 0 and u satisfying $I_\theta(u) < 0$. Indeed, we can show the existence of solutions of (QE) by the using Mountain Pass Theorem (e.g., see author's paper [18, Section 3]).

There is a close relationship between critical points of I_θ and a minimizer of $Q_{\lambda,K,\theta}$ defined in (2.2). For example, Struwe [22, pp.177-178] gives us the relationship between b_θ and $S_{\lambda,K,\theta}(\Omega)$.

Lemma 5.1. *Define* $S_{\lambda,K,\theta}(\Omega) := \inf\{Q_{\lambda,K,\theta}(u); u \in W^{1,p}(\theta,\Omega)\backslash\{0\}\}$. *Then*

$$b_\theta = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\theta(\gamma(t)) = \frac{1}{N}(S_{\lambda,K,\theta}(\Omega))^{N/p}. \tag{5.2}$$

It follows from Lemma 4.1 that (5.1) is equivalent to

$$S_{\lambda,K,\theta}(\Omega) < S_0\|V\|_{\infty,\Omega}^{-(N-p)/N}. \tag{5.3}$$

So we will show (5.3) instead of (5.1).

Let $\overline{\varphi}_0(x)$ be a cut-off funtion and define $w_\varepsilon(x) := e^{-\theta(x)}v_\varepsilon(x)\overline{\varphi}_0(x)$ where v_ε is a special function defined by (2.3). We may assume $\mathrm{dist}(x_0,\Omega) > 3$ without loss of generality. We observe that $w_\varepsilon \in W^{1,p}(\theta,\Omega)$ for all $\varepsilon > 0$. By using the technique of Brezis-Nirenberg [6], we can obtain the following Lemma.

Lemma 5.2. *Let* $s \in (p-2,p)$ *and* $N \geq p^2 - s(p-1)$. *Then there exists* $\varepsilon = \varepsilon(\lambda) > 0$ *such that* $Q_{\lambda,K,\theta}(w_\varepsilon) < S_0\|V\|_{\infty,\Omega}^{-(N-p)/N}$ *for every* $\lambda > 0$.

Lemma 5.3. *Let* $s = p-2$ *and* $N \geq 3p-2$. *There exist* $\lambda_0 = \lambda_0(p,N) > 0$, *defined in Theorem 1.2, such that if* $\lambda > \lambda_0$, *there exists* $\varepsilon = \varepsilon(\lambda) > 0$ *satisfying* $Q_{\lambda,K,\theta}(w_\varepsilon) < S_0\|V\|_{\infty,\Omega}^{-(N-p)/N}$.

6 Related problem corresponding to (QE)

Consider with the following quasilinear elliptic equation

$$\begin{cases} -\Delta_p u - p\nabla\theta(x)\cdot\nabla u|\nabla u|^{p-2} = \lambda a(x)|u|^{\gamma-2}u + K(x)|u|^{q-2}u & \text{in} \quad \Omega, \\ \qquad u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{6.1}$$

where $1 < p < N$, $1 < \gamma < p < q \leq p^* := Np/(N-p)$ and $\Omega \subset \mathbf{R}^N$ is an unbounded domain with smooth boundary $\partial\Omega$. Note that equation (QE) is a special case of (6.1) with $\gamma = p$.

If we put $\theta(x) = 0$, (6.1) is written as

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{\gamma-2}u + K(x)|u|^{q-2}u & \text{in} \quad \Omega, \\ \qquad u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{6.2}$$

This problem (6.2) is first studied by Ambrosetti-Brezis-Cerami [2] in case $p = 2$, $a(x) = K(x) \equiv 1$ and $\Omega \subset \mathbf{R}^N$ is bounded. They have shown the multiplicity of solutions of (6.2) by using supersolution-subsolution method and variational method. After their work, many authors have studied to clarify the structure of solutions of (6.2). See, e.g., Alves-Goncalves-Miyagaki [1], Ambrosetti-Garcia Azorero-Peral Alonso [3], Garcia Azorero-Peral Alonso [12] and Huang [13].

We are interested with case $\theta(x) \not\equiv 0$ in (6.1). We put assumptions on $a(x)$ as follows:

(B3) $b(x) := e^{(p-\gamma)\theta(x)}a(x) \in L^r(\Omega)$ for some $r \in (p^*/(p^*-\gamma), p/(p-\gamma)]$.
(B4) there exists $s > \{p(N-2)-\gamma(N-p)\}/p$ such that $b(x) = |x|^{-s} + o(|x|^{-s})$ as $|x| \to 0$.

Then we can get the following theorems in case $p = 2$.

Theorem 6.1 (case $q < 2^*$ [20]). *Assume* (A1), (B3) *and* (C1). *Then* (6.1) *admits at least two positive solutions* $u^*, u_* \in W^{1,2}(\theta, \Omega)$ *for sufficiently small* $\lambda > 0$.

Theorem 6.2 (case $q = 2^*$ [20]). *Let* $N(\gamma - 1) \geq 2\gamma - s$. *Assume* (A1), (B3), (B4) *and* (C2). *Then* (6.1) *admits at least two positive solutions* $u^*, u_* \in W^{1,2}(\theta, \Omega)$ *for sufficiently small* $\lambda > 0$.

Remark 6.1. In view of [2], the author guesses that (6.1) has the following properties

$$\begin{cases} \text{i)} & \text{there exsits } \Lambda > 0 \text{ such that (6.1) has at least two positive solutions} \\ & u^*, u_* \in W^{1,p}(\theta, \Omega) \text{ for every } \lambda \in (0, \Lambda), \\ \text{ii)} & \text{(6.1) has a positive solution } u^* \in W^{1,p}(\theta, \Omega) \text{ for every } \lambda = \Lambda, \\ \text{iii)} & \text{(6.1) has no positive solutions for } \lambda \geq \Lambda. \\ \text{iv)} & \text{The minimal solution of (6.1) converges to zero as } \gamma \to p. \end{cases}$$

However, we do not have proofs for the above properties.

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