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ソボレフ空間による関数の近似について

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Abstract. Let $H_K(E)$ be a reproducing kernel Hilbert space comprising complex-valued functions $\{f\}$ on E and L_j $(j = 1, 2, \ldots)$ be a bounded linear operator on $H_K(E)$ into a Hilbert space H_j. Then, for $d_j \in H_j$ we shall consider the simultaneous operator equations $L_j f = d_j$ $(j = 1, 2, \ldots)$ with the best approximation problem, for given $d_j \in H_j$

$$\inf_{f \in H_K(E)} \sum_j \|L_j f - d_j\|^2_{H_j}.$$

Furthermore we shall give a general idea and method for approximations of L_2 functions by Sobolev Hilbert spaces by using the Tikhonov regularization. We shall illustrate examples by figures for approximations of L_2 functions by the first and second order Sobolev Hilbert spaces.

Keywords: Reproducing kernel, operator equations, bounded linear operator, Tikhonov regularization, Sobolev space, best approximation, Green's function, simultaneous linear partial differential equation, generalized inverse.

1 Introduction and Background Theorems

We shall formulate our background theorem which has many concrete applications based on [2-6].

Let H_K be a Hilbert space comprising complex-valued functions $\{f\}$ on a set E admitting a reproducing kernel $K(x, y)$ and let L be a bounded linear operator on H_K into a Hilbert space H. We introduce the inner product in the space H_K, for any fixed $\lambda > 0$

$$\lambda(f_1, f_2)_{H_K} + (Lf_1, Lf_2)_H. \tag{1}$$

Then, it forms a Hilbert space and this Hilbert space $H_K(L; \lambda)$ admits a reproducing kernel $K_L(x, y; \lambda)$ on E. Then, we have the relation of $K(x, y)$ and $K_L(x, y; \lambda)$

$$K_L(x, y; \lambda) + \frac{1}{\lambda}(LK_L(., y; \lambda), LK(., x))_H = \frac{1}{\lambda}K(x, y). \tag{2}$$

Theorem 1 *The best approximation f^*_{λ,g,f_0} in the sense, for any $f_0 \in H_K$ and for any $g \in H$*

$$\inf_{f \in H_K} \left\{\lambda\|f - f_0\|^2_{H_K} + \|Lf - g\|^2_H\right\}$$

$$= \lambda\|f^*_{\lambda,g,f_0} - f_0\|^2_{H_K} + \|Lf^*_{\lambda,g,f_0} - g\|^2_H \tag{3}$$

exists uniquely and it is represented by

$$f^*_{\lambda,g,f_0}(x) = \lambda(f_0(\cdot), K_L(\cdot, x; \lambda))_{H_K} + (g(\cdot), LK_L(\cdot, x; \lambda))_H. \tag{4}$$

As simple and typical reproducing kernel Hilbert spaces, we shall consider the Sobolev Hilbert spaces H_{K_1} and H_{K_2} admitting the reproducing kernels

$$K_1(x, y) = \frac{1}{2}e^{-|x-y|} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\xi^2 + 1}d\xi \tag{5}$$

and

$$K_2(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\xi^4 + \xi^2 + 1}d\xi. \tag{6}$$

The norms in H_{K_1} and H_{K_2} are given by

$$\|f\|^2_{H_{K_1}} = \int_{-\infty}^{\infty}(|f'(x)|^2 + |f(x)|^2)dx$$

and

$$\|f\|^2_{H_{K_2}} = \int_{-\infty}^{\infty}(|f''(x)|^2 + |f'(x)|^2 + |f(x)|^2)dx,$$

respectively. We shall examine the best approximation in (3) for some typical Hilbert spaces H and bounded linear operators L. In general, we are interested in the behaviours of the best approximation functions for λ tending to zero from the viewpoint of the Tikhonov regularization. So, we wish to illustrate the behaviours of the best approximations for λ tending to zero.

2 Typical Examples

See [3] for many concrete reproducing kernel forms for which Theorem 1 is applied. We can see a general example and a general approach for simultaneous linear partial differential equations in N. Aronszajn [1] who discussed deeply Green's functions in connection with reproducing kernels. We shall give typical examples.

2.1 Let

$$G(x,y) = \frac{1}{2}e^{-|x-y|}. \tag{7}$$

Then $G(x,y)$ is the reproducing kernel for the Hilbert Sobolev space H_G comprising all absolutely continuous functions $f(x)$ on $\mathbf{R}$ with finite norms

$$\left\{\int_{-\infty}^{\infty}(|f'(x)|^2 + |f(x)|^2)dx\right\}^{\frac{1}{2}} < \infty. \tag{8}$$

Hence, we can examine the best approximation problem as follows:
For any given $F_1, F_2 \in L_2(\mathbf{R})$,

$$\inf_{f\in H_G}\int_{-\infty}^{\infty}(|F_1(x) - f'(x)|^2 + |F_2(x) - f(x)|^2)dx. \tag{9}$$

2.2 For the first order Sobolev Hilbert space H_{K_1} we shall consider the two bounded linear operators $L_1 : H_{K_1} \to L_1 f = f \in L_2(\mathbf{R})$ and $L_2 : H_{K_1} \to L_2 f = f' \in L_2(\mathbf{R})$. Then, the associated reproducing kernels $K_{1,1}(x,y;\lambda)$ and $K_{1,2}(x,y;\lambda)$ for the RKHSs with the norms

$$\lambda\|f\|^2_{H_{K_1}} + \|f\|^2_{L_2(\mathbf{R})}$$

and

$$\lambda\|f\|^2_{H_{K_1}} + \|f'\|^2_{L_2(\mathbf{R})}$$

are given by

$$K_{1,1}(x,y;\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\xi(x-y)}}{\lambda\xi^2 + (\lambda+1)}d\xi$$
$$= \frac{1}{2\sqrt{\lambda(\lambda+1)}}\exp\left\{-\sqrt{\frac{\lambda+1}{\lambda}}|x-y|\right\} \tag{10}$$

and

$$K_{1,2}(x,y;\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\xi(x-y)}}{(\lambda+1)\xi^2 + \lambda}d\xi$$

$$= \frac{1}{2\sqrt{\lambda(\lambda+1)}} \exp\left\{-\sqrt{\frac{\lambda}{\lambda+1}}|x-y|\right\}, \tag{11}$$

respectively. Hence, the best approximate functions $f^*_{1,1}(x;\lambda,g)$ and $f^*_{1,2}(x;\lambda,g)$ in the senses, for any $g \in L_2(\mathbf{R})$

$$\inf_{f\in H_{K_1}} \left\{\lambda\|f\|^2_{H_{K_1}} + \|f-g\|^2_{L_2(\mathbf{R})}\right\}$$

$$= \lambda\|f^*_{1,1}(\cdot;\lambda,g)\|^2_{H_{K_1}} + \|f^*_{1,1}(\cdot;\lambda,g)-g\|^2_{L_2(\mathbf{R})} \tag{12}$$

and

$$\inf_{f\in H_{K_1}} \left\{\lambda\|f\|^2_{H_{K_1}} + \|f'-g\|^2_{L_2(\mathbf{R})}\right\}$$

$$= \lambda\|f^*_{1,2}(\cdot;\lambda,g)\|^2_{H_{K_1}} + \|f^{*\prime}_{1,2}(\cdot;\lambda,g)-g\|^2_{L_2(\mathbf{R})} \tag{13}$$

are given by

$$f^*_{1,1}(x;\lambda,g) = \int_{-\infty}^{\infty} g(\xi)\frac{1}{2\sqrt{\lambda(\lambda+1)}} \exp\left\{-\sqrt{\frac{\lambda+1}{\lambda}}|\xi-x|\right\} d\xi \tag{14}$$

and

$$f^*_{1,2}(x;\lambda,g) = \int_{-\infty}^{\infty} g(\xi)\frac{1}{2\sqrt{\lambda(\lambda+1)}}\frac{\partial}{\partial\xi} \exp\left\{-\sqrt{\frac{\lambda}{\lambda+1}}|\xi-x|\right\} d\xi, \tag{15}$$

respectively. Note that $f^*_{1,2}(x;\lambda,g)$ can be considered as an approximate and generalized solution of the differential equation

$$y' = g(x) \quad \text{on} \quad \mathbf{R} \tag{16}$$

in the first order Sobolev Hilbert space H_{K_1}. See Figure 3.

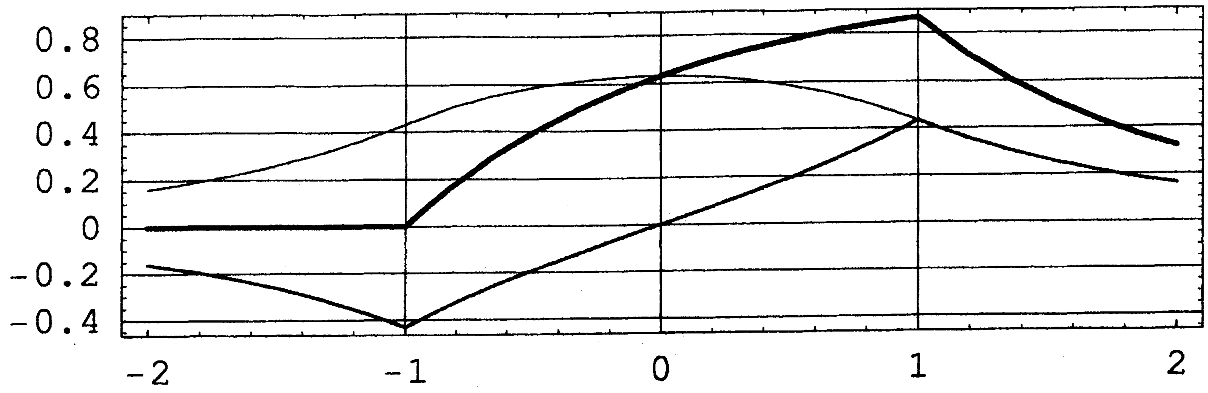


Figure 1: Examples of approximated functions in (9). (a) $F_1(x) = 0$ and $F_2(x) = \chi_{[-1,1]}$ (top thin curve). (b) $F_1(x) = F_2(x) = \chi_{[-1,1]}$ (middle bold curve). (c) $F_1(x) = \chi_{[-1,1]}$ and $F_2(x) = 0$ (bottom curve).

2.3 For the second order Sobolev Hilbert space H_{K_2} we shall consider the three bounded linear operators into $L_2(\mathbf{R})$ defined by

$$L_1 : f \longrightarrow f$$

$$L_2 : f \longrightarrow f'$$

and

$$L_3 : f \longrightarrow f''.$$

Then, the reproducing kernels $K_{2,1}(x, y; \lambda), K_{2,2}(x, y; \lambda)$ and $K_{2,3}(x, y; \lambda)$ for the Hilbert spaces with the norms

$$\lambda\|f\|^2_{H_{K_2}} + \|f\|^2_{L_2(\mathbf{R})},$$

$$\lambda\|f\|^2_{H_{K_2}} + \|f'\|^2_{L_2(\mathbf{R})},$$

and

$$\lambda\|f\|^2_{H_{K_2}} + \|f''\|^2_{L_2(\mathbf{R})},$$

are given by

$$K_{2,1}(x, y; \lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\lambda\xi^4 + \lambda\xi^2 + (\lambda + 1)} d\xi, \tag{17}$$

$$K_{2,2}(x, y; \lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\lambda\xi^4 + (\lambda + 1)\xi^2 + \lambda} d\xi, \tag{18}$$

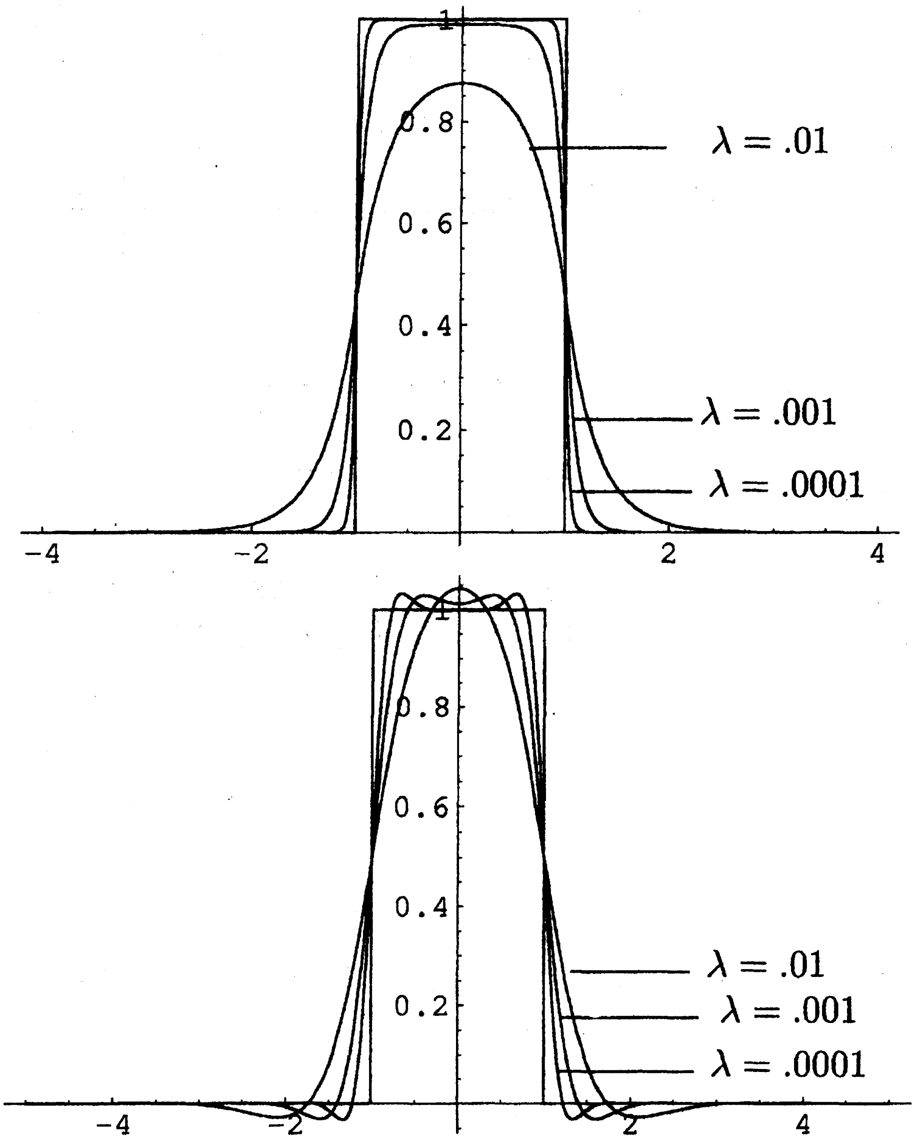
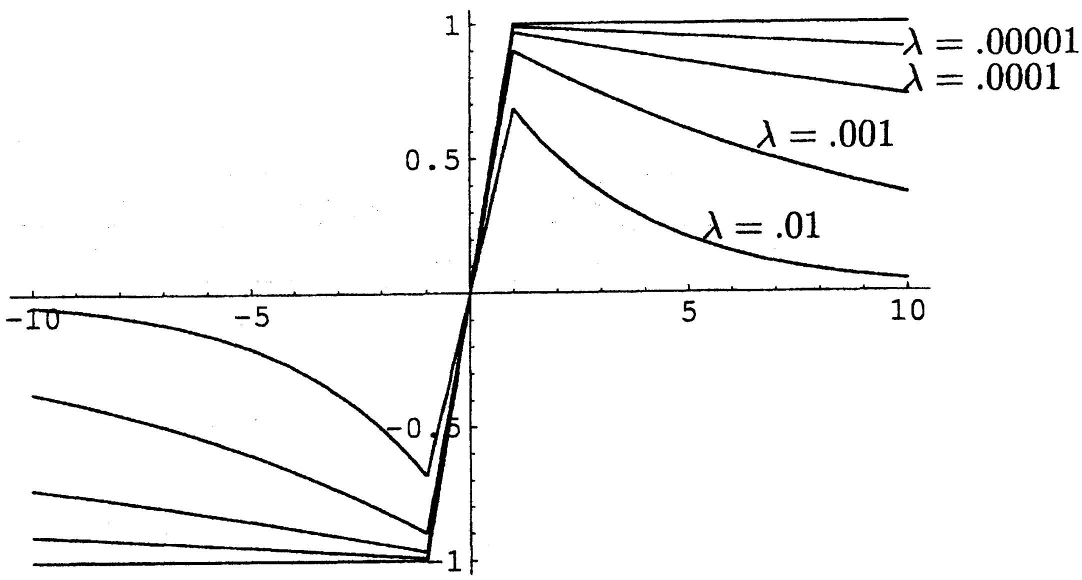


Figure 2: Graphs of $f^*_{1,1}(x; \lambda, g)$ in (14)(top) and $f^*_{2,1}(x; \lambda, g)$ in (20) (bottom) for $g(x) = \chi_{[-1,1]}$.



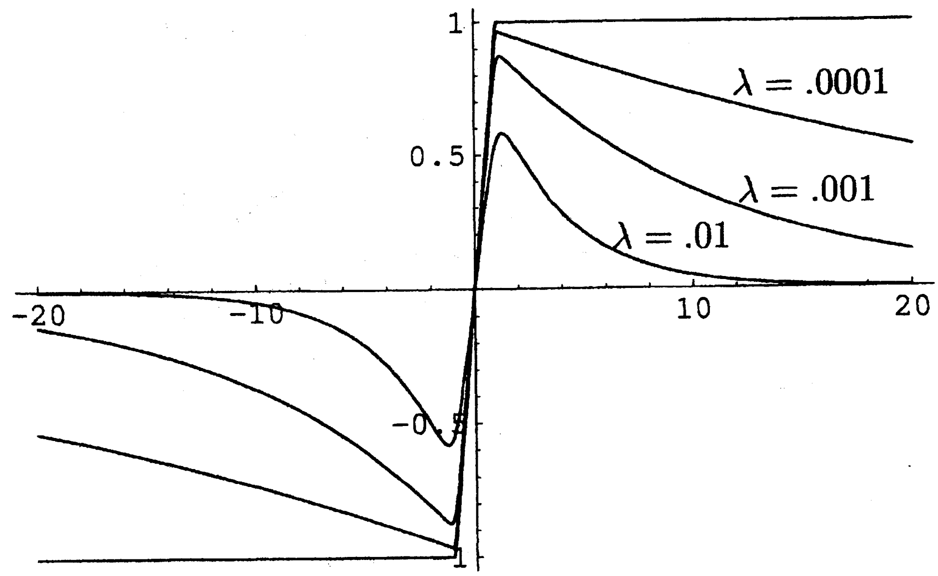


Figure 3: Graphs of $f^*_{1,2}(x;\lambda,g)$ in (15)(top) and $f^*_{2,2}(x;\lambda,g)$ in (21) (bottom) for $g(x)=\chi_{[-1,1]}$.

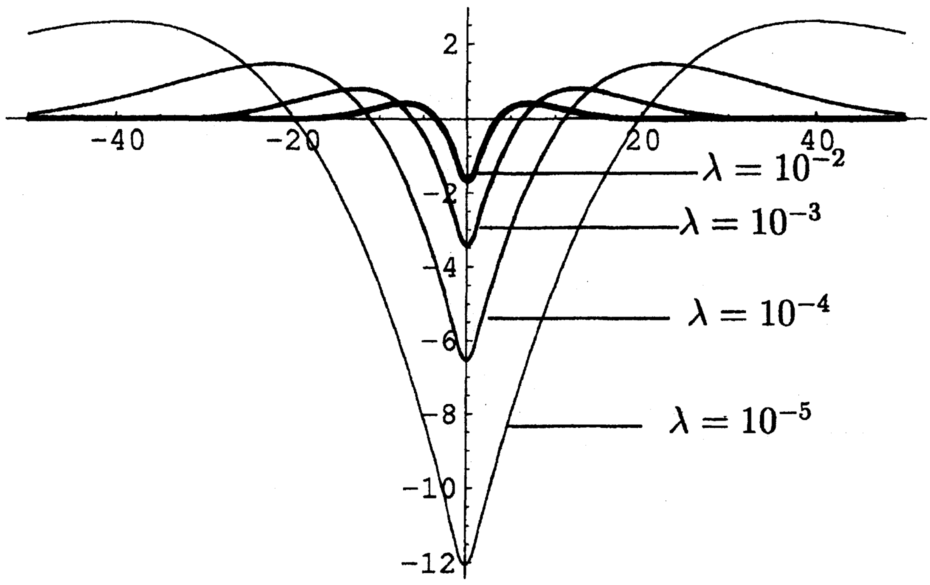


Figure 4: Graphs of $f^*_{2,3}(x; \lambda, g)$ in (22) for $g(x) = \chi_{[-1,1]}$.

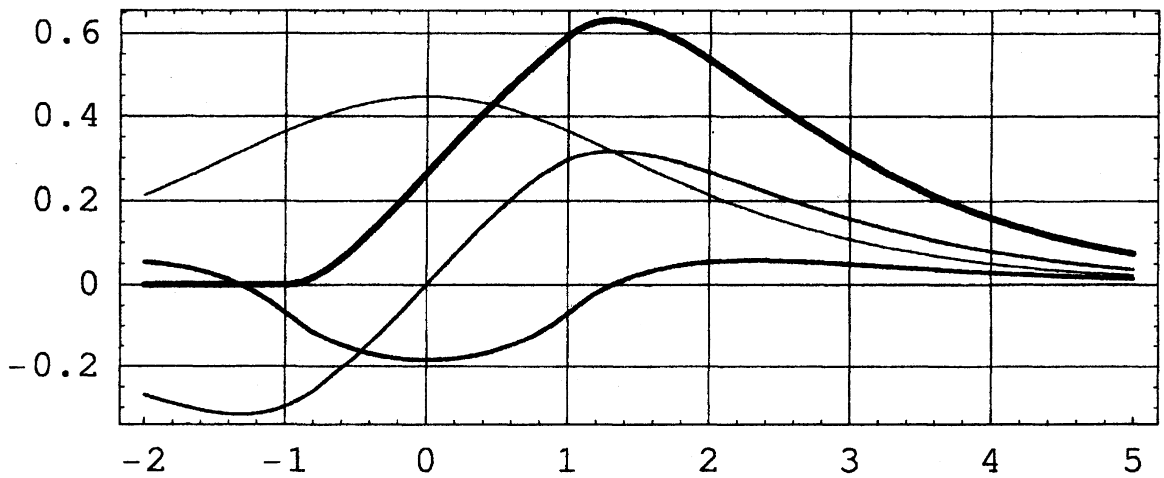


Figure 5: Examples of approximated functions in (25). (a) $F_1(x) = F_2(x) = F_3(x) = \chi_{[-1,1]}$ (top bold curve). (b) $F_1(x) = \chi_{[-1,1]}$ and $F_2(x) = F_3(x) = 0$ (the second bold curve) (c) $F_1(x) = F_2(x) = 0$ and $F_3(x) = \chi_{[-1,1]}$ (the thin curve). (d) $F_1(x) = F_3(x) = 0$ and $F_2(x) = \chi_{[-1,1]}$ (the rest curve).

and

$$K_{2,3}(x,y;\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{(\lambda+1)\xi^4+\lambda\xi^2+\lambda} d\xi. \tag{19}$$

Then, the corresponding best approximate functions $f_{2,1}^*(x;\lambda,g)$, $f_{2,2}^*(x;\lambda,g)$, and $f_{2,3}^*(x;\lambda,g)$ are given by, for any $g \in L_2(\mathbf{R})$

$$f_{2,1}^*(x;\lambda,g) = \int_{-\infty}^{\infty} g(\xi)d\xi \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\eta(\xi-x)}}{\lambda\eta^4+\lambda\eta^2+(\lambda+1)} d\eta, \tag{20}$$

$$f_{2,2}^*(x;\lambda,g) = \int_{-\infty}^{\infty} g(\xi)d\xi \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-i\eta\cdot e^{-i\eta(\xi-x)}}{\lambda\eta^4+(\lambda+1)\eta^2+\lambda} d\eta, \tag{21}$$

and

$$f_{2,3}^*(x;\lambda,g) = \int_{-\infty}^{\infty} g(\xi)d\xi \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-\eta^2\cdot e^{-i\eta(\xi-x)}}{(\lambda+1)\eta^4+\lambda\eta^2+\lambda} d\eta, \tag{22}$$

respectively. We shall give another type applications of Theorem 1. Note that

$$K(x,y) = \frac{1}{4}e^{-|x-y|}\{1+|x-y|\} \tag{23}$$

is the reproducing kernel of the Sobolev space H_K with finite norms

$$\left\{\int_{-\infty}^{\infty}(|f''(x)|^2+2|f'(x)|^2+|f(x)|^2)dx\right\}^{\frac{1}{2}} < \infty. \tag{24}$$

Therefore, we can examine the approximate problem as follows:

$$\inf_{f\in H_K}\int_{-\infty}^{\infty}(|F_1(x)-f''(x)|^2+2|F_2(x)-f'(x)|^2+|F_3(x)-f(x)|^2)dx. \tag{25}$$

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