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Interfacial energies in two dimensional phase field models and related variational problems

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1 Introduction

Let $n \in \mathbb{N}$, and let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz continuous boundary. For any given constant θ_*, let $\mathscr{F}_{\theta_*}$ be a functional from $L^2(\Omega)$ into $\overline{\mathbb{R}}$. Then, the following type of equation:

$$\nabla \mathscr{F}_{\theta_*}(w) = 0 \text{ in } L^2(\Omega); \tag{1.1}$$

is called as an Euler-Lagrange equation for the functional $\mathscr{F}_{\theta_*}$, where $\nabla \mathscr{F}_{\theta_*}$ is the derivative of the functional $\mathscr{F}_{\theta_*}$ in an appropriate sense.

Equation (1.1) often appears as a steady-state problem for a mathematical model of solid-liquid phase transitions (cf. [13, 14]). In the context, the constant θ_* is the (given) relative temperature, and the unknown function w is the so-called (nonconserved) order parameter that indicates the physical situation of the material.

As is well known, the solid-liquid phase transition is a phenomena of dramatic changes between solid and liquid states in a material (like H_2O), and it is said that such dramatic changes occur around a characteristic temperature, known as "critical temperature". Here, let us set the value 0 as the degree of the critical temperature, and indicate the physical situation in the following way:

$$w(x) \begin{cases} = 1, & \text{if the physical situation is liquid at } x \in \Omega, \\ = -1, & \text{if the physical situation is solid at } x \in \Omega, \\ \in (-1, 1), & \text{otherwise.} \end{cases}$$

The functional $\mathscr{F}_{\theta_*}$ is usually called as a free energy, and in most cases it is given by the following style:

$$\mathscr{F}_{\theta_*}(w) := \mathscr{I}(\nabla w) + \int_\Omega f_{\theta_*}(w)\, dx.$$

Here, the term $\mathscr{I}(\nabla w)$ is called as an interfacial energy, and it mainly depends on the variation (gradient) of the parameter w. On the other hand, the second integral is called as a bulk energy, and the density f_{θ_*} is usually set by a double-well function satisfying the following conditions:

(dw1) f_{θ_*} has two global minimizers 1 and -1 when $\theta_* = 0$;

(dw2) f_{θ_*} has a (unique) global minimizer 1 (resp. -1) when $\theta_* > 0$ (resp. $\theta_* < 0$).

The above conditions imply the stronger stability of the liquid (resp. solid) phase than the another one, when the temperature is higher (resp. lower) than the critical temperature. So, they are important conditions to characterize the dynamics of solid-liquid phase transitions.

Recently, the authors of [13, 14] introduced the following type of functional as one of possible choices of the free energy:

$$w \in L^2(\Omega) \mapsto \sigma_0 \int_\Omega |\nabla w| + \int_\Omega \left\{ I_{[-1,1]}(w) - \frac{1}{2}w^2 - \theta_* w \right\} dx. \tag{1.2}$$

In this free energy, the interfacial energy is given by the total variation functional with a small positive constant σ_0. The total variation energy is introduced to represent the contribution from the surface tension on the interface. In the mathematical framework, the contribution is represented by a function which characterize the curvature of level curves of the parameter w, and such a function is derived from the calculation of the first variation of the interfacial energy. On the other hand, the density of the bulk energy is given by the sum of convex and concave functions. Here, $I_{[-1,1]}(\,\cdot\,)$ is the so-called indicator function on the closed interval $[-1, 1]$, that is defined as follows:

$$I_{[-1,1]}(\tau) := \begin{cases} 0, \text{ if } \tau \in [-1, 1], \\ +\infty, \text{ otherwise.} \end{cases}$$

Since the indicator function constrains the range of parameters onto the closed interval $[-1, 1]$, the density of the bulk energy is certainly a double well function satisfying conditions (dw1) and (dw2) in the above.

In this case, the corresponding Euler-Lagrange equation is calculated by a variational inequality associated with the total variation functional. In recent years, some structural results of solutions of the variational inequality have been reported in some papers. For example, the authors of [13] studied the structure of one-dimensional solutions, and showed that any one-dimensional solution is a piecewise constant function having at most a finite number of discontinuities. Also, the structure of multi-dimensional solutions was studied in [14]. The authors of [14] considered only piecewise constant steady-state solutions, and characterized the shapes of interfaces by spheres with sufficiently large radii. The idea of the characterization by spheres was referred to the result in [5], and we would see from [5, 14] that the interfaces should have the regularity of Hölder continuity in $C^{1,1}$-class. Moreover, it is shown in [14] that the stability of (steady-state) solutions is also characterized on the basis of spheres having sufficiently large radii. But, this result also implies that the anisotropy of materials is not assumed in this free energy. The main objective of this paper is to propose an interfacial energy involving the anisotropic effects, and investigate the structure of steady-state solutions from the geometric viewpoint.

In this paper, we shall try to represent the anisotropic effects by indefinite surface tension coefficients. More precisely, for any fixed nonnegative and Lipschitz continuous function σ on $\overline{\Omega}$, we take the following functional as the interfacial energy:

$$z \in L^1(\Omega) \mapsto \bar{V}_\sigma(z) := \inf \left\{ \liminf_{i \to +\infty} \int_\Omega \sigma |\nabla z_i| \, dx \;\middle|\; \begin{array}{l} \{z_i\} \subset W^{1,1}(\Omega) \text{ and} \\ z_i \to z \text{ in } L^1(\Omega) \\ \text{as } i \to +\infty \end{array} \right\}; \tag{1.3}$$

and give the free energy $\mathscr{F}_{\theta_*}(\,\cdot\,)$ by putting:

$$\mathscr{F}_{\theta_*}(w) := \bar{V}_\sigma(w) + \int_\Omega \left\{ I_{[-1,1]}(w) - \frac{1}{2}w^2 - \theta_* w \right\} dx, \; w \in L^2(\Omega); \tag{1.4}$$

with the same density of the bulk energy as in (1.2). Then the corresponding Euler-Lagrange equation is formulated as the following variational inequality:

$$\begin{aligned} &\bar{V}_\sigma(w) + \int_\Omega \{I_{[-1,1]}(w) - (w + \theta_*)w\} \, dx \\ &\le \bar{V}_\sigma(z) + \int_\Omega \{I_{[-1,1]}(z) - (w + \theta_*)z\} \, dx \;\; \text{for any } z \in L^2(\Omega). \end{aligned} \tag{1.5}$$

Here, let us consider the convex part of the free energy given in (1.4). Then, we notice that sublevel sets of the convex part may be not compact in general. In this study, the lack of the compactness is a serious problem, because we need it to characterize the large time behavior for corresponding evolution systems by the variational inequality (1.5).

In order to escape such a problem, it is typically assumed that σ is (strictly) positive on $\overline{\Omega}$. In fact, since the interfacial energy of this case dominates the total variation of the parameter, the compactness of sublevel sets immediately follows from the embedding theorem of $BV(\Omega) \cap L^\infty(\Omega)$ into $L^2(\Omega)$. But, the interfacial energy of this case, as well as that in (1.2), makes the shapes of interfaces be smooth. It implies that we have to give up to represent interfaces having corners, like snow crystals.

In the former part of this paper, we shall investigate fundamental properties of the interfacial energy as in (1.3), and introduce some special conditions such that:

$$\left\{ \begin{array}{l} \bullet \text{ the set } \sigma^{-1}(0) \text{ of zero points of } \sigma \text{ is nonempty,} \\ \bullet \text{ the interfacial energy as in (1.3) has compact sublevel sets.} \end{array} \right. \tag{1.6}$$

Then, some characterizations of solutions of (1.5) will be shown as one of the main results.

In the latter part of this paper, we will consider the case that $\Omega \subset \mathbb{R}^2$ (namely $n = 2$) and σ is piecewise linear, to show some examples of solutions of (1.5). Consequently, it will be seen that the interface may be more variable around zero-points of σ.

2 Preliminaries

For any abstract Banach space X, we denote by $|\cdot|_X$ the norm of X.

Let $n \in \mathbb{N}$. Throughout this paper, we denote by $\mathscr{L}^n$ the n-dimensional Lebesgue measure, and use this measure when it is specified nothing particular. Also, let us denote by $\mathscr{H}^n$ the n-dimensional Hausdorff measure.

Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary $\Gamma := \partial\Omega$, and let $\mathscr{B}(\Omega)$ be the class of all Borel subsets in Ω.

For any $m \in \mathbb{N}$ and any $\mathbb{R}^m$-valued Radon measure $\boldsymbol{\mu}$ in Ω, we denote by $|\boldsymbol{\mu}|$ the total variation of the Radon measure $\boldsymbol{\mu}$, that is defined as

$$|\boldsymbol{\mu}|(B) := \sup \left\{ \sum_{i=1}^{+\infty} |\boldsymbol{\mu}(B_i)| \;\middle|\; \{B_i\} \subset \mathscr{B}(\Omega) : \text{ pairwise disjoint family, } B = \bigcup_{i=1}^{+\infty} B_i \right\}.$$

As is well known, $\boldsymbol{\mu}$ is absolutely continuous with respect to $|\boldsymbol{\mu}|$. So, by Radon-Nikodým's theorem (cf. [1, Theorem 1.28 and Corollary 1.29]), there exists a unique $|\boldsymbol{\mu}|$-measurable function $\frac{\boldsymbol{\mu}}{|\boldsymbol{\mu}|} : \Omega \longrightarrow \mathbb{R}^m$ such that

$$\left| \frac{\boldsymbol{\mu}}{|\boldsymbol{\mu}|}(x) \right| = 1, \ |\boldsymbol{\mu}|\text{-a.e. } x \in \Omega \text{ and } \int_B d\boldsymbol{\mu} = \int_B \frac{\boldsymbol{\mu}}{|\boldsymbol{\mu}|} \, d|\boldsymbol{\mu}| \ \text{ for any } B \in \mathscr{B}(\Omega). \tag{2.1}$$

The $|\boldsymbol{\mu}|$-measurable function $\frac{\boldsymbol{\mu}}{|\boldsymbol{\mu}|}$ is known as the Radon-Nikodým density of $\boldsymbol{\mu}$ with respect to $|\boldsymbol{\mu}|$, and it is easily seen from (2.1) that

$$\begin{gathered} \left| \int_B \boldsymbol{f}(x) \cdot d\boldsymbol{\mu} \right| \le \int_B |\boldsymbol{f}(x)| \, d|\boldsymbol{\mu}| \ \text{ for any } B \in \mathscr{B}(\Omega) \\ \text{and any } |\boldsymbol{\mu}|\text{-integrable (summable) function } \boldsymbol{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m. \end{gathered} \tag{2.2}$$

Moreover,

$$\int_\Omega d|\boldsymbol{\mu}| = \sup \left\{ \int_\Omega \boldsymbol{\varphi} \cdot d\boldsymbol{\mu} \;\middle|\; \boldsymbol{\varphi} \in C_0(\Omega)^m \text{ satisfying } |\boldsymbol{\varphi}| \le 1 \text{ on } \overline{\Omega} \right\}. \tag{2.3}$$

For any function $f \in C(\Omega)$, we denote by spt f the support of f, and denote by $C_0(\Omega)$ the space of all continuous functions having compact supports in Ω. Also, for any $m \in \mathbb{N} \cup \{\infty\}$, we denote by $C_0^m(\Omega)$ the space of all functions in C^m-class having compact supports in Ω.

For each nonnegative and Lipschitz continuous function σ, let us define a functional V_σ on $L^1(\Omega)$ by putting:

$$V_\sigma(z) := \sup \left\{ \int_\Omega z \operatorname{div}(\sigma\boldsymbol{\varphi}) \, dx \;\middle|\; \boldsymbol{\varphi} \in C_0^1(\Omega)^n \text{ with } |\boldsymbol{\varphi}| \le 1 \text{ on } \overline{\Omega} \right\},$$

and denote by $D(V_\sigma)$ the effective domain of V_σ, namely

$$D(V_\sigma) := \{ \ z \in L^1(\Omega) \mid V_\sigma(z) < +\infty \ \}.$$

As is easily checked, if $\sigma \equiv 1$ on $\overline{\Omega}$, then the corresponding functional V_1 coincides with the so-called total variation functional.

Remark 2.1 The functional V_σ is proper l.s.c. and convex in $L^1(\Omega)$ whenever σ is nonnegative. In fact, since $V_\sigma(0) = 0$, the functional V_σ is proper. Also, the convexity immediately follows from the definition of V_σ.

For the check of the lower semicontinuity, let us take any $z \in L^1(\Omega)$ and any sequence $\{z_i\} \subset D(V_\sigma)$ satisfying $z_i \to z$ in $L^1(\Omega)$. Then, since σ is Lipschitz continuous,

$$\int_\Omega z \operatorname{div}(\sigma\varphi)\, dx = \lim_{i\to+\infty} \int_\Omega z_i \operatorname{div}(\sigma\varphi)\, dx \le \liminf_{i\to+\infty} V_\sigma(z_i)$$
$$\text{for any } \varphi \in C_0^1(\Omega)^n \text{ satisfying } |\varphi| \le 1 \text{ on } \overline{\Omega}.$$

Thus, taking the supremum with respect to φ, we conclude the lower semicontinuity of the functional V_σ.

The functional V_σ has a measure theoretical representation, stated as in the following lemma.

Lemma 2.1 *(Representation of V_σ by Radon measures) Let σ be a nonnegative and Lipschitz continuous function on $\overline{\Omega}$. Then, for any $z \in D(V_\sigma)$, there exists a unique $\mathbb{R}^n$-valued Radon measure $D_\sigma z$ such that:*

(i) $\displaystyle\int_\Omega z \operatorname{div}(\sigma\varphi)\, dx = -\int_\Omega \varphi(x)\cdot D_\sigma z$ *for any* $\varphi \in C_0^1(\Omega)^n$;

(iii) $\displaystyle V_\sigma(z) = \int_\Omega |D_\sigma z| = \sup\left\{ \int_\Omega \varphi(x)\cdot D_\sigma z \;\middle|\; \varphi \in C_0(\Omega)^n \text{ and } |\varphi| \le 1 \text{ on } \overline{\Omega} \right\}.$

Proof. Let us fix any $z \in D(V_\sigma)$, and define a linear functional from $C_0^1(\Omega)^n$ into $\mathbb{R}$ by putting

$$L_z(\varphi) := \int_\Omega z \operatorname{div}\varphi\, dx \quad \text{for any } \varphi \in C_0^1(\Omega)^n.$$

Then, by the definition of V_σ,

$$|L_z(\varphi)| \le V_\sigma(z)|\varphi|_{C(\overline{\Omega})} \quad \text{for any } \varphi \in C_0^1(\Omega)^n.$$

It implies that L_z can be extended to a continuous and linear functional on $C_0(\Omega)^n$. So, applying Riesz's representation theorem (cf. [1, Theorem 1.54] or [7, section 1.8]), we find a unique $\mathbb{R}^n$-valued Radon measure $D_\sigma z$ which satisfies the assertion (i). Furthermore, by (2.3) and the definition of V_σ,

$$V_\sigma(z) \le \sup\left\{ \int_\Omega \varphi(x)\cdot D_\sigma z \;\middle|\; \varphi \in C_0(\Omega)^n \text{ and } |\varphi| \le 1 \text{ on } \overline{\Omega} \right\} = \int_\Omega |D_\sigma z|.$$

Now, let us show the converse inequality. Let us take a sequence $\{\varphi_i\} \subset C_0^1(\Omega)^n$ to satisfy

$$|\varphi_i(x)| \le 1 \text{ and } \varphi_i(x) \to \tfrac{D_\sigma z}{|D_\sigma z|}(x) \text{ as } i \to +\infty,\ |D_\sigma z|\text{-a.e. } x \in \Omega.$$

Then, it follows from (2.1) and Lebesgue's convergence theorem that

$$\begin{aligned}\int_\Omega |D_\sigma z| &= \lim_{i\to+\infty} \int_\Omega \varphi_i(x)\cdot \tfrac{D_\sigma z}{|D_\sigma z|}(x)\ |D_\sigma z| = \lim_{i\to+\infty} \int_\Omega \varphi_i(x)\cdot D_\sigma z \\ &\le \sup_{i\in\mathbb{N}} \int_\Omega z \operatorname{div}(\sigma(-\varphi_i))\, dx \le V_\sigma(z).\end{aligned}$$

Thus we conclude the lemma. ■

Remark 2.2 The equality as in the assertion (i) of Lemma 2.1 holds for any Lipschitz continuous test function having a compact support. In fact, since σ is Lipschitz continuous, for any Lipschitz continuous function $\hat{\varphi} \in C_0(\Omega)^n$, there is a sequence $\{\varphi_i\} \subset C_0^1(\Omega)^n$ such that

$$\varphi_i \to \hat{\varphi} \text{ in } C(\overline{\Omega})^n \text{ and } \mathrm{div}\,(\sigma\varphi_i) \to \mathrm{div}\,(\sigma\hat{\varphi}) \text{ weakly } * \text{ in } L^\infty(\Omega) \text{ as } i \to +\infty.$$

So, we immediately calculate that

$$\int_\Omega z \,\mathrm{div}\,(\sigma\hat{\varphi})\, dx = \lim_{i\to+\infty} \int_\Omega z\,\mathrm{div}\,(\sigma\varphi_i)\,dx = -\lim_{i\to+\infty}\int_\Omega \varphi_i(x)\cdot D_\sigma z = -\int_\Omega \hat{\varphi}(x)\cdot D_\sigma z.$$

As well as, we also obtain that

$$\int_\Omega z\,\mathrm{div}\,\hat{\varphi}\,dx = -\int_\Omega \hat{\varphi}(x)\cdot\nabla z \text{ for any } z \in BV(\Omega),$$
and any Lipschitz continuous function $\hat{\varphi} \in C_0(\Omega)^n$.

Remark 2.3 In general, we easily see that $D(V_\sigma) \supset BV(\Omega)$. In fact, according to the approximation theorem of BV-functions (cf. [1, Theorem 3.9] or [7, section 5.2] or [10, 1.17 Theorem]), for any $z \in BV(\Omega)$ there is a sequence $\{\zeta_i\} \subset C^\infty(\Omega) \cap BV(\Omega)$ of smooth functions such that

$$\zeta_i \to z \text{ in } L^1(\Omega) \text{ and } \int_\Omega |\nabla\zeta_i| \to \int_\Omega |\nabla z| \text{ as } i \to +\infty.$$

So, It follows from Remark 2.2 and the lower semicontinuity of V_σ that

$$\begin{aligned}
V_\sigma(z) &\le \liminf_{i\to+\infty} V_\sigma(\zeta_i)\\
&= \liminf_{i\to+\infty}\ \sup\left\{ \int_\Omega \zeta_i\,\mathrm{div}\,(\sigma\varphi)\,dx \;\middle|\; \varphi \in C_0^1(\Omega)^n \text{ and } |\varphi| \le 1 \text{ on } \overline{\Omega} \right\}\\
&\le \liminf_{i\to+\infty}\ \sup\left\{ \int_\Omega \zeta_i\,\mathrm{div}\,\hat{\varphi}\,dx \;\middle|\; \begin{array}{l}\hat{\varphi} \in C_0(\Omega)^n \text{ Lipschitz continuous,}\\ \text{and } |\hat{\varphi}| \le \sigma \text{ on } \overline{\Omega}\end{array} \right\}\\
&\le |\sigma|_{C(\overline{\Omega})} \int_\Omega |\nabla z| < +\infty.
\end{aligned}$$

But in some cases, the effective domain $D(V_\sigma)$ does not coincides with the space $BV(\Omega)$. In fact, putting

$$\Omega := \{\ x \in \mathbb{R}^2 \mid |x| < 1\ \} \text{ and } \left\{ \begin{array}{l} \sigma(x) := |x|,\\ f(x) := \dfrac{1}{|x|^{1+\alpha}}, \end{array}\right. \quad \text{for any } x \in \Omega$$

with a fixed constant $0 < \alpha < 1$, we can see that σ is nonnegative and Lipschitz continuous on $\overline{\Omega}$ and $f \in D(V_\sigma)$, but $f \notin BV(\Omega)$.

Lemma 2.2 *Let σ be a nonnegative and Lipschitz continuous function on $\overline{\Omega}$. If $z \in BV(\Omega)$, then $D_\sigma z = \sigma \nabla z$ in $\mathscr{B}(\Omega)$, in particular*

$$V_\sigma(z) = \int_\Omega \sigma(x)\ |\nabla z|.$$

Proof. Let us take any function $z \in BV(\Omega)$. Then, by Lemma 2.1 and Remark 2.2,

$$\int_\Omega \hat{\varphi}(x) \cdot D_\sigma z = -\int_\Omega z \ \mathrm{div}\ (\sigma \hat{\varphi})\ dx = \int_\Omega \sigma(x)\, \hat{\varphi}(x) \cdot \nabla z$$

for any Lipschitz continuous function $\hat{\varphi} \in C_0(\Omega)^n$.

Thus, by the uniqueness of the Radon measure $D_\sigma z$, we have $D_\sigma z = \sigma \nabla z$ in $\mathscr{B}(\Omega)$. ■

Lemma 2.3 *(Approximation by smooth functions) Let $1 \leq p < +\infty$, and let σ be a nonnegative and Lipschitz continuous function on $\overline{\Omega}$. If $z \in BV(\Omega) \cap L^p(\Omega)$, then there exists a sequence $\{\zeta_i\} \subset C^\infty(\Omega) \cap BV(\Omega) \cap L^p(\Omega)$ such that*

$$\zeta_i \to z \text{ in } L^p(\Omega), \ \int_\Omega |\nabla \zeta_i| \to \int_\Omega |\nabla z| \text{ and } V_\sigma(\zeta_i) \to V_\sigma(z) \text{ as } i \to +\infty.$$

Proof. We consider only the case of $|\sigma|_{C(\overline{\Omega})} > 0$, since the another case is obtained just as in [1, Theorem 3.9] or [7, section 5.2] or [10, 1.17 Theorem].

The proof is a modified version of that of [7, THEOREM 2 in section 5.2] or [6, Theorem 2.7].

Let us fix any $z \in D(V_\sigma)$ and any small positive number ε. Let $\{\Delta_k\}$ be an open covering of Ω, defined as

$$\Delta_1 := \Omega_2 \text{ and } \Delta_k := \Omega_{k+1} \setminus \overline{\Omega_{k-1}},\ k = 2, 3, 4, \cdots,$$

where $\Omega_k := \left\{ x \in \Omega \,\middle|\, \mathrm{dist}(x, \Gamma) > \dfrac{1}{k + m_\varepsilon} \right\}$, $k = 0, 1, 2, 3, \cdots$, with a sufficiently large number $m_\varepsilon \in \mathbb{N}$ satisfying

$$\int_{\Omega \setminus \overline{\Omega_0}} |D_\sigma z| \leq |\sigma|_{C(\overline{\Omega})} \int_{\Omega \setminus \overline{\Omega_0}} |\nabla z| < \frac{\varepsilon}{2}. \tag{2.4}$$

Let $\{\eta_k\} \subset C_0^\infty(\Omega)$ be the partition of unity subordinate to $\{\Delta_k\}$, and let $\{\varepsilon_k\}$ be a sequence of positive numbers such that

$$0 < \varepsilon_k < \frac{\varepsilon}{2^{k+1}}, \tag{2.5}$$

$$\int_\Omega |\rho_{\varepsilon_k} * (z\eta_k) - z\eta_k|^p \ dx < \frac{\varepsilon}{2^{k+1}}, \tag{2.6}$$

$$\int_\Omega |\rho_{\varepsilon_k} * (z\nabla\eta_k) - z\nabla\eta_k| \ dx < \frac{\varepsilon}{2^{k+1}}, \tag{2.7}$$

and

$$\mathrm{spt}\ (\rho_{\varepsilon_k} * (z\eta_k)) \subset \Delta_k,\ k = 1, 2, 3, \cdots, \tag{2.8}$$

where ρ_{ε_k} is the usual mollifier on $\mathbb{R}^n$.

Here, let us define

$$\zeta_\varepsilon(x) := \sum_{k=1}^{+\infty} \rho_{\varepsilon_k} * (z\eta_k)(x) \quad \text{for any } x \in \Omega.$$

Then, we see from (2.6) and the lower semicontinuity of the total variation and the functional V_σ that

$$\zeta_\varepsilon \to z \text{ in } L^p(\Omega) \text{ as } \varepsilon \searrow 0,$$
$$\liminf_{\varepsilon \searrow 0} \int_\Omega |\nabla \zeta_\varepsilon| \geq \int_\Omega |\nabla z| \quad \text{and} \quad \liminf_{\varepsilon \searrow 0} V_\sigma(\zeta_\varepsilon) \geq V_\sigma(z).$$

Next, let us take any $\boldsymbol{\varphi} \in C_0^1(\Omega)^n$ satisfying $|\boldsymbol{\varphi}| \leq 1$ on $\overline{\Omega}$. Then, since spt $\boldsymbol{\varphi}$ is compact, we see from (2.8) and Fubini's theorem that

$$\int_\Omega \zeta_\varepsilon \operatorname{div}(\sigma\boldsymbol{\varphi})\, dx = \sum_{k=1}^{+\infty} \int_\Omega z\eta_k \operatorname{div}(\rho_{\varepsilon_k} * (\sigma\boldsymbol{\varphi}))\, dx = I_0 + I_1 + I_2, \tag{2.9}$$

where

$$I_0 := -\sum_{k=1}^{+\infty} \int_\Omega \sigma\boldsymbol{\varphi} \cdot (\rho_{\varepsilon_k} * (z\nabla\eta_k) - z\nabla\eta_k)\, dx,$$

$$I_1 := \int_\Omega z \operatorname{div}(\eta_1\rho_{\varepsilon_1} * (\sigma\boldsymbol{\varphi}))\, dx \quad \text{and} \quad I_2 := \sum_{k=2}^{+\infty} \int_\Omega z \operatorname{div}(\eta_k\rho_{\varepsilon_k} * (\sigma\boldsymbol{\varphi}))\, dx.$$

Here, by (2.7),

$$|I_0| \leq \sum_{k=1}^{+\infty} \int_\Omega |\sigma\boldsymbol{\varphi} \cdot (\rho_{\varepsilon_k} * (z\nabla\eta_k) - z\nabla\eta_k)|\, dx < |\sigma|_{C(\overline{\Omega})}\varepsilon. \tag{2.10}$$

On the other hand,

$$\begin{aligned} |\rho_{\varepsilon_k} * (\sigma\boldsymbol{\varphi})(x)| &\leq \left| \int_{\mathbb{R}^n} \rho_{\varepsilon_k}(x-y)\sigma(x)\boldsymbol{\varphi}(y)\, dy \right| + \left| \int_{\mathbb{R}^n} \rho_{\varepsilon_k}(x-y)(\sigma(y)-\sigma(x))\boldsymbol{\varphi}(y)\, dy \right| \\ &\leq \sigma(x) + M_\sigma\varepsilon_k, \quad k = 1, 2, 3, \cdots, \end{aligned}$$

where M_σ is the Lipschitz constant of the function σ. So, we see from (2.4) and (2.5) that

$$\begin{aligned} |I_1| &= \left| \int_\Omega z \operatorname{div}(\eta_k\rho_{\varepsilon_1} * (\sigma\boldsymbol{\varphi}))\, dx \right| \\ &\leq \sup\left\{ \int_\Omega z \operatorname{div}((\sigma + M_\sigma\varepsilon_1)\hat{\boldsymbol{\varphi}})\, dx \;\middle|\; \begin{array}{l} \hat{\boldsymbol{\varphi}} \in C_0(\Omega)^n\text{: Lipschitz continuous,} \\ \text{and } |\hat{\boldsymbol{\varphi}}| \leq 1 \text{ on } \overline{\Omega} \end{array} \right\} \\ &\leq V_\sigma(z) + M_\sigma\varepsilon_1 \int_\Omega |\nabla z| < V_\sigma(z) + \frac{M_\sigma\varepsilon}{2} \int_\Omega |\nabla z|, \end{aligned} \tag{2.11}$$

as well as

$$
\begin{aligned}
|I_2| &\le \sum_{k=2}^{+\infty}\left(\int_{\Delta_k}|D_\sigma z| + M_\sigma \varepsilon_k \int_{\Delta_k}|\nabla z|\right)\\
&\le \sum_{\ell=1}^{+\infty}\int_{\Delta_{2\ell}}|D_\sigma z| + \sum_{\ell=1}^{+\infty}\int_{\Delta_{2\ell+1}}|D_\sigma z| + \frac{M_\sigma \varepsilon}{2}\int_\Omega |\nabla z|\\
&\le 2|\sigma|_{C(\overline{\Omega})}\int_{\Omega\setminus\overline{\Omega_0}}|\nabla z| + \frac{M_\sigma\varepsilon}{2}\int_\Omega|\nabla z| < \varepsilon + \frac{M_\sigma\varepsilon}{2}\int_\Omega|\nabla z|. \qquad (2.12)
\end{aligned}
$$

On account of (2.9)~(2.12),

$$V_\sigma(\zeta_\varepsilon) \le V_\sigma(z) + \left(1 + |\sigma|_{C(\overline{\Omega})} + M_\sigma\int_\Omega |\nabla z|\right)\varepsilon.$$

Also, by a similar way to obtain (2.9)~(2.12) (replacing the function σ to the constant 1), we have

$$\int_\Omega |\nabla\zeta_\varepsilon| \le \int_\Omega |\nabla z| + \left(2 + \int_\Omega|\nabla z|\right)\varepsilon.$$

Thus, letting $\varepsilon \searrow 0$ yields that

$$\limsup_{\varepsilon\searrow 0}\int_\Omega |\nabla\zeta_\varepsilon| \le \int_\Omega|\nabla z| \quad\text{and}\quad \limsup_{\varepsilon\searrow 0} V_\sigma(\zeta_\varepsilon) \le V_\sigma(z).$$

■

Remark 2.4 In this paper, we may assume that the approximation sequence $\{\zeta_i\}$ as in Lemma 2.3 belongs to the class $C^\infty(\overline{\Omega})$. In fact, since $\{\zeta_i\} \subset C^\infty(\Omega)\cap BV(\Omega) \subset W^{1,1}(\Omega)$ and $\Gamma = \partial\Omega$ is Lipschitz, for any $i \in \mathbb{N}$ there is a sequence $\{\tilde{\zeta}_j^{(i)}\} \in C^\infty(\overline{\Omega})$ such that $\tilde{\zeta}_j^{(i)} \to \zeta_i$ in $W^{1,1}(\Omega)$ as $j \to +\infty$ (cf. [7, section 4.2]). Now, we can construct a sequence in $C^\infty(\overline{\Omega})$ by a standard diagonal argument applied to $\{\tilde{\zeta}_j^{(i)}\}$.

Lemma 2.4 *(The strictly positive case of σ) Let σ be a Lipschitz continuous function on $\overline{\Omega}$. If there exists a positive constant δ_0 such that $\sigma \ge \delta_0$ on $\overline{\Omega}$, then $D(V_\sigma) = BV(\Omega)$. Therefore, by Lemma 2.2,*

$$V_\sigma(z) = \int_\Omega \sigma(x)\,|\nabla z| \quad \textit{for any } z \in L^1(\Omega).$$

Proof. It is sufficient to show that $D(V_\sigma) \subset BV(\Omega)$, since the converse inclusion always follows from Remark 2.3.

Let us take any $z \in D(V_\sigma)$. Then, by Remark 2.2 and the assumption of the strict positiveness for σ,

$$
\begin{aligned}
V_\sigma(z) &= \sup\left\{ \int_\Omega z \operatorname{div}(\sigma\hat{\varphi})\,dx \;\middle|\; \begin{array}{l}\hat{\varphi} \in C_0(\Omega)^n \text{ Lipschitz continuous,}\\ \text{and } |\hat{\varphi}| \le 1 \text{ on } \overline{\Omega}\end{array}\right\}\\
&= \sup\left\{ \int_\Omega z \operatorname{div}\hat{\varphi}\,dx \;\middle|\; \begin{array}{l}\hat{\varphi} \in C_0(\Omega)^n \text{ Lipschitz continuous,}\\ \text{and } |\hat{\varphi}| \le \sigma \text{ on } \overline{\Omega}\end{array}\right\} \ge \delta_0\int_\Omega|\nabla z|,
\end{aligned}
$$

which implies $z \in BV(\Omega)$. Thus we conclude the lemma. ■

3 Euler-Lagrange equations

Let $n \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary $\Gamma := \partial\Omega$, and let σ be a nonnegative and Lipschitz continuous function on $\overline{\Omega}$. Let V_σ be the proper l.s.c. and convex function in $L^1(\Omega)$ as in the previous section.

Let $\bar{V}_\sigma$ be a functional on $L^1(\Omega)$, defined as

$$\bar{V}_\sigma(z) := \inf \left\{ \liminf_{i \to +\infty} V_\sigma(z_i) \;\middle|\; \begin{array}{l} \{z_i\} \subset W^{1,1}(\Omega) \text{ and} \\ z_i \to z \text{ in } L^1(\Omega) \text{ as } i \to +\infty \end{array} \right\}$$

Remark 3.1 (Fundamental properties of the functional $\bar{V}_\sigma$) The functional $\bar{V}_\sigma$ is known as a natural extension of the functional:

$$W_\sigma(z) := \int_\Omega \sigma |\nabla z| \, dx \quad \text{for } z \in W^{1,1}(\Omega);$$

onto the space $L^1(\Omega)$. Here, by the definition of the functional $\bar{V}_\sigma$, we easily see the following items.

(i) $\bar{V}_\sigma$ is a proper l.s.c. and convex function on $L^1(\Omega)$ such that $\bar{V}_\sigma(z) = W_\sigma(z)$ for any $z \in W^{1,1}(\Omega)$.

(ii) If a lower semicontinuous functional $F : L^1(\Omega) \longrightarrow \overline{\mathbb{R}}$ satisfies that $F(z) \le W_\sigma(z)$ for any $z \in W^{1,1}(\Omega)$, then $\bar{V}_\sigma(z) \ge F(z)$ for any $z \in L^1(\Omega)$.

(iii) Let us denote by $D(\bar{V}_\sigma)$ the effective domain of $\bar{V}_\sigma$, namely

$$D(\bar{V}_\sigma) := \{ \; z \in L^1(\Omega) \mid \bar{V}_\sigma(z) < +\infty \; \}.$$

Then, for any $z \in D(\bar{V}_\sigma)$, there exists a sequence $\{\hat{\zeta}_i\} \subset C^\infty(\overline{\Omega})$ such that

$$\hat{\zeta}_i \to z \text{ in } L^1(\Omega) \text{ and } W_\sigma(\hat{\zeta}_i) \to \bar{V}_\sigma(z) \text{ as } i \to +\infty.$$

Lemma 3.1 *Let V_σ and $\bar{V}_\sigma$ be functionals on $L^1(\Omega)$ as in the above. Then,*

$$BV(\Omega) \subset D(\bar{V}_\sigma) \subset D(V_\sigma) \quad \textit{and} \quad \bar{V}_\sigma(z) = V_\sigma(z) = \int_\Omega \sigma(x) \, |\nabla z| \quad \textit{for any } z \in BV(\Omega).$$

Proof. By Lemma 2.2 and (ii) of Remark 3.1,

$$V_\sigma(z) \le \bar{V}_\sigma(z) \quad \text{for any } z \in L^1(\Omega), \tag{3.1}$$

which implies $D(\bar{V}_\sigma) \subset D(V_\sigma)$.

Next, let us assume $z \in BV(\Omega)$. Then, by Lemma 2.3 and Remark 2.4, we find a sequence $\{\tilde{\zeta}_i\} \subset C^\infty(\overline{\Omega})$ such that

$$\tilde{\zeta}_i \to z \text{ in } L^1(\Omega) \text{ and } \int_\Omega \sigma |\nabla \tilde{\zeta}_i| \, dx \to V_\sigma(z) \text{ as } i \to +\infty.$$

Here, we see from (i) of Remark 3.1 and the lower semicontinuity of $\bar{V}_\sigma$ that

$$\bar{V}_\sigma(z) \leq \liminf_{i\to+\infty} \bar{V}_\sigma(\tilde{\zeta}_i) = \lim_{i\to+\infty} \int_\Omega \sigma|\nabla\tilde{\zeta}_i|\, dx = V_\sigma(z). \tag{3.2}$$

Combining (3.1) and (3.2), we conclude that

$$BV(\Omega) \subset D(\bar{V}_\sigma) \text{ and } \bar{V}_\sigma(z) = V_\sigma(z) = \int_\Omega \sigma(x)\ |\nabla z| \ \text{ for any } z \in BV(\Omega).$$

■

Let Φ_σ on $L^2(\Omega)$ be a functional on $L^2(\Omega)$, defined as

$$\Phi_\sigma(z) := \begin{cases} \bar{V}_\sigma(z), \text{ if } |z| \leq 1, \text{ a.e. in } \Omega, \\ +\infty, \text{ otherwise.} \end{cases}$$

As is easily seen, the functional Φ_σ is proper l.s.c. and convex in $L^2(\Omega)$, and it corresponds to the convex part of the free energy as in (1.4).

For any constant $\theta_* \in \mathbb{R}$, let us consider the following variational inequality, denoted by $(P_\sigma)_{\theta_*}$:

$$(P_\sigma)_{\theta_*} \quad \Phi_\sigma(w) - \int_\Omega (w+\theta_*)w\, dx \leq \Phi_\sigma(z) - \int_\Omega (w+\theta_*)z\, dx \ \text{ for any } z \in D(\Phi_\sigma),$$

where $D(\Phi_\sigma)$ is the effective domain of the functional Φ_σ, namely

$$D(\Phi_\sigma) := \left\{\ z \in D(\bar{V}_\sigma)\ \middle|\ |z| \leq 1, \text{ a.e. in } \Omega\ \right\}.$$

The problem $(P_\sigma)_{\theta_*}$ is the variational inequality (1.5), that is motivated by the steady-state problem for solid-liquid phase transitions. As is mentioned in the introduction, the compactness of sublevel sets of Φ_σ is very important to characterize the large-time behavior of evolution systems by the variational inequality $(P_\sigma)_{\theta_*}$. Hence, the strict positiveness for σ as in Lemma 2.4 used to be assumed in several papers (e.g. [6, 13, 14]), because the functional V_σ of this case is essentially the same with the total variation. Here, we would like to consider the functionals $\bar{V}_\sigma$ and Φ_σ in more general settings for σ. But, if we do not assume anything except for the nonnegativeness and the Lipschitz continuity of σ, then it is not enough to guarantee the compactness of sublevel sets. So, in this paper, we add the following assumption for the function σ:

(s1) $\mathscr{L}^n(\sigma^{-1}(0)) = 0$.

Furthermore, since the function σ is nonnegative and Lipschitz continuous, we may also assume that:

(s2) there exists a sequence $\{U_k\}$ of open subsets in Ω such that

$$U_k \subset\subset U_{k+1} \subset\subset \Omega\setminus\sigma^{-1}(0),\ k = 1,2,3,\cdots, \text{ and } \ \Omega\setminus\sigma^{-1}(0) = \bigcup_{k=1}^{+\infty} U_k;$$

(s3) there exists a sequence $\{\delta_k\}$ of positive numbers such that:

$$\sigma \geq \delta_k \text{ for any } x \in \overline{U_k},\ k = 1, 2, 3, \cdots, \quad \text{and} \quad \delta_k \searrow 0 \text{ as } k \to +\infty.$$

Now, let us check the compactness of sublevel sets of V_σ under assumptions in the above.

Proposition 3.1 *(Compactness) Let σ be a nonnegative and Lipschitz continuous function satisfying the condition (s1). Then, for any $r > 0$, the sublevel set*

$$L(r; V_\sigma) := \{\ z \in L^1(\Omega) \mid |z|_{L^1(\Omega)} \leq r \text{ and } V_\sigma(z) \leq r\ \}$$

of the functional V_σ is compact in $L^1(\Omega)$.

Proof. Let us take any $r > 0$ and any sequence $\{z_i\} \subset L(r; V_\sigma)$. Let $\{U_k\}$ be the sequence of open sets as in (s2). Then, by (s3) and Lemma 2.4, we immediately have $\{z_i\}$ is bounded in $BV(U_k)$, so that $\{z_i\}$ is relatively compact in $L^1(U_k)$, $k = 1, 2, 3, \cdots$.

First, let us choose a subsequence $\{z_i^{(1)}\} \subset \{z_i\}$ and a function $\bar{z}_1 \in L^1(U_1)$ to satisfy

$$|z_i^{(1)} - \bar{z}_1|_{L^1(U_1)} < \frac{1}{i} \quad \text{for } i = 1, 2, 3, \cdots.$$

Secondly, let us choose a subsequence $\{z_i^{(2)}\} \subset \{z_i^{(1)}\}$ ($\subset \{z_i\}$) and a function $\bar{z}_2 \in L^1(U_2)$ to satisfy

$$|z_i^{(2)} - \bar{z}_2|_{L^1(U_2)} < \frac{1}{i} \quad \text{for } i = 1, 2, 3, \cdots.$$

Then, we notice that $\bar{z}_2$ ($= \bar{z}_2|_{U_1}$) $= \bar{z}_1$ in $L^1(U_1)$.

Generally, for any $k \in \mathbb{N}$, we can choose a subsequence $\{z_i^{(k)}\} \subset \{z_i\}$ and a function $\bar{z}_k \in L^1(U_k)$ to satisfy

$$\left\{ \begin{array}{l} \{z_i^{(k+1)}\} \subset \{z_i^{(k)}\} \subset L(r; V_\sigma),\ \bar{z}_{k+1} = \bar{z}_k \text{ in } L^1(U_k), \\ \text{and } |z_i^{(k)} - \bar{z}_k|_{L^1(U_k)} < \dfrac{1}{i} \quad \text{for } i = 1, 2, 3, \cdots. \end{array} \right. \tag{3.3}$$

Here, putting

$$\bar{z}(x) := \left\{ \begin{array}{l} \bar{z}_k(x), \text{ if } x \in U_k,\ k = 1, 2, 3, \cdots, \\ 0, \text{ otherwise}, \end{array} \right. \quad \text{for a.e. } x \in \Omega,$$

we see from (3.3) and Fatou's lemma that $|\bar{z}|_{L^1(\Omega)} \leq r$. Thus, $\bar{z} \in L^1(\Omega)$.

Now, by (s1) and (s2), for any $\ell \in \mathbb{N}$ we can take a number $k_\ell \in \mathbb{N}$ such that

$$k_\ell \geq 2\ell \quad \text{and} \quad \int_{\Omega \setminus U_{k_\ell}} |\bar{z}|\, dx < \frac{1}{2\ell}$$

Then, putting

$$\bar{\zeta}_\ell(x) := \left\{ \begin{array}{l} z_{k_\ell}^{(k_\ell)}(x), \text{ if } x \in U_{k_\ell}, \\ 0, \text{ otherwise}, \end{array} \right. \quad \text{for a.e. } x \in \Omega,$$

it follows from (3.3) and the lower semicontinuity of V_σ that

$$|\bar{\zeta}_\ell - \bar{z}|_{L^1(\Omega)} < \frac{1}{k_\ell} + \frac{1}{2\ell} \le \frac{1}{\ell} \to 0 \text{ as } \ell \to +\infty \quad \text{and} \quad V_\sigma(\bar{z}) \le \liminf_{\ell \to +\infty} V_\sigma(\bar{\zeta}_\ell) \le r.$$

Thus, we conclude that the sublevel set $L(r; V_\sigma)$ is compact in $L^1(\Omega)$. ■

Remark 3.2 Under the same assumption as in Proposition 3.1, it is easily seen that for any $r > 0$ the sublevel set $L(r; \bar{V}_\sigma) := \{ \ z \in L^1(\Omega) \mid |z|_{L^1(\Omega)} \le r \text{ and } \bar{V}_\sigma(z) \le r \ \}$ of the functional $\bar{V}_\sigma$ is also compact in $L^1(\Omega)$. In fact, since $\bar{V}_\sigma$ is a lower semicontinuous function satisfying (3.1), the set $L(r; \bar{V}_\sigma)$ is a closed subset in the compact set $L(r; V_\sigma)$. Thus, the sublevel set $L(r; \bar{V}_\sigma)$ is also compact in $L^1(\Omega)$.

Corollary 3.1 *Let σ be the same as in Proposition 3.1. Then, for any $r > 0$, the sublevel set*

$$L(r; \Phi_\sigma) := \{ \ z \in D(\Phi_\sigma) \mid \Phi_\sigma(z) \le r \ \}$$

of the functional Φ_σ is compact in $L^2(\Omega)$.

Proof. Since $L(r; \Phi_\sigma) \subset L(r + \mathscr{L}^n(\Omega); \bar{V}_\sigma)$, for any sequence $\{z_i\} \subset L(r; \Phi_\sigma)$ we find a subsequence $\{\bar{\zeta}_\ell\} \subset \{z_i\}$, that converges to a limit $\bar{z} \in L(r + \mathscr{L}^n(\Omega); \bar{V}_\sigma)$ in the topology of $L^1(\Omega)$. Here, since it is easily seen that $|\bar{\zeta}_\ell| \le 1$ and $|\bar{z}| \le 1$, a.e. in Ω, the convergence can be replaced to that in the topology of $L^2(\Omega)$. Now, we see from the lower semicontinuity of Φ_σ that $\bar{z} \in L(r; \Phi_\sigma)$. Therefore, the sublevel set $L(r; \Phi_\sigma)$ is compact in $L^2(\Omega)$. ■

The next concept is concerned with an useful tool to calculate the first variation of the functional Φ_σ.

Definition 3.1 (Producted distribution) Let σ be a nonnegative and Lipschitz continuous function, and let $\boldsymbol{\nu} \in L^\infty(\Omega)^n$ be a bounded $\mathbb{R}^n$-valued function such that $\sigma\boldsymbol{\nu}$ is Lipschitz continuous on $\overline{\Omega}$. Then, for any $z \in D(V_\sigma)$, we define a distribution $\boldsymbol{\nu} \cdot D_\sigma z$ by putting

$$\langle \boldsymbol{\nu} \cdot D_\sigma z, \varphi \rangle := - \int_\Omega z \operatorname{div}(\sigma \boldsymbol{\nu} \varphi) \, dx \quad \text{for any } \varphi \in C_0^\infty(\Omega).$$

The concept as the above is a modified version of a distribution which was proposed in [4]. The author of [4] introduced some (sufficient) conditions that the distribution may be regarded as a Radon measure, and also gave some measure theoretical characterizations for the Radon measure. Now, on the basis of the theory obtained in [4], we also have similar characterization results for the distribution $\boldsymbol{\nu} \cdot D_\sigma z$.

Lemma 3.2 *Let σ be a nonnegative and Lipschitz continuous function on $\overline{\Omega}$, and let $\boldsymbol{\nu} \in L^\infty(\Omega)^n$ be a bounded $\mathbb{R}^n$-valued function such that $\sigma\boldsymbol{\nu}$ is Lipschitz continuous on $\overline{\Omega}$. If $z \in D(\bar{V}_\sigma)$, then the distribution $\boldsymbol{\nu} \cdot D_\sigma z$ is a Radon measure such that*

$$\left| \int_\Omega \varphi(x) \ \boldsymbol{\nu} \cdot D_\sigma z \right| \le |\varphi|_{C(\overline{\Omega})} |\boldsymbol{\nu}|_{L^\infty(\Omega)^n} \bar{V}_\sigma(z) \quad \text{for any } \varphi \in C_0(\Omega). \tag{3.4}$$

Proof. For any $z \in D(\bar{V}_\sigma)$, let $\{\hat{\zeta}_i\} \subset C^\infty(\overline{\Omega})$ be the sequence of the approximation as in (iii) of Remark 3.1. Then, for any $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned}|\langle \boldsymbol{\nu} \cdot D_\sigma \hat{\zeta}_i, \varphi\rangle| &= \left| - \int_\Omega \hat{\zeta}_i \operatorname{div}(\sigma \boldsymbol{\nu} \varphi)\, dx \right| = \left| \int_\Omega \varphi(\sigma \boldsymbol{\nu}) \cdot \nabla \hat{\zeta}_i \, dx \right| \\ &\le |\varphi|_{C(\overline{\Omega})} |\boldsymbol{\nu}|_{L^\infty(\Omega)^n} \int_\Omega \sigma |\nabla \hat{\zeta}_i|\, dx, \quad i = 1, 2, 3, \cdots.\end{aligned}$$

So, letting $i \to +\infty$ yields that

$$|\langle \boldsymbol{\nu} \cdot D_\sigma z, \varphi\rangle| \le |\varphi|_{C(\overline{\Omega})} |\boldsymbol{\nu}|_{L^\infty(\Omega)^n} \bar{V}_\sigma(z) \quad \text{for any } \varphi \in C_0^\infty(\Omega).$$

Thus, the distribution $\boldsymbol{\nu} \cdot D_\sigma z$ can be regarded as a Radon measure in Ω, satisfying the inequality (3.4). ■

Remark 3.3 Let σ and $\boldsymbol{\nu}$ be the same as in Lemma 3.2. Then, combining (2.2), (2.3) and (3.4), we also have

$$\left| \int_\Omega \boldsymbol{\nu} \cdot D_\sigma z \right| \le \int_\Omega |\boldsymbol{\nu} \cdot D_\sigma z| \le |\boldsymbol{\nu}|_{L^\infty(\Omega)^n} \bar{V}_\sigma(z) \quad \text{for any } z \in D(\bar{V}_\sigma).$$

Lemma 3.3 *Let σ and $\boldsymbol{\nu}$ be the same as in Lemma 3.2. If $z \in BV(\Omega)$, then $\boldsymbol{\nu} \cdot D_\sigma z = (\sigma \boldsymbol{\nu}) \cdot \nabla z$ in $\mathscr{B}(\Omega)$.*

Proof. Let us take any function $z \in BV(\Omega)$. Then, we see from Remark 2.2 that

$$\int_\Omega \varphi(\sigma \boldsymbol{\nu}) \cdot \nabla z = - \int_\Omega z \operatorname{div}(\sigma \boldsymbol{\nu} \varphi)\, dx = \int_\Omega \varphi(x)\, \boldsymbol{\nu} \cdot D_\sigma z \quad \text{for any } \varphi \in C_0^\infty(\Omega).$$

So, by the uniqueness of the Radon measure, $\boldsymbol{\nu} \cdot D_\sigma z = (\sigma \boldsymbol{\nu}) \cdot \nabla z$ in $\mathscr{B}(\Omega)$. ■

Lemma 3.4 *(Gauss-Green type formula) Let σ be a nonnegative and Lipschitz continuous function on $\overline{\Omega}$, and let $\boldsymbol{\nu}$ be a bounded $\mathbb{R}^n$-valued function such that $\sigma \boldsymbol{\nu}$ is Lipschitz continuous on $\overline{\Omega}$. If the support of $\sigma \boldsymbol{\nu}$ is compact, then*

$$\int_\Omega \boldsymbol{\nu} \cdot D_\sigma z = - \int_\Omega \operatorname{div}(\sigma \boldsymbol{\nu})\, z \, dx \quad \textit{for any } z \in D(\bar{V}_\sigma).$$

Proof. Let us take any function $z \in D(\bar{V}_\sigma)$, and a sequence $\{\varphi_i\} \subset C_0^\infty(\Omega)$ of smooth functions to satisfy:

$$\begin{cases} |\varphi_i| \le 1 \text{ on } \overline{\Omega}, \ \varphi_i \equiv 1 \text{ on spt}(\sigma \boldsymbol{\nu}), \ i = 1, 2, 3, \cdots, \\ \varphi_i(x) \to 1 \text{ for any } x \in \Omega \text{ as } i \to +\infty. \end{cases} \tag{3.5}$$

Then, it is easily seen that

$$\begin{cases} |z(x)\varphi_i(x) \operatorname{div}(\sigma \boldsymbol{\nu})(x)| \le |\operatorname{div}(\sigma \boldsymbol{\nu})|_{L^\infty(\Omega)} |z(x)|, \ i = 1, 2, 3, \cdots, \\ z(x)\varphi_i(x) \operatorname{div}(\sigma \boldsymbol{\nu})(x) \to z(x) \operatorname{div}(\sigma \boldsymbol{\nu})(x) \text{ as } i \to +\infty, \ \text{ for a.e. } x \in \Omega. \end{cases} \tag{3.6}$$

Here, since $(\sigma\boldsymbol{\nu}) \cdot \nabla\varphi_i \equiv 0$ on $\overline{\Omega}$,

$$\begin{aligned}\int_\Omega \varphi_i(x)\ \boldsymbol{\nu} \cdot D_\sigma z &= -\int_\Omega z\varphi_i \operatorname{div}(\sigma\boldsymbol{\nu})\ dx - \int_\Omega z(\sigma\boldsymbol{\nu}) \cdot \nabla\varphi_i\ dx \\ &= -\int_\Omega z\varphi_i \operatorname{div}(\sigma\boldsymbol{\nu})\ dx \quad \text{for } i = 1, 2, 3, \cdots.\end{aligned}$$

Therefore, the required inequality follows from (3.5), (3.6) and Lebesgue's dominated convergence theorem, as $i \to +\infty$. ■

Now, we are on the stage to characterize solutions of $(P_\sigma)_{\theta_*}$. As is observed in several papers (cf. [5, 8, 12, 13, 14]), variational inequalities, associated with total variation energies, admit a lot of piecewise constant solutions having strong stability for the corresponding total variation flow. Here, we can expect similar situation for our problem $(P_\sigma)_{\theta_*}$, since the convex part Φ_σ of the free energy is given as an extended version of the total variation functional. The next theorem is concerned with the sufficient condition for piecewise constant functions to be solutions of the variational inequality $(P_\sigma)_{\theta_*}$.

Theorem 3.1 *(Characterization for solutions of $(P_\sigma)_{\theta_*}$) Let σ be a nonnegative and Lipschitz continuous function on $\overline{\Omega}$. Let $D \subset\subset \Omega$ be an open set with a Lipschitz boundary ∂D, and let χ_D and $\chi_{\Omega\setminus D}$ be characteristic functions of D and $\Omega \setminus D$, respectively. Let c be a constant either 1 or -1. Then, a piecewise constant function given as:*

$$w_D(x) := c\{\chi_D(x) - \chi_{\Omega\setminus D}(x)\} = \begin{cases} c, & \text{if } x \in D, \\ -c, & \text{otherwise,} \end{cases} \quad \text{a.e. } x \in \Omega; \tag{3.7}$$

is a solution of $(P_\sigma)_{\theta_*}$, if there exists a $\mathbb{R}^n$-valued function $\boldsymbol{\nu}_D \in L^\infty(\Omega)^n$ such that:

(a) $|\boldsymbol{\nu}_D| \leq 1$, a.e. $x \in \Omega$;

(b) for $\mathscr{H}^{n-1}$-a.e. $x \in \partial D$, the vector $\boldsymbol{\nu}_D(x) \in \mathbb{R}^n$ is defined to satisfy $\boldsymbol{\nu}_D(x)\cdot\boldsymbol{n}_{\partial D}(x) = c$, where $\boldsymbol{n}_{\partial D}$ is the unit inner normal vector on ∂D;

(c) $\sigma\boldsymbol{\nu}_D$ is Lipschitz continuous, and spt $(\sigma\boldsymbol{\nu})$ *is compact in Ω;*

(d) $$-\operatorname{div}(\sigma\boldsymbol{\nu}_D)(x) \begin{cases} \leq 1 + \theta_*, & \text{if } w_D(x) = 1, \\ \geq -1 + \theta_*, & \text{if } w_D(x) = -1, \end{cases} \quad \text{a.e. } x \in \Omega.$$

Proof. Let us take any $z \in D(\Phi_\sigma)$. Then, since $w_D \in BV(\Omega)$, we see from Lemmas 3.1~3.3 and Remark 3.3 that

$$\begin{aligned}&\Phi_\sigma(z) - \Phi_\sigma(w_D) = \bar{V}_\sigma(z) - \int_\Omega \sigma(x)\ |\nabla w_D| \\ \geq\ & \int_\Omega \boldsymbol{\nu}_D \cdot D_\sigma z - \int_{\partial D} \sigma\ (2c\boldsymbol{\nu}_D \cdot \boldsymbol{n}_{\partial D})\ d\mathscr{H}^{n-1} = \int_\Omega \boldsymbol{\nu}_D \cdot D_\sigma z - \int_\Omega (\sigma\boldsymbol{\nu}_D)(x) \cdot \nabla w_D \\ =\ & \int_\Omega \boldsymbol{\nu}_D \cdot D_\sigma z - \int_\Omega \boldsymbol{\nu}_D \cdot D_\sigma w_D.\end{aligned}$$

Now, on account of the condition (b) and the Gauss-Green type formula as in Lemma 3.4, we obtain that

$$\Phi_\sigma(z) - \Phi_\sigma(w_D) \geq - \int_\Omega \mathrm{div}\,(\sigma \boldsymbol{\nu}_D)(z - w_D)\, dx \geq \int_\Omega (w_D + \theta_*)(z - w_D)\, dx.$$

■

4 Examples of solutions

In this section, some piecewise constant functions will be shown as examples of solutions of $(P_\sigma)_{\theta_*}$. First, let us consider the constant case of solutions.

Lemma 4.1 *(Higher or lower cases of the temperature) Let $n \in \mathbb{N}$, and let Ω be a bounded domain with a Lipschitz boundary $\Gamma := \partial\Omega$. Let σ be a nonnegative and Lipschitz continuous function on $\overline{\Omega}$. If a constant $\theta_* \in \mathbb{R}$ satisfies $|\theta_*| \geq 1$, then any solution of the variational inequality $(P_\sigma)_{\theta_*}$ is constant on $\overline{\Omega}$.*

Proof. It is sufficient to consider only the case of $\theta_* \geq 1$, since the another case is similarly obtained. Let us assume that there is a nonconstant solution $\tilde{w}$ under the assumption. Then, since $\tilde{w} + \theta_* \geq 0$, $|\tilde{w}| \leq 1$, a.e. in Ω, and $\tilde{w}$ is nonconstant,

$$\Phi_\sigma(\tilde{w}) \geq 0 = \Phi_\sigma(1) \quad \text{and} \quad \int_\Omega (\tilde{w} + \theta_*)(1 - \tilde{w})\, dx > 0,$$

so that

$$\Phi_\sigma(\tilde{w}) - \int_\Omega (\tilde{w} + \theta_*)\tilde{w}\, dx > \Phi_\sigma(1) - \int_\Omega (\tilde{w} + \theta_*) \cdot 1\, dx.$$

It contradicts that $\tilde{w}$ is a solution of $(P_\sigma)_{\theta_*}$. ■

Proposition 4.1 *(Constant solutions) Let Ω and σ be the same as in Lemma 4.1, and let $\theta_* \in \mathbb{R}$ be any constant. Then, a constant function $\bar{w} : \Omega \longrightarrow \mathbb{R}$ is a solution of $(P_\sigma)_{\theta_*}$, if and only if:*

$$\begin{cases} \bar{w} \equiv 1 \ (\textit{resp. } \bar{w} \equiv -1) \textit{ on } \overline{\Omega}, \textit{ when } \theta_* > 1 \ (\textit{resp. } \theta_* < -1); \\ \bar{w} \equiv 1 \textit{ or } \bar{w} \equiv -\theta_* \textit{ or } \bar{w} \equiv -1 \textit{ on } \overline{\Omega}, \textit{ when } |\theta_*| \leq 1. \end{cases}$$

Proof. We consider only the case of $|\theta_*| \leq 1$, since proofs of other cases are similar. Let us take any constant solution $\bar{w}$ of $(P_\sigma)_{\theta_*}$. If $-\theta_* < \bar{w} < 1$ (resp. $-1 < \bar{w} < -\theta_*$), then

$$\Phi_\sigma(\bar{w}) - \int_\Omega (\bar{w} + \theta_*)\bar{w}\, dx = - \int_\Omega (\bar{w} + \theta_*)\bar{w}\, dx$$
$$> \Phi_\sigma(1) - \int_\Omega (\bar{w} + \theta_*) \cdot 1\, dx \ \left(\text{resp. } > \Phi_\sigma(-1) - \int_\Omega (\bar{w} + \theta_*) \cdot (-1)\, dx\right)$$

It contradicts that $\bar{w}$ is a solution of $(P_\sigma)_{\theta_*}$.

Conversely, if $\bar{w} \equiv 1$ or $\bar{w} \equiv -\theta_*$ or $\bar{w} \equiv -1$ on $\overline{\Omega}$, then it is easily checked that

$$0 = \Phi_\sigma(\bar{w}) \leq \Phi_\sigma(z) \quad \text{and} \quad \int_\Omega (\bar{w} + \theta_*)(z - \bar{w})\, dx \begin{cases} \leq 0, & \text{if } \bar{w} \equiv 1 \text{ or } \bar{w} \equiv -1 \text{ on } \overline{\Omega}, \\ = 0, & \text{if } \bar{w} \equiv -\theta_* \text{ on } \overline{\Omega}, \end{cases}$$

for any $z \in D(\Phi_\sigma)$. Thus, adding the both sides of the above inequalities, we conclude that $\bar{w}$ is a solution of $(P_\sigma)_{\theta_*}$. ■

On account of Lemma 4.1 and Proposition 4.1, we notice that the variational inequality $(P_\sigma)_{\theta_*}$ has only trivial (constant) solutions when $|\theta_*| \geq 1$.

Now, our next interest is nonconstant (but piecewise constant) solutions, so that we assume $|\theta_*| < 1$ in the rest. In the observation of such solutions, geometric information of graphs of functions will be needed to construct the vector field $\boldsymbol{\nu}_D$ that appeared in Theorem 3.1. Therefore, for a simplicity, we consider only the case of $\Omega \subset \mathbb{R}^2$ (namely $n = 2$), and show examples of two-dimensional solutions under concrete settings of the domain Ω and the function σ.

Example 4.1 (The constant case of σ) Let θ_* be a constant satisfying $|\theta_*| < 1$, and let c be a constant either 1 or -1. Let L, r and σ_0 be positive numbers such that $L > 2r$ and $(1 - |\theta_*|)r \geq 2\sigma_0$. Let us set

$$\Omega := (-L, L) \times (-L, L),$$
$$\sigma \equiv \sigma_0 \text{ on } \overline{\Omega} \text{ and}$$
$$D := \{\ x \in \mathbb{R}^2 \mid |x| < r\ \}.$$

Then, the piecewise constant function w_D given as in (3.7) is a solution of the variational inequality $(P_\sigma)_{\theta_*}$ (see Fig. 4.1).

The keypoint of the proof is to give the explicit expression of the vector field $\boldsymbol{\nu}_D$ that satisfies all conditions (a)$\sim$ (d) as in Theorem 3.1. In this case, putting

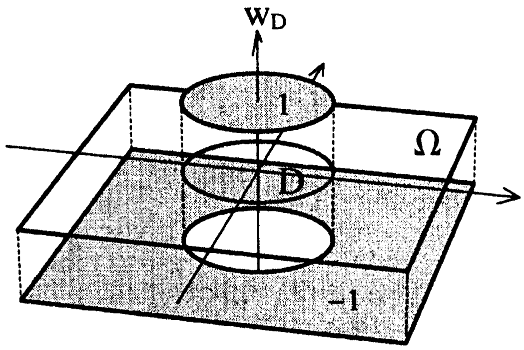

Fig. 4.1 (Profile of w_D)

$$\boldsymbol{\nu}_D(x) := \begin{cases} -\dfrac{c}{r}x, & \text{if } 0 \leq |x| < r, \\[2ex] c\left(\dfrac{1}{r}x - \dfrac{2}{|x|}x\right), & \text{if } r \leq |x| < 2r, \\[2ex] \mathbf{0}, & \text{otherwise,} \end{cases}$$

we can check all conditions (a)$\sim$(d) by quite fundamental calculations.

Example 4.2 (Variable interfaces in constant cases of σ) Let L, r and σ_0 be the same as in Example 4.1. Let us set

$$\Omega := (-L, L) \times (-L, L) \quad \text{and} \quad \sigma \equiv \sigma_0 \text{ on } \overline{\Omega}.$$

Let $D \subset\subset \Omega$ be any open set with a C^2-boundary ∂D such that

$$\partial D(r) := \{\ x \in \Omega \mid \text{dist}(x, \partial D) \le r\ \} \subset \Omega,$$

$$D = \bigcup_{\substack{x \in D \\ B_r(x) \subset D}} B_r(x) \quad \text{and} \quad \Omega \setminus \overline{D} = \bigcup_{\substack{x \in \Omega\setminus\overline{D} \\ B_r(x)\cap\Omega \subset \Omega\setminus\overline{D}}} B_r(x) \cap \Omega.$$

Then, a piecewise constant function w_D given as in (3.7) is a solution of the variational inequality $(P_\sigma)_{\theta_*}$ (see Fig. 4.2).

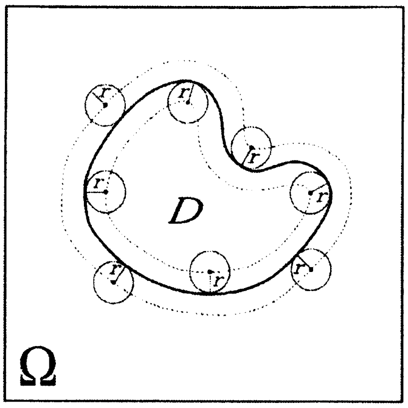

Fig. 4.2

This example has already reported in [14, Example 3.4]. According to [14], the required vector field $\boldsymbol{\nu}_D$ is given as follows.

$$\boldsymbol{\nu}_D(x) := \begin{cases} \dfrac{c(r - \text{dist}(x, \partial D))}{r} \nabla \text{dist}(x, \partial D), & \text{if } x \in D \cap \partial D(r), \\ -\dfrac{c(r - \text{dist}(x, \partial D))}{r} \nabla \text{dist}(x, \partial D), & \text{if } x \in \partial D(r) \setminus \overline{D}, \\ c\, \boldsymbol{n}_{\partial D}(x), & \text{if } x \in \partial D, \\ \boldsymbol{0}, & \text{otherwise.} \end{cases}$$

Next, let us consider nonconstant cases of σ. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be the usual base in $\mathbb{R}^2$, namely $\boldsymbol{e}_1 := {}^t(1, 0)$ and $\boldsymbol{e}_2 := {}^t(0, 1)$.
For any $\alpha \in \mathbb{R}$, let $R(\alpha)$ be the 2×2-matrix of the rotation, defined as

$$R(\alpha) := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

and let $\boldsymbol{\rho}(\alpha)$ be a vector given as

$$\boldsymbol{\rho}(\alpha) := R(\alpha)\boldsymbol{e}_1.$$

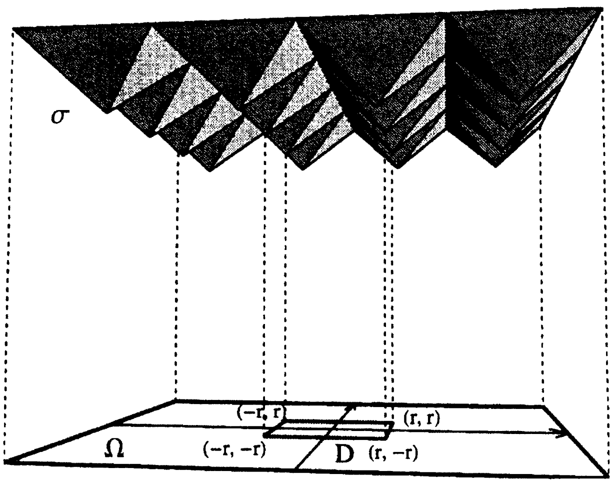

Fig. 4.3

Example 4.3 Let θ_* be a constant satisfying $|\theta_*| < 1$, and let c be a constant either -1 or 1. Let L, r and σ_0 be positive numbers satisfying $L \ge 4r$ and $(1 - |\theta_*|)r \ge 2\sigma_0$. Let us set

$$\Omega := (-L, L) \times (-L, L), \quad D := (-r, r) \times (-r, r), \quad \text{and}$$

$$\sigma(x) := \begin{cases} \dfrac{\sigma_0}{r} \max\{\ |x_i - (2k_i - 1)r| \mid i = 1,2\ \}, \\ \quad \text{if } x = {}^t(x_1, x_2) \in \overline{\Omega} \text{ and } \max\{\ |x_i - (2k_i-1)r| \mid i = 1,2\ \} < r \\ \quad \text{for some } (k_1, k_2) \in \mathbb{Z}^2, \\ \sigma_0, \text{ otherwise } \ (\text{for any } x = {}^t(x_1, x_2) \in \overline{\Omega}). \end{cases}$$

Then, the piecewise constant function w_D given as in (3.7) is a solution of the variational inequality $(P_\sigma)_{\theta_*}$ (see Fig. 4.3). In fact, putting $\mathbb{R}_+ := \{\ x \in \mathbb{R} \mid x \geq 0\ \}$,

$$n_0(x) := \begin{cases} \bullet\ -c\,{}^t\!\left(\dfrac{r - x_2}{(r-x_1)+(r-x_2)}, \dfrac{r-x_1}{(r-x_1)+(r-x_2)}\right), \\ \quad \text{if } x = {}^t(x_1,x_2) \in D \cap \mathbb{R}^2_+ \text{ and } (r-x_1)+(r-x_2) < r, \\ \bullet\ -\dfrac{c}{r}x, \text{ if } x = {}^t(x_1,x_2) \in D \cap \mathbb{R}^2_+ \text{ and } (r-x_1)+(r-x_2) \geq r, \\ \bullet\ -\dfrac{c}{r}(2r - x_i)e_i, \text{ if } x = {}^t(x_1,x_2) \notin D, \\ \quad r \leq x_i < 2r,\ 0 \leq x_j < r, \text{ and } (i,j) \in \{(1,2),(2,1)\}, \\ \bullet\ -\dfrac{c}{r}(r - |x - r(e_1+e_2)|)\dfrac{x - r(e_1+e_2)}{|x - r(e_1+e_2)|}, \text{ if } x \notin D \cup \{{}^t(r,r)\}, \\ \quad x - r(e_1+e_2) \in \mathbb{R}^2_+ \text{ and } |x - r(e_1+e_2)| < r \\ \bullet\ \mathbf{0}, \text{ otherwise}, \quad \text{for any } x = {}^t(x_1,x_2) \in \mathbb{R}^2_+, \end{cases}$$

and

$$\nu_D(x) := R(\tfrac{\pi}{2}i)\, n_0(R(-\tfrac{\pi}{2}i)\,x), \text{ if } R(-\tfrac{\pi}{2}i)\,x \in \mathbb{R}^2_+,\ i = 0,1,2,3, \text{ for any } x \in \Omega,$$

we easily see that all conditions (a)$\sim$(d) as in Theorem 3.1 are fulfilled for the vector field ν_D.

The above example suggests us that the singularity at corners of the interface can be canceled by multiplying the zero of the coefficient σ. It also implies that we can represent various shapes of interfaces by choosing appropriate functions as the coefficient σ. The next example is concerned with a piecewise constant solution which can represent more variable patterns of interfaces.

Example 4.4 Let θ_* be a constant satisfying $|\theta_*| < 1$, and let c be a constant either -1 or 1. Let L, r and σ_0 be positive numbers satisfying $L \geq 8r$ and $(1 - |\theta_*|)r \geq 2\sigma_0$. Let us set

$$\Omega := (-L, L) \times (-L, L),$$

$$D := \left\{\ x = (x_1, x_2) \in \mathbb{R}^2 \ \middle|\ \begin{array}{l} \max\{\ \rho(\frac{\pi}{3}(2i+1)) \cdot x \mid i = 0,1,2\ \} < 3r \\ \text{or } \max\{\ \rho(\frac{\pi}{3}(2i)) \cdot x \mid i = 0,1,2\ \} < 3r \end{array} \right\}$$

$$\Xi_1^+ := \{\ ((6k_1+1)r, (2k_2+1)\sqrt{3}r) \mid (k_1,k_2) \in \mathbb{Z}^2\ \},$$

$$\Xi_1^- := \{\ ((6k_1-1)r, (2k_2-1)\sqrt{3}r) \mid (k_1,k_2) \in \mathbb{Z}^2\ \},$$

$$\Xi_2^+ := \left\{\ ((6k_1+2)r, (2k_2+2)\sqrt{3}r)\ \middle|\ (k_1,k_2) \in \mathbb{Z}^2\ \right\}$$
$$\Xi_2^- := \left\{\ ((6k_1-2)r, (2k_2-2)\sqrt{3}r)\ \middle|\ (k_1,k_2) \in \mathbb{Z}^2\ \right\},$$

and

$$\sigma(x) := \begin{cases} \sigma_0 - \dfrac{\sigma_0}{r} \max\left\{\ \rho(\frac{\pi}{3}(2i+1)) \cdot (x-\xi)\ \middle|\ i = 0,1,2\ \right\}, \\ \quad \text{if } \max\left\{\ \rho(\frac{\pi}{3}(2i+1)) \cdot (x-\xi)\ \middle|\ i=0,1,2\ \right\} < r \text{ for some } \xi \in \Xi_1^+ \cup \Xi_2^-, \\ \sigma_0 - \dfrac{\sigma_0}{r} \max\left\{\ \rho(\frac{\pi}{3}(2i)) \cdot (x-\xi)\ \middle|\ i = 0,1,2\ \right\}, \\ \quad \text{if } \max\left\{\ \rho(\frac{\pi}{3}(2i)) \cdot (x-\xi)\ \middle|\ i=0,1,2\ \right\} < r \text{ for some } \xi \in \Xi_1^- \cup \Xi_2^+, \\ 0, \text{ otherwise, for any } x \in \overline{\Omega}. \end{cases}$$

Then, the piecewise constant function w_D given as in (3.7) is a solutions of the variational inequality $(P_\sigma)_{\theta_*}$ (see Fig. 4.4). In fact, putting

$$Y_1^+ := \left\{\ y = (y_1,y_2) \in \mathbb{R}^2\ \middle|\ 0 \le y_2 \le y_1 \tan(\tfrac{\pi}{6})\ \right\},$$
$$Y_1^- := \left\{\ y = (y_1,y_2) \in \mathbb{R}^2\ \middle|\ -y_1 \tan(\tfrac{\pi}{6}) \le y_2 \le 0\ \right\},$$

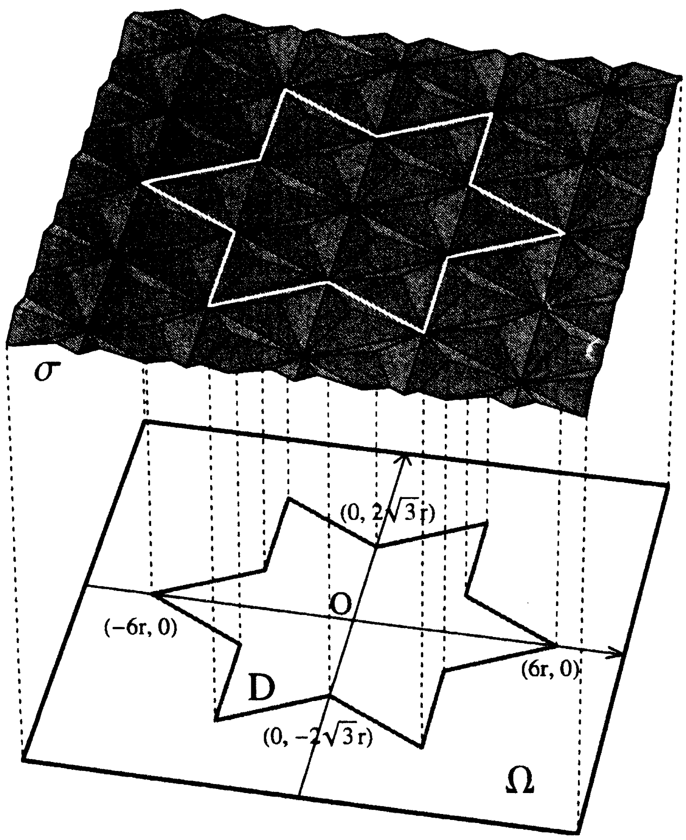

Fig. 4.4

$$
\boldsymbol{n}_1^+(x) := \begin{cases}
\bullet\ -c\,{}^t\!\left(\cos(\frac{\pi}{3}), \dfrac{\sqrt{3}x_2\sin(\frac{\pi}{3})}{6r-x_1}\right), \text{ if } x \in \overline{D} \text{ and } 4r + r\cos(\frac{\pi}{3}) \le x_1 < 6r, \\
\bullet\ -\dfrac{c}{r}(x - 4r\boldsymbol{e}_1), \text{ if } x \in D,\ 4r \le x_1 < 4r + r\cos(\frac{\pi}{3}) \\
\quad \text{and } 0 \le x_2 < x_1\tan(\frac{\pi}{3}), \\
\bullet\ -\dfrac{c}{r}R(\frac{\pi}{3})(\boldsymbol{e}_1 \cdot R(-\frac{\pi}{3})(x - 4r\boldsymbol{e}_1))\boldsymbol{e}_1, \text{ if } x \in D \\
\quad \text{and } R(-\frac{\pi}{3})(x - 4r\boldsymbol{e}_1) \in (0, r] \times [0, \sqrt{3}r], \\
\bullet\ \dfrac{c}{r}(r - |x - 2\sqrt{3}r\boldsymbol{\rho}(\frac{\pi}{6})|)\dfrac{x - 2\sqrt{3}r\boldsymbol{\rho}(\frac{\pi}{6})}{|x - 2\sqrt{3}r\boldsymbol{\rho}(\frac{\pi}{6})|}, \text{ if } x \in D \setminus \{2\sqrt{3}r\boldsymbol{\rho}(\frac{\pi}{6})\} \\
\quad |x - 2\sqrt{3}r\boldsymbol{\rho}(\frac{\pi}{6})| < r \text{ and } \frac{x - 2\sqrt{3}r\boldsymbol{\rho}(\frac{\pi}{6})}{|x - 2\sqrt{3}r\boldsymbol{\rho}(\frac{\pi}{6})|} = \boldsymbol{\rho}(\alpha) \\
\quad \text{for some } -\frac{5}{6}\pi \le \alpha < -\frac{2}{3}\pi, \\
\bullet\ -cR(-\frac{\pi}{3})\,{}^t\!\left(\dfrac{\boldsymbol{e}_1 \cdot R(\frac{\pi}{3})(x - 2\sqrt{3}r\boldsymbol{\rho}(\frac{\pi}{6}))}{\boldsymbol{e}_2 \cdot R(\frac{\pi}{3})(x - 2\sqrt{3}r\boldsymbol{\rho}(\frac{\pi}{6}))}\cot(\frac{\pi}{3})\cos(\frac{2}{3}\pi), \sin(\frac{2}{3}\pi)\right), \\
\quad \text{if } x \notin D, \text{ and} \\
\quad 0 \le 2\boldsymbol{e}_1 \cdot R(\frac{\pi}{3})(x - 2\sqrt{3}r\boldsymbol{\rho}(\frac{\pi}{6})) < 2\sqrt{3}\,\boldsymbol{e}_2 \cdot R(\frac{\pi}{3})(x - 2\sqrt{3}r\boldsymbol{\rho}(\frac{\pi}{6})) < r, \\
\bullet\ \dfrac{c}{r}(x - 8r\tan(\frac{\pi}{6})\boldsymbol{\rho}(\frac{\pi}{6})), \text{ if } x \notin D, \\
\quad \boldsymbol{e}_1 \cdot R(-\frac{\pi}{6})(x - 8r\tan(\frac{\pi}{6})\boldsymbol{\rho}(\frac{\pi}{6})) < \sqrt{3}\boldsymbol{e}_2 \cdot R(-\frac{\pi}{6})(x - 8r\tan(\frac{\pi}{6})\boldsymbol{\rho}(\frac{\pi}{6})) \le 0 \\
\quad \text{and } -\sqrt{3}r \le 2\boldsymbol{e}_1 \cdot R(-\frac{\pi}{6})(x - 8r\tan(\frac{\pi}{6})\boldsymbol{\rho}(\frac{\pi}{6})) < 0, \\
\bullet\ -\dfrac{c}{r}R(\frac{\pi}{3})(r - \boldsymbol{e}_1 \cdot R(-\frac{\pi}{3})(x - 6r\boldsymbol{e}_1))\boldsymbol{e}_1, \\
\quad \text{if } x \notin D \text{ and } R(-\frac{\pi}{3})(x - 6r\boldsymbol{e}_1) \in (0, r) \times [0, \frac{5}{\sqrt{3}}r], \\
\bullet\ -\dfrac{c}{r}(r - |x - 6r\boldsymbol{e}_1|)\dfrac{x - 6r\boldsymbol{e}_1}{|x - 6r\boldsymbol{e}_1|}, \text{ if } x \notin D \cup \{6r\boldsymbol{e}_1\},\ |x - 6r\boldsymbol{e}_1| < r, \\
\quad \text{and } \frac{x - 6r\boldsymbol{e}_1}{|x - 6r\boldsymbol{e}_1|} = \boldsymbol{\rho}(\alpha) \text{ for some } 0 \le \alpha < \frac{\pi}{3}, \\
\bullet\ \boldsymbol{0}, \text{ otherwise}, \quad \text{for any } x = {}^t(x_1, x_2) \in Y_1^+,
\end{cases}
$$

$$
A_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{n}_1^-(x) := A_1\boldsymbol{n}_1^+(A_1x), \text{ for any } x \in Y_1^-,
$$

and

$$
\boldsymbol{\nu}_D(x) := \begin{cases}
R(\frac{\pi}{3}i)\,\boldsymbol{n}_1^+(R(-\frac{\pi}{3}i)\,x), \text{ if } R(-\frac{\pi}{3}i)\,x \in Y_1^+,\ i = 0,1,2,3,4,5, \\
R(\frac{\pi}{3}i)\,\boldsymbol{n}_1^-(R(-\frac{\pi}{3}i)\,x), \text{ if } R(-\frac{\pi}{3}i)\,x \in Y_1^-,\ i = 0,1,2,3,4,5,
\end{cases} \quad \text{for any } x \in \Omega,
$$

it is not so difficult to see that all conditions (a)$\sim$(d) as in Theorem 3.1 are fulfilled for the vector field $\boldsymbol{\nu}_D$.

On the basis of the above example, we obtain the following theorem.

Theorem 4.1 *Let c, r, Ω and σ be the same as in Example 4.4. Let $D \subset\subset \Omega$ be any open set such that*

$$D \text{ has a Lipschitz boundary } \partial D,\ \partial D \subset \sigma^{-1}(0) \text{ and } \inf_{\substack{x\in\partial D \\ y\in\Gamma}} |x-y| \geq r \tag{4.1}$$

Then, the piecewise constant function w_D given as in (3.7) is a solution of the variational inequality $(P_\sigma)_{\theta_*}$.

Remark 4.1 As is easily seen, the class of all open sets satisfying (4.1) includes a lot of domains which have piecewise linear boundaries. Here, let us notice that the domain illustrated in Fig. 4.5 can be one of examples of such open sets.

Finally, we prove a theorem which would give us useful information in the stability analysis for solutions of $(P_\sigma)_{\theta_*}$ as in Theorem 4.1.

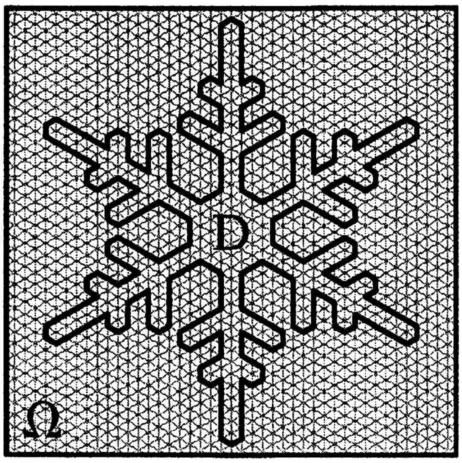

Fig. 4.5

Theorem 4.2 *(Minimizers of the free energy) Let $\mathscr{F}_0$ be a functional on $L^2(\Omega)$, defined as:*

$$\mathscr{F}_0(z) := \Phi_\sigma(z) - \frac{1}{2}\int_\Omega |z|^2\, dx \quad \textit{for any } z \in L^2(\Omega).$$

Then, any solution w_D of $(P_\sigma)_{\theta_*}$ as in Theorem 4.1 is a (global) minimizer of $\mathscr{F}_0$. Here, let us recall that the functional $\mathscr{F}_0$ is the free energy $\mathscr{F}_{\theta_*}$ given as in (1.4) of the case that $\theta_* = 0$.

Proof. We see from (4.1) and the definition of the functional Φ_σ that

$$\Phi_\sigma(w_D) = \int_\Omega \sigma(x)\ |\nabla w_D| = 2\int_{\partial D} \sigma(x)\ d\mathscr{H}^1 = 0 \leq \Phi_\sigma(z) \text{ for any } z \in D(\Phi_\sigma).$$

On the other hand, since $|w_D| = 1$, a.e. in Ω,

$$-\frac{1}{2}\int_\Omega |w_D|^2\, dx \leq -\frac{1}{2}\int_\Omega |z|^2\, dx \text{ for any } z \in D(\Phi_\sigma).$$

Adding the both sides of two inequalities in the above, we conclude that

$$\mathscr{F}_0(w_D) = \Phi_\sigma(w_D) - \frac{1}{2}\int_\Omega |w_D|^2\, dx \leq \Phi_\sigma(z) - \frac{1}{2}\int_\Omega |z|^2\, dx = \mathscr{F}_0(z) \text{ for any } z \in L^2(\Omega).$$

■

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