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Nonlinear Stability of Strong Rarefaction Waves for Compressible Navier-Stokes Equations

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1 Introduction and main results

Consider the one-dimensional compressible Navier-Stokes equations in the Lagrangian coordinates

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \left(\mu \frac{u_x}{v}\right)_x, \\ \left(e + \frac{u^2}{2}\right)_t + (up)_x = \left(\kappa \frac{\theta_x}{v} + \mu \frac{u u_x}{v}\right)_x, \end{cases} \tag{1.1}$$

where the unknowns $v > 0, u, \theta > 0, p, e$, and s represent the specific volume, the velocity, the absolute temperature, the pressure, the internal energy, and the entropy of the gas respectively. The coefficients of viscosity and heat-conductivity, μ and κ, are assumed to be positive constants. We assume, as is usual in thermodynamics, that by any given two of the five thermodynamical variables, v, p, e, θ, and s, the remaining three variables are expressed.

The second law of thermodynamics asserts that

$$\theta ds = de + pdv,$$

from which, if we choose (v,θ), (v,s), or (v,e) as independent variables and write $(p,e,s) = (p,e,s)(v,\theta)$, or $(p,e,\theta) = (\tilde{p},\tilde{e},\tilde{\theta})(v,s)$, or $(p,s,\theta) = (\hat{p},\hat{s},\hat{\theta})(v,e)$ respectively, then we can deduce that

$$\begin{cases} s_v(v,\theta)=p_\theta(v,\theta),\\ s_\theta(v,\theta)=\frac{e_\theta(v,\theta)}{\theta},\\ e_v(v,\theta)=\theta p_\theta(v,\theta)-p(v,\theta),\end{cases}\tag{1.2}$$

$$\begin{cases} \tilde{e}_v(v,s)=-p(v,\theta), & \tilde{e}_s(v,s)=\theta,\\ \tilde{p}_v(v,s)=p_v(v,\theta)-\frac{\theta(p_\theta(v,\theta))^2}{e_\theta(v,\theta)}, & \tilde{p}_s(v,s)=\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)},\\ \tilde{\theta}_v(v,s)=-\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}, & \tilde{\theta}_s(v,s)=\frac{\theta}{e_\theta(v,\theta)},\end{cases}\tag{1.3}$$

or

$$\begin{cases} \hat{s}_e(v,e)=\frac{1}{\theta}, & \hat{s}_v(v,e)=\frac{p(v,\theta)}{\theta},\\ \hat{p}_e(v,e)=\frac{p_\theta(v,\theta)}{e_\theta(v,\theta)}, & \hat{p}_v(v,e)=\left(p_v(v,\theta)-\frac{\theta(p_\theta(v,\theta))^2}{e_\theta(v,\theta)}\right)+\frac{p(v,\theta)p_\theta(v,\theta)}{e_\theta(v,\theta)},\\ \hat{\theta}_e(v,e)=\frac{1}{e_\theta(v,\theta)}, & \hat{\theta}_v(v,e)=\frac{p(v,\theta)-\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}.\end{cases}\tag{1.4}$$

From (1.3) and (1.4), we get that

$$\tilde{p}_v(v,s)=\hat{p}_v(v,e)-p(v,\theta)\hat{p}_e(v,e).\tag{1.5}$$

What we are interested in this paper is to show that the strong expansion waves for (1.1) are nonlinear stable. For this, it is convenient to work with the equations for the entropy s and the absolute temperature θ, *i.e.*

$$s_t=\kappa\left(\frac{\theta_x}{v\theta}\right)_x+\kappa\frac{\theta_x^2}{v\theta^2}+\mu\frac{u_x^2}{v\theta}\tag{1.6}$$

and

$$\theta_t+\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}u_x=\frac{\kappa}{e_\theta(v,\theta)}\left(\frac{\theta_x}{v}\right)_x+\frac{\mu}{e_\theta(v,\theta)}\frac{u_x^2}{v}.\tag{1.7}$$

In fact, for smooth solutions, equations $(1.1)_1$, $(1.1)_2$, $(1.1)_3$ are equivalent to equations $(1.1)_1$, $(1.1)_2$, (1.6) or $(1.1)_1$, $(1.1)_2$, (1.7). In what follows, we will consider $(1.1)_1$, $(1.1)_2$, (1.6) with the initial data

$$(v,u,s)(t,x)|_{t=0}=(v_0,u_0,s_0)(x)\to(v_\pm,u_\pm,s_\pm)\ \text{ as }\ x\to\pm\infty.\tag{1.8}$$

Here $v_\pm>0$, $u_\pm$, $s_\pm$ are constants. Since we will focus on the expansion waves to (1.1), we assume that $s_+=s_-=\bar{s}$ in the rest of this paper.

For expansion waves, the right-hand side of (1.1) decays faster than each term on the left-hand side. Therefore, the compressible Navier-Stokes equations (1.1) may be approximated, time-asymptotically, by the compressible Euler equations

$$\begin{cases} v_t-u_x=0,\\ u_t+\tilde{p}(v,s)_x=0,\\ s_t=0.\end{cases}\tag{1.9}$$

There are two families of expansion (rarefaction) waves for (1.9) which are solutions of the compressible Euler equations (1.9) with Riemann data $\left(v_0^R, u_0^R, s_0^R\right)(x)$ (*cf.* [1]),

$$(v,u,s)(t,x)|_{t=0} = \left(v_0^R, u_0^R, s_0^R\right)(x) = \begin{cases} (v_-, u_-, s_-), & x<0, \\ (v_+, u_+, s_+), & x>0. \end{cases} \tag{1.10}$$

For illustration, we only consider the 1-rarefaction wave $\left(V^R, U^R, S^R\right)(t,x)$, which is characterized by

$$\begin{cases} S^R(t,x) = \bar{s}, \\ U^R(t,x) - \int^{V^R(t,x)} \sqrt{-\tilde{p}_v(z,\bar{s})}dz = u_\pm - \int^{v_\pm} \sqrt{-\tilde{p}_v(z,\bar{s})}dz, \\ \lambda_{1x}\left(V^R(t,x), S^R(t,x)\right) > 0, \quad \lambda_1(v,s) = -\sqrt{-\tilde{p}_v(v,s)}. \end{cases} \tag{1.11}$$

The case for the 3-rarefaction wave can be discussed similarly.

Before stating the main results, we first list the assumptions on the pressure function $p(v,\theta)$ and the internal energy $e(v,\theta)$ used throughout this paper:

$$\text{(H}_1\text{)} \qquad p_v(v,\theta) = \frac{\partial p(v,\theta)}{\partial v} < 0, \quad e_\theta(v,\theta) = \frac{\partial e(v,\theta)}{\partial \theta} > 0$$

and

$$\text{(H}_2\text{)} \qquad \tilde{p}_{vv}(v,s) = \frac{\partial^2 \tilde{p}(v,s)}{\partial v^2} > 0 \text{ and } \tilde{p}(v,s) \text{ is convex with respect to } (v,s).$$

From (1.3) and (H$_1$), we can deduce that

$$\tilde{p}_v(v,s) = p_v(v,\theta) - \frac{\theta(p_\theta(v,\theta))^2}{e_\theta(v,\theta)} < 0, \tag{1.12}$$

$$\begin{cases} \tilde{e}_{ss}(v,s) = \frac{\theta}{e_\theta(v,\theta)} > 0, \\ \tilde{e}_{vs}(v,s) = \frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}, \\ \tilde{e}_{vv}(v,s) = -p_v(v,\theta) + \frac{\theta(p_\theta(v,\theta))^2}{e_\theta(v,\theta)} > 0, \end{cases}$$

and

$$\tilde{e}_{ss}(v,s)\tilde{e}_{vv}(v,s) - (\tilde{e}_{vs}(v,s))^2 = -\frac{\theta p_v(v,\theta)}{e_\theta(v,\theta)} > 0. \tag{1.13}$$

Equation (1.13) implies that $\tilde{e}(v,s)$ is convex with respect to v and s. Consequently, $\tilde{e}(v,s) + \frac{1}{2}u^2$ is a strictly convex function of (v,u,s). Now we can construct the following normalized entropy $\eta(v,u,s;V,U,S)$ around $(V,U,S)(t,x)$, which is the smooth approximation of the 1-rarefaction waves $\left(V^R, U^R, S^R\right)(t,x)$,

$$\begin{aligned} \eta(v,u,s;V,U,S) = &\left(e(v,\theta) + \tfrac{u^2}{2}\right) - \left(e(V,\Theta) + \tfrac{U^2}{2}\right) \\ &- \Big\{ -p(V,\Theta)(v-V) + U(u-U) + \Theta(s-S) \Big\}. \end{aligned} \tag{1.14}$$

Here we have used the fact that $\tilde{e}_v(v,s) = -p(v,\theta)$, $\tilde{e}_s(v,s) = \theta$. And the approximate rarefaction waves $V(t,x), U(t,x), S(t,x)$, and $\Theta(t,x)$ are constructed as follows (*cf.* [20]).

Given a suitably small but fixed constant $\varepsilon > 0$, let $w(t,x)$ be the unique global smooth solution to the Cauchy problem

$$\begin{cases} w_t + ww_x = 0, \\ w(t,x)|_{t=0} = w_0(x) := \frac{\lambda_1(v_-,\overline{s})+\lambda_1(v_+,\overline{s})}{2} + \frac{\lambda_1(v_+,\overline{s})-\lambda_1(v_-,\overline{s})}{2}\tanh(\varepsilon x), \end{cases} \tag{1.15}$$

then, $V(t,x), U(t,x), S(t,x)$, and $\Theta(t,x)$ are defined by

$$\begin{cases} \lambda_1(V(t,x),\overline{s}) = -\sqrt{-\tilde{p}_v(V(t,x),\overline{s})} = w(t,x), \\ U(t,x) = u_\pm + \int_{v_\pm}^{V(t,x)} \sqrt{-\tilde{p}_v(z,\overline{s})}dz, \\ S(t,x) = \overline{s}, \\ \Theta(t,x) = \tilde{\theta}(V(t,x),\overline{s}). \end{cases} \tag{1.16}$$

Under the above preparation, for the general gas, our stability result on strong rarefaction waves $\left(V^R, U^R, S^R\right)(t,x)$ can be stated as in the following.

Theorem 1.1 (Local Stability Result for General Gas) *Assume that* $\left(V^R, U^R, S^R\right)(t,x)$ *is the 1-rarefaction wave solution to the Riemann problem of the compressible Euler equations (1.9), (1.10) and that the initial data* $(v_0, u_0, s_0)(x)$ *of the compressible Navier-Stokes equations* $(1.1)_1$, $(1.1)_2$, *(1.6) satisfies (1.8),*

$$\begin{cases} 0 < 2\underline{V} \le v_0(x), V(t,x) \le \frac{1}{2}\overline{V}, \\ 0 < 2\underline{\Theta} \le \theta_0(x), \Theta(t,x) \le \frac{1}{2}\overline{\Theta} \end{cases} \tag{1.17}$$

for all $(t,x) \in \mathbf{R}_+ \times \mathbf{R}$ *and some positive constants* $\underline{V}, \overline{V}, \underline{\Theta}$, *and* $\overline{\Theta}$, *and*

$$N(0) = \|(v_0(x) - V(0,x), u_0(x) - U(0,x), s_0(x) - \overline{s})\|_{H^2(\mathbf{R})}$$

is sufficiently small. Then the Cauchy problem (1.1), (1.8) admits a unique global smooth solution $(v,u,s)(t,x)$ *satisfying*

$$\lim_{t\to+\infty} \sup_{x\in\mathbf{R}} \left\{ \left| \left(v(t,x) - V^R(t,x), u(t,x) - U^R(t,x), s(t,x) - \overline{s}\right) \right| \right\} = 0. \tag{1.18}$$

Note that the essential meaning of nonlinear stability of rarefaction waves to the compressible Navier-Stokes equations (1.1), (1.8) in [12], [15], [20], [21], [22] is that if $(v_0, u_0, s_0)(x)$ is a (small or large) perturbation of $(V(0,x), U(0,x), \overline{s})$, the smooth approximation of the rarefaction wave solutions $\left(V^R(t,x), U^R(t,x), \overline{s}\right)$, then the Cauchy problem of the compressible Navier-Stokes equations (1.1), (1.8) admits a unique global smooth solution $(v,u,s)(t,x)$ which tends time-asymptotically to $\left(V^R(t,x),\ U^R(t,x), \overline{s}\right)$. In this sense, the result obtained in Theorem 1.1 does imply the nonlinear stability of strong rarefaction waves for the compressible Navier-Stokes equations. But, due to the assumption that the initial perturbation $(v_0(x) - V(0,x), u_0(x) - U(0,x), s_0(x) - \overline{s})$ should be small, the nonlinear stability result obtained in Theorem 1.1 is essentially local. Then a natural question of importance and interest is how to get the global stability result which is for large perturbation. Our second purpose is to devote to this problem and show that, for the ideal polytropic gas, such a global stability result indeed holds for γ near 1 without the weakness of the rarefaction waves. To state the result precisely, we recall that for the ideal polytropic gas, $(p,e)(v,\theta)$ have the following special constitutive relations

$$p(v,\theta)=\frac{R\theta}{v}=Av^{-\gamma}\exp\left(\frac{\gamma-1}{R}s\right),\qquad e(v,\theta)=\frac{R\theta}{\gamma-1},\tag{1.19}$$

where $R>0$ is the gas constant, $\gamma>1$ the adiabatic constant, and A a positive constant.

Our second result is stated as follows.

Theorem 1.2 (Global Stability Result for the Ideal Polytropic Gas) *Assume that $\left(V^R(t,x),\ U^R(t,x),\bar{s}\right)$ is the 1-rarefaction wave solution of the Riemann problem of the compressible Euler equations (1.9), (1.10) and that $(p,e)(v,\theta)$ satisfy the constitutive relations (1.19). Then for any $(v_0(x)-V(0,x),u_0(x)-U(0,x),s_0(x)-\bar{s})\in H^2(\mathbf{R})$ satisfying (1.17) and its $H^1(\mathbf{R})-$norm to be bounded by a constant independent of $\frac{1}{\varepsilon}$, the corresponding Cauchy problem (1.1), (1.8) admits a unique global smooth solution $(v,u,s)(t,x)$ satisfying (1.18) provided that $\gamma-1$ is sufficiently small.*

In the proof of Theorem 1.2, the assumption that γ is close to 1 is used for obtaining the *a priori* assumption $0<\underline{\Theta}<\theta(t,x)<\overline{\Theta}$ for $(t,x)\in[0,\infty]\times\mathbf{R}$ so that $\theta(t,x)-\Theta(t,x)$ is small. Hence, one can image that for the isentropic polytropic gas, such a smallness assumption can be removed and this has been obtained by A. Matsumura and K. Nishihara in [21], [22] by cleverly introducing another type of smooth approximation of the rarefaction wave solution. That is, $w_0(x)$ in $(1.15)_2$ is replaced by

$$w(t,x)|_{t=0}=\underline{w}_0(x)=\frac{\lambda_1(v_-,\bar{s})+\lambda_1(v_+,\bar{s})}{2}+\frac{\lambda_1(v_+,\bar{s})-\lambda_1(v_-,\bar{s})}{2}K_q\int_0^{\varepsilon x}(1+y^2)^{-q}dy,\tag{1.20}$$

where $K_q>0$ is a constant satisfying

$$K_q\int_0^{+\infty}(1+y^2)^{-q}dy=1\tag{1.21}$$

for some suitably large constant $q>0$.

Our third purpose is to show the global stability result on strong rarefaction waves for $p-$system with viscosity with a general pressure $p=p(v)$. To state this result, we recall that the isentropic compressible Navier-Stokes equations in Lagrangian Coordinates can be written as

$$\begin{cases} v_t-u_x=0,\\ u_t+p(v)_x=\mu\left(\frac{u_x}{v}\right)_x,\end{cases}\tag{1.22}$$

with the initial data

$$(v,u)(t,x)|_{t=0}=(v_0,u_0)(x)\to(v_\pm,u_\pm)\quad\text{as}\quad x\to\pm\infty.\tag{1.23}$$

Here $v_\pm>0$ and $u_\pm$ are given constants so that the Riemann problem of the isentropic compressible Euler equations

$$\begin{cases} v_t-u_x=0,\\ u_t+p(v)_x=0,\end{cases}\tag{1.24}$$

with the Riemann data

$$(v,u)(t,x)|_{t=0}=\left(\overline{v}_0^R,\overline{u}_0^R\right)(x)=\begin{cases}(v_-,u_-), & x<0,\\ (v_+,u_+), & x>0,\end{cases} \tag{1.25}$$

is assumed to admit a unique 1-rarefaction wave solution $\left(\overline{V}^R,\overline{U}^R\right)(t,x)$.

We only assume that $p(v)$ is a positive smooth function for $v>0$ and satisfies

$$p'(v)<0,\quad p''(v)>0 \text{ for } v>0. \tag{1.26}$$

Under the above assumptions, we have the following theorem.

Theorem 1.3 (Global Stability Result for General Isentropic Gas) *Assume that the Riemann problem (1.24), (1.25) to the compressible Euler equations admits a unique 1-rarefaction wave solution* $\left(\overline{V}^R,\overline{U}^R\right)(t,x)$ *and that* $\left(\overline{V},\overline{U}\right)(t,x)$ *is a smooth approximation of the Riemann solution* $\left(\overline{V}^R,\overline{U}^R\right)(t,x)$ *constructed by*

$$\begin{cases}\overline{V}(t,x)=\lambda_1^{-1}\left(\overline{w}(t,x)\right), \quad \lambda_1(v)=-\sqrt{-p'(v)},\\ \overline{U}(t,x)=u_\pm+\displaystyle\int_{v_\pm}^{\overline{V}(t,x)}\sqrt{-p'(s)}ds.\end{cases} \tag{1.27}$$

Here $\overline{w}(t,x)$ *is the unique smooth solution to the following Cauchy problem*

$$\begin{cases} w_t+ww_x=0,\\ w(t,x)|_{t=0}=\overline{w}_0(x)=\frac{\lambda_1(v_-)+\lambda_1(v_+)}{2}+\frac{\lambda_1(v_+)-\lambda_1(v_-)}{2}\tanh(\varepsilon x).\end{cases} \tag{1.28}$$

Then for any $p(v)$ *satisfying (1.26) and* $\left(v_0(x)-\overline{V}(0,x),u_0(x)-\overline{U}(0,x)\right)\in H^2(\mathbf{R})$ *satisfying* $0<2\underline{V}\le v_0(x),\overline{V}(t,x)\le\frac{1}{2}\overline{V}$ *for all* $(t,x)\in\mathbf{R}_+\times\mathbf{R}$ *and some positive constants* $\underline{V}$, $\overline{V}$ *and with its* $H^1(\mathbf{R})-$*norm bounded by a constant independent of the quantity* $\frac{1}{\varepsilon}$, *the Cauchy problem (1.22), (1.23) admits a unique global smooth solution* $(v,u)(t,x)$ *satisfying*

$$\lim_{t\to+\infty}\sup_{x\in\mathbf{R}}\left\{\left|\left(v-\overline{V}^R,u-\overline{U}^R\right)(t,x)\right|\right\}=0. \tag{1.29}$$

Remark 1.1 *In [21] and [22], the assumption that* $p(v)=v^{-\gamma}(\gamma\ge1)$ *plays an essential role in the analysis and it is worth to pointing out that even by using their smooth approximation of the Riemann solutions, their arguments can not be applied to the case when* $p(v)$ *satisfies only (1.26). However we have assumed that the* $H^1(\mathbf{R})-$*norm of the initial perturbation is bounded by a constant independent of* $\frac{1}{\varepsilon}$ *with small fixed number* $\varepsilon>0$. *This implies that the data* $(v_0,u_0)(x)$ *for (1.23) is initially rather flat though* $\left(v_0(x),\overline{V}(0,x),u_0(x)-\overline{U}(0,x)\right)$ *may be large. So, we should seek for the global solution and its behavior for any data* $(v_0,u_0)(x)$ *with* $\|(v_0(x)-v_\pm,u_0(x)-u_\pm)\|_{H^1(\mathbf{R}_\pm)}$ *bounded. This will be done under some additional assumptions on* $p(v)$ *in Theorem 1.4.*

In Theorem 1.1, 1.2, and 1.3, we assume that the solutions to the corresponding Riemann problem of the compressible Euler equations consists of only one rarefaction wave. In fact such a restriction can be removed by suitably modifying the arguments used in the proof of the theorems. To simplify the presentation, we use the isentropic compressible Navier-Stokes equations to explain this. Suppose that the solution $\left(\overline{V}^R, \overline{U}^R\right)(t,x)$ to the Riemann problem (1.24), (1.25) consists of one 1-rarefaction wave $\left(\overline{V}_1^R, \overline{U}_1^R\right)(t,x)$ and one 2-rarefaction wave $\left(\overline{V}_2^R, \overline{U}_2^R\right)(t,x)$. That is, there exists a unique constant state $(\overline{v}, \overline{u}) \in \mathbf{R}^2$ such that (v_-, u_-) and $(\overline{v}, \overline{u})$ are connected by one 1-rarefaction wave $\left(\overline{V}_1^R, \overline{U}_1^R\right)(t,x)$, *i.e.*, $(\overline{v}, \overline{u}) \in R_1(v_-, u_-)$, while $(\overline{v}, \overline{u})$ and (v_+, u_+) are connected by one 2-rarefaction wave $\left(\overline{V}_2^R, \overline{U}_2^R\right)(t,x)$, *i.e.*, $(v_+, u_+) \in R_2(\overline{v}, \overline{u})$. Here

$$\begin{cases} R_1(v_-, u_-) = \left\{ (v,u) \,\middle|\, u = u_- + \displaystyle\int_{v_-}^{v} \sqrt{-p'(s)}ds, \ u \geq u_- \right\}, \\ R_2(\overline{v}, \overline{u}) = \left\{ (v,u) \,\middle|\, u = \overline{u} - \displaystyle\int_{\overline{v}}^{v} \sqrt{-p'(s)}ds, \ u \geq \overline{u} \right\}. \end{cases} \tag{1.30}$$

Consequently

$$\left(\overline{V}^R, \overline{U}^R\right)(t,x) = \left(\overline{V}_1^R(t,x) + \overline{V}_2^R(t,x) - \overline{v}, \overline{U}_1^R(t,x) + \overline{U}_2^R(t,x) - \overline{u}\right). \tag{1.31}$$

Let $\overline{w}_i(t,x)(i = 1,2)$ be the unique global smooth solution to the following Cauchy problem

$$\begin{cases} \overline{w}_{it} + \overline{w}_i \overline{w}_{ix} = 0, \\ \overline{w}_i(t,x)|_{t=0} = \overline{w}_{i0}(x) = \frac{\overline{w}_{i-} + \overline{w}_{i+}}{2} + \frac{\overline{w}_{i+} - \overline{w}_{i-}}{2} \tanh(\varepsilon x), \quad i = 1,2, \end{cases} \tag{1.32}$$

then, as in [20], the smooth approximate solution $\left(\overline{V}, \overline{U}\right)(t,x)$ of $\left(\overline{V}^R, \overline{U}^R\right)(t,x)$ is constructed as follows:

$$\left(\overline{V}, \overline{U}\right)(t,x) = \left(\overline{V}_1(t,x) + \overline{V}_2(t,x) - \overline{v}, \overline{U}_1(t,x) + \overline{U}_2(t,x) - \overline{u}\right), \tag{1.33}$$

where $\left(\overline{V}_1, \overline{U}_1\right)(t,x)$ $\left(\text{resp. } \left(\overline{V}_2, \overline{U}_2\right)(t,x)\right)$ is defined by

$$\begin{cases} \lambda_1\left(\overline{V}_1(t,x)\right) = \overline{w}_1(t,x), \left(\text{resp. } \lambda_2\left(\overline{V}_2(t,x)\right) = \overline{w}_2(t,x)\right) \\ \overline{U}_1 = u_- + \displaystyle\int_{v_-}^{\overline{V}_1(t,x)} \sqrt{-p'(s)}ds, \quad \left(\text{resp. } \overline{U}_2(t,x) = \overline{u} - \displaystyle\int_{\overline{v}}^{\overline{V}_2(t,x)} \sqrt{-p'(s)}ds\right) \end{cases} \tag{1.34}$$

and $\overline{w}_1(t,x)$ (resp. $\overline{w}_2(t,x)$) is the solution of (1.32) with $\overline{w}_{1-} = \lambda_1(v_-)$ and $\overline{w}_{1+} = \lambda_1(\overline{v})$, (resp. $\overline{w}_{2-} = \lambda_2(\overline{v})$ and $\overline{w}_{2+} = \lambda_2(v_+)$).

It is easy to deduce that the smooth functions $\left(\overline{V}, \overline{U}\right)(t,x)$ satisfies the system

$$\begin{cases} \overline{V}_t - \overline{U}_x = 0, \\ \overline{U}_t + p\left(\overline{V}\right)_x = g\left(\overline{V}\right)_x, \end{cases} \tag{1.35}$$

where $g\left(\overline{V}\right) = p\left(\overline{V}\right) - p\left(\overline{V}_1\right) - p\left(\overline{V}_2\right) + p\left(\overline{v}\right)$. Hence, we only need to control $g\left(\overline{V}(t,x)\right)_x$ suitably in this case. Notice that from the properties on the smooth approximation of the rarefaction wave solution stated in [19], we only have

$$\int_0^t \left\| g\left(\overline{V}(\tau)\right)_x \right\|_{L^p(\mathbf{R})} d\tau \leq O(1)\varepsilon^{-\frac{1}{p}}. \tag{1.36}$$

From this observation together with the fact that, in deducing our main results, we need the smallness of ε, a quantity introduced in the construction of the smooth approximation to the rarefaction wave solutions, to close the energy estimates, it seems hopeless to use our method to deal with the nonlinear stability of the superposition of rarefaction waves of different families.

We note, however, that $g\left(\overline{V}(t,x)\right)_x$ satisfies the following estimate(*cf.* [20]): There exist constants $C > 0, \alpha > 0$ such that for $t \geq 0, x \in \mathbf{R}$

$$\left| g\left(\overline{V}(t,x)\right)_x \right| \leq C\varepsilon \exp(-\alpha\varepsilon(|x| + t)). \tag{1.37}$$

From (1.37), we can see that, like those for the study of nonlinear stability of travelling wave solutions to dissipative hyperbolic systems of conservation laws, if we give the smooth approximation $\overline{V}(t,x)$ a shift, that is, if we let $\overline{V}'(t,x) = \overline{V}(t+t_0,x)$ with $t_0 > 0$ being a suitably chosen fixed constant, then we have for $\overline{V}'(t,x)$ that

$$\int_0^t \left\| g\left(\overline{V}'(\tau)\right)_x \right\|_{L^p(\mathbf{R})} \leq O(1)\varepsilon^{-\frac{1}{p}} \exp(-\alpha\varepsilon t_0). \tag{1.38}$$

If we let for example $t_0 = \varepsilon^{-2}$, the right-hand of (1.38) is controlled by $O(1)\varepsilon^{-\frac{1}{p}} \exp\left(-\frac{\alpha}{\varepsilon}\right)$ which can be as small as we wanted if we choose $\varepsilon > 0$ sufficiently small. Consequently, our method can indeed be applied directly to deal with the nonlinear stability of the superposition of rarefaction waves of different families provided that we approximate the rarefaction wave solutions by $\overline{V}'(t,x)$ (Note that in this case, the initial data $(v_0,u_0)(x)$ of the compressible Navier-Stokes equations (1.24) is a perturbation of $\left(\overline{V},\overline{U}\right)(t_0,x)$.)

In Theorems 1.2 and 1.3, we assume that the H^1–norm of the initial perturbation is bounded by a constant independent of $\frac{1}{\varepsilon}$, which is excluded under additional assumption

$$\begin{cases} p(v) \geq C_1^{-1}v^{-1},\ C_1 p(v) \geq v|p'(v)| = -vp'(v) \geq C_1^{-1} & (0 < v \leq 1), \\ -p'(v) \geq C_1^{-1}v^{-C_1} & (v \geq 1) \end{cases} \tag{1.39}$$

for arbitrarily fixed constant $C_1 > 2$. Note that (1.39) derives

$$\begin{cases} C_1^{-1}v^{-1} \leq p(v) \leq p(1)v^{-C_1} & (0 < v \leq 1), \\ p(v) \geq p(\infty) + \frac{v^{1-C_1}}{C_1(C_1-1)} & (v \geq 1). \end{cases} \tag{1.40}$$

Hence, though (1.40) is not sufficient condition for (1.39), the assumption (1.39), roughly speaking, seems to be reasonable including the typical pressure model $p(v) = v^{-\gamma} (\gamma \geq 1)$. Then we have the final theorem.

Theorem 1.4 *Assume that $p(v)$ satisfies (1.26) and (1.39) and that the solution $\left(\overline{V}^R, \overline{U}^R\right)(t,x)$ to the Riemann problem (1.24), (1.25) is given by (1.31). Let $\left(\overline{V},\overline{U}\right)(t,x)$ be a smooth approximation of the Riemann solution $\left(\overline{V}^R,\ \overline{U}^R\right)(t,x)$ constructed by (1.33)-(1.34) with $\overline{w}_{i0}(x)$ in (1.32) being replaced by*

$$\frac{\overline{w}_{i-} + \overline{w}_{i+}}{2} + \frac{\overline{w}_{i+} - \overline{w}_{i-}}{2} K_q \int_0^{\varepsilon x} (1+y^2)^{-q} dy$$

for $q > \frac{3}{2}$ and K_q satisfying (1.21).

Then for any $\left(v_0(x) - \overline{\overline{V}}(0,x), u_0(x) - \overline{\overline{U}}(0,x)\right) \in H^2(\mathbf{R})$ satisfying $0 < 2\underline{V} \le v_0(x)$, $\overline{\overline{V}}(t,x) \le \frac{1}{2}\overline{V}$ for all $(t,x) \in \mathbf{R}_+ \times \mathbf{R}$ and some positive constants $\underline{V}$, $\overline{V}$, the Cauchy problem (1.22), (1.23) admits a unique global smooth solution $(v,u)(t,x)$ satisfying (1.29)

Now we outline the main ideas we used in proving our main results. The main new ingredient in our analysis is to introduce two quantities ε and t_0 in the construction of the smooth approximation of the rarefaction wave solutions to control the possible growth caused by the nonlinearity of the systems and by the interactions of waves from different families respectively. As to the global stability results, the key point is to get the uniform lower bound for $v(t,x)$ and our main observation for the isentropic case is that if $p(v)$ satisfies (1.26), then we can deduce that there exists a positive conctant $C_2 > 0$ such that

$$\Phi(V,z) \ge C_2 \frac{z^2}{z+2V}. \tag{1.41}$$

Such an estimates plays an important role in our proving Theorem 1.3 and Theorem 1.4. Here $\Phi(V,z) = p(V)z - \int_V^{V+z} p(s)ds$.

Remark 1.2 *It is worth to pointing out that the large time behavior of solutions to the compressible Navier-Stokes equations (1.1), (1.8) has been studied by many people, cf. [1-24] and the references cited therein. When the initial data $(v_0, u_0, s_0)(x)$ is a small perturbation of a non-vacuum constant state, i.e., $v_- = v_+ > 0, u_- = u_+, s_- = s_+$, quite perfect results have been obtained, cf. [10] and [17]. In the case when the far fields of the the initial data are different, i.e., $(v_-, u_-, s_-) \neq (v_+, u_+, s_+)$, many interesting results have been obtained: When the solutions to the corresponding Riemann problem consist in only shock waves, the nonlinear stability of travelling wave solutions has been established by [11], [14], and [19], etc. While, when the solutions to the corresponding Riemann problem consist in only rarefaction waves, the corresponding nonlinear stability results are obtained by [12], [15], [21], and [22].*

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