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Stability analysis of stationary solutions for surface diffusion flow equation

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1 Introduction

The geometrical evolution law

$$V = -\Delta\kappa$$

was derived by Mullins [7] to model the motion of interfaces in the case that the motion of interfaces is governed purely by mass diffusion within the interfaces (for simplicity we set the diffusion constant to 1). Here V is the normal velocity of the evolving interface, Δ is the Laplace-Beltrami operator and κ is the mean curvature of the interface where we use the sign convention that a sphere with the normal pointing to the inside has positive curvature.

In this paper we study the following problem. Given an open bounded domain $\Omega \subset \mathbb{R}^2$ we look for evolving curves $\Gamma = \{\Gamma_t\}_{t>0}$ (for a definition, see Gurtin [4]), which lies in Ω and satisfies $\partial\Gamma_t \subset \partial\Omega$, with the properties for $t > 0$:

$$\begin{cases} V = -\kappa_{ss} & \text{on} \quad \Gamma_t, \\ \angle(\partial\Omega, \Gamma_t) = \pi/2 & \text{at} \quad \partial\Omega \cap \Gamma_t, \\ \kappa_s = 0 & \text{at} \quad \partial\Omega \cap \Gamma_t, \end{cases} \tag{1.1}$$

where a subscript s denotes the differentiation with respect to the arc-length parameter. Then we observe that the problem (1.1) has the basic properties:

$$\frac{d}{dt}L_\Gamma(t) \leq 0, \qquad \frac{d}{dt}A_\Gamma(t) = 0.$$

Here we denote by $A_\Gamma(t)$ the area enclosed by the curve and $\partial\Omega$ at time t and by $L_\Gamma(t)$ the length of Γ at time t.

Our goal in this paper is to derive a linearized stability criterion based on the work of [2], [3], [6] which deal with the mean curvature flow. The analysis in the case of the

*Speaker in the talk on 17 Oct. 2003 [e-mail: kohsaka@mmm.muroran-it.ac.jp] surface diffusion flow is more difficult because the surface diffusion flow is the gradient flow with respect to the H^{-1}-inner product (see [8]) in contrast to the case of motion by the mean curvature flow which is a gradient flow with respect to the L^2-inner product. Here, for the convenience of readers, we show some typical differences between the mean curvature flow and the surface diffusion flow.

- The mean curvature flow: $V = \kappa$
  - The gradient flow of the length with respect to the L^2-inner product.
  - Not area-preserving.
  - Stationary solutions are the line segments.
  - A singular limit of Allen-Cahn equation.
- The surface diffusion flow: $V = -\kappa_{ss}$
  - The gradient flow of the length with respect to the H^{-1}-inner product.
  - Area-preserving.
  - Stationary solutions are the line segments and the circular arcs.
  - A singular limit of Cahn-Hilliard equation.

We remark that our results also have some relevance to isoperimetric problems which give stability or instability for critical points of the length functional of curves that enclose a fixed area. Since the surface diffusion flow reduces the length conserving the area at the same time, the stability analysis for the evolution problem can be reduced to the study of critical points of the length functional under an area constraint .

This paper is a survey of the article [5]. If readers are interested in the details of this paper, refer to [5].

2 Parameterization and linearization

For a smooth function $\psi : \mathbb{R}^2 \to \mathbb{R}$ with $\nabla\psi(x) \neq 0$ if $\psi(x) = 0$, set

$$\Omega = \{x \in \mathbb{R}^2 \mid \psi(x) < 0\}, \quad \partial\Omega = \{x \in \mathbb{R}^2 \mid \psi(x) = 0\}.$$

Let Γ_* be a stationary solution, which is a part of circle or a line segment, and let σ be the arc-length parameter of Γ_*. Then we denote an arc-length parameterization of Γ_* as

$$\Gamma_* = \{\Phi_*(\sigma) \mid \sigma \in [-l, l]\}.$$

Note that we can extend Γ_* naturally either to the full circle when Γ_* is a part of circle or to the straight line when Γ_* is a line segment. Also note that the curvature κ_* of Γ_* is a constant. We denote

$$\bar{l} := \begin{cases} \pi/|\kappa_*|, & \kappa_* \neq 0, \\ +\infty, & \kappa_* = 0. \end{cases}$$

That is, $\bar{l}$ is the length of the extension of Γ_* to a full circle (if $\kappa_* \neq 0$). Define

$$\begin{cases} \xi_+(q) = \max\{\sigma \in (-\bar{l}, \bar{l}) \mid \Phi_*(\sigma) + qN_*(\sigma) \in \Omega\}, \\ \xi_-(q) = \min\{\sigma \in (-\bar{l}, \bar{l}) \mid \Phi_*(\sigma) + qN_*(\sigma) \in \Omega\}. \end{cases}$$

where $q \in [-d, d]$ for a small $d > 0$, and $N_*(\sigma)$ is a unit normal vector of Γ_* at σ and is obtained by rotating the unit tangent vector $T_*(\sigma)$ of Γ_* with $\pi/2$. Then it holds $\psi(\Phi_*(\xi_\pm(q)) + qN_*(\xi_\pm(q))) = 0$. In addition, we have $\xi_\pm(0) = \pm l$. Using the implicit function theorem, we see that $\xi_+(q)$ and $\xi_-(q)$ are smooth. Let

$$\Psi(\sigma, q) := \Phi_*(\xi(\sigma, q)) + qN_*(\xi(\sigma, q))$$

with

$$\xi(\sigma, q) := \xi_-(q) + \frac{\sigma + l}{2l}(\xi_+(q) - \xi_-(q)).$$

Note that $\xi(\pm l, q) = \xi_\pm(q)$ and $\xi(\sigma, 0) = \sigma$.

Let Γ be curves in the neighbourhood of Γ_*, which touch the boundary $\partial\Omega$ and are contained in Ω. For some functions $\rho : [-l, l] \to [-d, d]$, we define $\Phi(\sigma) := \Psi(\sigma, \rho(\sigma))$ for $\sigma \in [-l, l]$, which denotes a parameterization of such curves Γ. Thus we set

$$\Gamma_t := \{\Phi(\sigma, t) \mid \sigma \in [-l, l]\} \tag{2.1}$$

with $\Phi(\sigma, t) := \Psi(\sigma, \rho(\sigma, t))$ for a function ρ depending on σ and t. We remark that $\rho \equiv 0$ means that curves Γ coincide with a stationary curve Γ_*.

Let us derive the representation of (1.1) to the parameterization (2.1). For the arc-length parameter s of Γ, we have

$$\frac{ds}{d\sigma} = |\Phi_\sigma| = \sqrt{|\Psi_\sigma|^2 + 2(\Psi_\sigma, \Psi_q)_{\mathbb{R}^2}\rho_\sigma + |\Psi_q|^2\rho_\sigma^2} \ (=: J(\rho)). \tag{2.2}$$

Here and hereafter $(\cdot, \cdot)_{\mathbb{R}^2}$ denotes the inner product in $\mathbb{R}^2$. Then we find

$$T = \frac{1}{J(\rho)}\Phi_\sigma, \quad N = \frac{1}{J(\rho)}R\Phi_\sigma,$$

where T and N are the unit tangent and normal vector of Γ respectively, and R is the rotation matrix with $\pi/2$. The normal velocity V of Γ_t is denoted by

$$V = (\Phi_t, N)_{\mathbb{R}^2} = \frac{1}{J(\rho)}(\Phi_t, R\Phi_\sigma)_{\mathbb{R}^2} = \frac{1}{J(\rho)}(\Psi_q, R\Psi_\sigma)_{\mathbb{R}^2}\rho_t.$$

Moreover, since (2.2) gives

$$\partial_s^2 = \frac{1}{J(\rho)}\partial_\sigma\left(\frac{1}{J(\rho)}\partial_\sigma\right) = \frac{1}{(J(\rho))^2}\partial_\sigma^2 + \frac{1}{J(\rho)}\left(\partial_\sigma\frac{1}{J(\rho)}\right)\partial_\sigma \ (=: \Delta(\rho)), \tag{2.3}$$

the curvature κ of Γ_t is written by

$$\begin{aligned}
\kappa(\rho) &= (\Delta(\rho)\Phi, N)_{\mathbb{R}^2} \\
&= \frac{1}{(J(\rho))^3}(\Phi_{\sigma\sigma}, R\Phi_\sigma)_{\mathbb{R}^2} \\
&= \frac{1}{(J(\rho))^3}\Big[(\Psi_q, R\Psi_\sigma)_{\mathbb{R}^2}\rho_{\sigma\sigma} + \big\{2(\Psi_{\sigma q}, R\Psi_\sigma)_{\mathbb{R}^2} + (\Psi_{\sigma\sigma}, R\Psi_q)_{\mathbb{R}^2}\big\}\rho_\sigma \\
&\quad + \big\{(\Psi_{qq}, R\Psi_\sigma)_{\mathbb{R}^2} + 2(\Psi_{\sigma q}, R\Psi_q)_{\mathbb{R}^2} + (\Psi_{qq}, R\Psi_q)_{\mathbb{R}^2}\rho_\sigma\big\}\rho_\sigma^2 \\
&\quad + (\Psi_{\sigma\sigma}, R\Psi_\sigma)_{\mathbb{R}^2}\Big]. \qquad (2.4)
\end{aligned}$$

Thus the surface diffusion flow equation is described by

$$\rho_t = -\Lambda(\rho)\Delta(\rho)\kappa(\rho), \qquad (2.5)$$

where

$$\Lambda(\rho) := \frac{1}{(\Psi_q, R\Psi_\sigma)_{\mathbb{R}^2}} J(\rho). \qquad (2.6)$$

Let us derive the representation of the boundary conditions which are the Neumann boundary condition and the no-flux condition $\kappa_s = 0$ on $\partial\Omega$. Since the Neumann boundary condition $(\Phi_\sigma, T_{\partial\Omega})_{\mathbb{R}^2} = 0$ is equivalent to $(R\Phi_\sigma, \nabla\psi(\Phi))_{\mathbb{R}^2} = 0$, we have

$$(R\Psi_\sigma + R\Psi_q\rho_\sigma, \nabla\psi(\Psi))_{\mathbb{R}^2} = 0.$$

By (2.2) and (2.4) the no-flux condition $\kappa_s = 0$ is denoted by

$$\partial_\sigma\kappa(\rho) = 0.$$

Consequently we have the following proposition.

Proposition 2.1 *For a parameterization (2.1), the problem (1.1) is denoted by*

$$\begin{cases} \rho_t = -\Lambda(\rho)\Delta(\rho)\kappa(\rho) & \text{for } \sigma \in (-l, l),\ t > 0, \\ (R\Psi_\sigma + R\Psi_q\rho_\sigma, \nabla\psi(\Psi))_{\mathbb{R}^2} = 0 & \text{at } \sigma = \pm l, \\ \partial_\sigma\kappa(\rho) = 0 & \text{at } \sigma = \pm l, \end{cases} \qquad (2.7)$$

where $\Lambda(\rho)$, $\Delta(\rho)$ and $\kappa(\rho)$ are defined by (2.6), (2.3) and (2.4) respectively.

To study the linearized stability of a stationary solution Γ_*, the curvature κ_* of which is a constant, we linearize (2.7) around $\rho \equiv 0$. For this purpose we need the properties of Ψ at $q = 0$ as follows:

$$\begin{cases} \Psi(\sigma, 0) = \Phi_*(\sigma), \quad \Psi_\sigma(\sigma, 0) = T_*(\sigma), \quad \Psi_q(\sigma, 0) = N_*(\sigma), \\ \Psi_{\sigma\sigma}(\sigma, 0) = \kappa_* N_*(\sigma), \quad \Psi_{\sigma q}(\sigma, 0) = -\kappa_* T_*(\sigma), \quad \Psi_{\sigma\sigma q}(\sigma, 0) = -\kappa_*^2 N_*(\sigma). \end{cases} \qquad (2.8)$$

Let us consider the linearization of (2.7). Set

$$\begin{cases} A(\rho) := -\Lambda(\rho)\Delta(\rho)\kappa(\rho), \\ B_1(\rho) := (R\Psi_\sigma, \nabla\psi(\Psi))_{\mathbb{R}^2} + (R\Psi_q, \nabla\psi(\Psi))_{\mathbb{R}^2}\rho_\sigma, \\ B_2(\rho) := \partial_\sigma\kappa(\rho), \end{cases}$$

and denote $x_*^\pm := \Phi_*(\pm l)$. Then we define

$$\mathcal{A} := \partial A(0),$$
$$\mathcal{B} := \begin{pmatrix} \partial B_1(0)/(\mp|\nabla\psi(x_*^\pm)|) \\ \partial B_2(0) \end{pmatrix} \quad \text{at } \sigma = \pm l$$

where $\partial A(0)$, $\partial B_1(0)$ and $\partial B_2(0)$ are the Fréchet derivatives of A, B_1 and B_2 at 0, respectively. By using (2.8), we have the following representations of $\mathcal{A}$ and $\mathcal{B}$.

Lemma 2.2 *(i) It holds*

$$\mathcal{A} = -\partial_\sigma^2(\partial_\sigma^2 + \kappa_*^2).$$

(ii) Let $h_\pm$ be the curvatures of $\partial\Omega$ at $x_*^\pm \in \Gamma_* \cap \partial\Omega$, respectively (where we use the sign convention that $h_\pm < 0$ if Ω is convex). Then

$$\mathcal{B} = \begin{pmatrix} \partial_\sigma \pm h_\pm \\ \partial_\sigma(\partial_\sigma^2 + \kappa_*^2) \end{pmatrix} \quad \textit{at } \sigma = \pm l.$$

By the Lemmas 2.2, we see the linearization of (2.7) around $\rho \equiv 0$.

Theorem 2.3 *The linearization of (2.7) around $\rho \equiv 0$ is as follows:*

$$\begin{cases} \rho_t = -\partial_\sigma^2(\partial_\sigma^2 + \kappa_*^2)\rho & \textit{for} \quad \sigma \in (-l, l),\ t > 0, \\ (\partial_\sigma \pm h_\pm)\rho = 0 & \textit{at} \quad \sigma = \pm l, \\ \partial_\sigma(\partial_\sigma^2 + \kappa_*^2)\rho = 0 & \textit{at} \quad \sigma = \pm l. \end{cases} \tag{2.9}$$

Remark 2.4 *The linearization of the area-preserving property is*

$$\int_{-l}^{l} \rho \, d\sigma = 0 \tag{2.10}$$

(see Section A). Since the original problem (1.1) has the area-preserving property, we need to analyze the linearized problem (2.9) for functions ρ satisfying (2.10).

3 Gradient flow structure

The surface diffusion flow can be interpreted as the H^{-1}-gradient flow of the length functional in $\mathbb{R}^2$ (see [8]). In this section we demonstrate that the linearized problem (2.9) can also be interpreted as a gradient flow. This observation will be important for our stability analysis.

In what follows we need the duality pairing $\langle \cdot, \cdot \rangle$ between $(H^1(-l,l))'$ and $(H^1(-l,l))$; and the following weak formulation.

Definition 3.1 *We say that $u_v \in H^1(-l,l)$ for a given $v \in (H^1(-l,l))'$ with $\langle v, 1\rangle = 0$ is a weak solution of*

$$\begin{cases} -\partial_\sigma^2 u_v = v & \text{for} \quad \sigma \in (-l,l)\,, \\ \partial_\sigma u_v = 0 & \text{at} \quad \sigma = \pm l \end{cases} \tag{3.1}$$

if u_v satisfies

$$\langle v, \xi \rangle = \int_{-l}^{l} \partial_\sigma u_v \partial_\sigma \xi$$

for all $\xi \in H^1(-l,l)$.

Definition 3.2 *For a given $v \in (H^1(-l,l))'$ with $\langle v, 1\rangle = 0$, we say that $\rho \in H^3(-l,l)$ with $\int_{-l}^{l} \rho = 0$ is a weak solution of the boundary value problem*

$$\begin{cases} v = -\partial_\sigma^2(\partial_\sigma^2 + \kappa_*^2)\rho & \text{for} \quad \sigma \in (-l,l)\,, \\ (\partial_\sigma \pm h_\pm)\rho = 0 & \text{at} \quad \sigma = \pm l\,, \\ \partial_\sigma(\partial_\sigma^2 + \kappa_*^2)\rho = 0 & \text{at} \quad \sigma = \pm l \end{cases} \tag{3.2}$$

if ρ satisfies

$$\langle v, \xi \rangle = \int_{-l}^{l} \partial_\sigma(\partial_\sigma^2 + \kappa_*^2)\rho\, \partial_\sigma \xi, \quad \text{and} \quad (\partial_\sigma \pm h_\pm)\rho = 0 \quad \text{at} \quad \sigma = \pm l$$

for all $\xi \in H^1(-l,l)$.

In addition we also need the symmetric bilinear form on $H^1(-l,l)$

$$I(\rho_1, \rho_2) := \int_{-l}^{l} \{\partial_\sigma \rho_1 \partial_\sigma \rho_2 - \kappa_*^2 \rho_1 \rho_2\} d\sigma + h_+ \rho_1(l)\rho_2(l) + h_- \rho_1(-l)\rho_2(-l) \tag{3.3}$$

and the inner product

$$(\rho_1, \rho_2)_{-1} := \int_{-l}^{l} \partial_\sigma u_{\rho_1} \partial_\sigma u_{\rho_2}$$

where $u_{\rho_i} \in H^1(-l,l)$ for a given $\rho_i \in (H^1(-l,l))'$ with $\langle \rho_i, 1\rangle = 0$ is defined as the weak solution of (3.1). The bilinear form I is defined on $H^1(-l,l)$ and the inner product

$(\cdot\,,\cdot)_{-1}$ is defined for all pairs of elements in $(H^1(-l,l))'$ with $\langle \rho_i, 1\rangle = 0$. We remark that by Definition 3.1

$$(\rho_1, \rho_2)_{-1} = \langle \rho_1, u_{\rho_2}\rangle \tag{3.4}$$

holds for $\rho_i \in (H^1(-l,l))'$ with $\langle \rho_i, 1\rangle = 0$.

Remark 3.3 *If $\rho \equiv 0$ is the extremal value of the length functional under the area constraint, the bilinear form I is derived from the second variation of such a functional (see Section B). This means that our linearized stability analysis has a close relation to isoperimetric problems which give a criterion for the stability of critical points of the length functional of curves that come into contact with the outer boundary and enclose a fixed area.*

Now we are going to show that the linearized problem (2.9) is the gradient flow of $E(\rho) := I(\rho,\rho)/2$ with respect to the inner product $(\cdot,\cdot)_{-1}$. Let us review the concept of gradient flows. For a given functional E on a linear space X and an inner product $(\cdot,\cdot)_X$ on X we say that a time dependent function ρ with values in X is a solution of the gradient flow equation to E and $(\cdot,\cdot)_X$ if and only if

$$(\rho_t(t), \xi)_X = -\partial E(\rho(t))(\xi)$$

holds for all $\xi \in X$ and all t. Here $\partial E(\rho(t))(\xi)$ denotes the derivative of E at the point $\rho(t)$ in the direction ξ. The fact that the linearized problem (2.9) is the gradient flow of $I(\rho,\rho)/2$ with respect to the inner product $(\cdot,\cdot)_{-1}$ follows from the following lemma. This is true since the derivative of $E(\rho) = I(\rho,\rho)/2$ in a direction ξ is given by $I(\rho,\xi)$.

Lemma 3.4 *Let $v \in (H^1(-l,l))'$ with $\langle v, 1\rangle = 0$ be given. Then a function $\rho \in H^3(-l,l)$ with $\int_{-l}^{l} \rho = 0$ is a weak solution of (3.2) if and only if*

$$(v,\xi)_{-1} = -I(\rho,\xi)$$

holds for all $\xi \in H^1(-l,l)$ with $\int_{-l}^{l} \xi = 0$.

4 Eigenvalue problem

In this section, we study the eigenvalue problem corresponding to the linearized problem (2.9). By choosing an appropriate domain of definition, the linearized operator of (2.9) is given by

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \to H, \qquad \langle \mathcal{A}\rho, \xi\rangle := \int_{-l}^{l} \partial_\sigma(\partial_\sigma^2 + \kappa_*^2)\rho\, \partial_\sigma \xi$$

with

$$\begin{cases} \mathcal{D}(\mathcal{A}) = \{\rho \in H^3(-l,l) \mid (\partial_\sigma \pm h_\pm)\rho = 0 \text{ at } \sigma = \pm l \text{ and } \int_{-l}^{l} \rho = 0\}, \\ H = \{\rho \in (H^1(-l,l))' \mid \langle \rho, 1\rangle = 0\}. \end{cases}$$

Then it follows from this definition and Lemma 3.4 that

$$(\mathcal{A}\rho, \xi)_{-1} = -I(\rho, \xi)$$

for all $\xi \in H^1(-l, l)$ with $\int_{-l}^{l} \xi = 0$.

Let us analyze the spectrum of $\mathcal{A}$ in order to decide on the stability behaviour of the linearized problem (2.9). Using classical principles of the variational calculus, we can describe the spectrum of $\mathcal{A}$ with the help of the inner product $(\cdot\,,\cdot)_{-1}$ and I. In fact, if ρ is an eigenfunction to the eigenvalue λ, it holds

$$\lambda(\rho, \xi)_{-1} = (\mathcal{A}\rho, \xi)_{-1} = -I(\rho, \xi)\,.$$

We remark that eigenvalues $\lambda \neq 0$ always correspond to eigenfunctions which have the mean value zero. In what follows we will only study eigenvalues which have eigenfunctions with mean value zero. This is a natural request for the linearized problem (see Remark 2.4). First we have the following lemma for the operator $\mathcal{A}$.

Lemma 4.1 *(i) The operator $\mathcal{A}$ is self-adjoint with respect to the inner product $(\cdot, \cdot)_{-1}$.*
(ii) The spectrum of $\mathcal{A}$ contains a countable system of real eigenvalues.

In addition, we have the following lemmas for the eigenvalues of $\mathcal{A}$.

Lemma 4.2 *Let*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$$

be the eigenvalues of $\mathcal{A}$ (taking the multiplicity into account).
(i) Then it holds for all $n \in \mathbf{N}$

$$-\lambda_n = \inf_{W \in \Sigma_n} \sup_{\rho \in W \setminus \{0\}} \frac{I(\rho, \rho)}{(\rho, \rho)_{-1}},$$

$$-\lambda_n = \sup_{W \in \Sigma_{n-1}} \inf_{\rho \in W^{\perp} \setminus \{0\}} \frac{I(\rho, \rho)}{(\rho, \rho)_{-1}}.$$

Here Σ_n is the collection of n-dimensional subspaces of V and $W^{\perp}$ is the orthogonal complement with respect to the inner product $(\cdot\,,\cdot)_{-1}$.
(ii) The eigenvalues λ_n depend continuously on h_+, h_- and κ_*^2; and are monotone decreasing in each of the parameters h_+, h_- and $(-\kappa_*^2)$.

Lemma 4.3 *(i) Assume $\kappa_* \neq 0$ and $\kappa_* l < \pi$. Then the operator $\mathcal{A}$ has a zero eigenvalue if and only if*

$$\frac{a}{c} + \frac{b}{c}(h_+ + h_-) + h_+ h_- = 0 \tag{4.1}$$

where

$$\begin{aligned} a &= -2\kappa_*^2 l \sin(\kappa_* l)\cos(\kappa_* l)\,, \\ b &= \kappa_* l(\cos^2(\kappa_* l) - \sin^2(\kappa_* l)) - \sin(\kappa_* l)\cos(k_* l)\,, \\ c &= 2\Big\{-\frac{1}{\kappa_*}\sin^2(\kappa_* l) + l\sin(\kappa_* l)\cos(\kappa_* l)\Big\}\,. \end{aligned}$$

Furthermore, it holds the inequality

$$\frac{b^2}{c^2} - \frac{a}{c} > 0\,. \tag{4.2}$$

(ii) Assume that $\kappa_* = 0$. Then the operator $\mathcal{A}$ has a zero eigenvalue if and only if

$$\frac{3}{l^2} + \frac{2}{l}(h_+ + h_-) + h_+ h_- = 0\,. \tag{4.3}$$

(iii) If we interpret a, b, and c as functions of κ_*, we obtain

$$\frac{a}{c} \longrightarrow \frac{3}{l^2} \quad and \quad \frac{b}{c} \longrightarrow \frac{2}{l} \quad as \quad \kappa_* \longrightarrow 0\,.$$

(iv) The multiplicity of a zero eigenvalue is equal to one for all h_+, h_-, and κ_*.

Set

$$\mathcal{D}(h_+, h_-, \kappa_*) = \frac{a}{c} + \frac{b}{c}(h_+ + h_-) + h_+ h_-$$

for all h_+, h_-, and κ_*. Note that the extension to $\kappa_* = 0$ is well defined by the above lemma.

Remark 4.4 *The equations (4.1) and (4.3) define hyperbolas in the (h_-, h_+)-plane (see Figures 1-5). The hyperbolas are symmetric with respect to the $h_- = h_+$ line and the inequality (4.2) implies that the line defined by $h_+ = h_-$ always has two intersection points with the hyperbolas.*

5 Main result

To obtain a linearized stability result for stationary solutions of (2.7), it is enough to show that $I(\rho, \rho)$ is positive for all $\rho \in V \setminus \{0\}$. Then $\lambda_1 < 0$ which implies stability. This is true since λ_1 allows the characterization

$$-\lambda_1 = \inf_{\rho \in V\setminus\{0\}} \frac{I(\rho,\rho)}{(\rho,\rho)_{-1}}$$

and the infimum is in fact a minimum. Therefore it is enough to show the positivity of I pointwise. The following lemma shows that for given κ_* the stationary solution is always stable provided h_+, h_- are large enough.

Lemma 5.1 *Let $\kappa_* l < \pi$. Then there exists a constant $K > 0$ such that*

$$I(\rho, \rho) > 0 \quad \textit{for all} \quad \rho \in V \setminus \{0\}$$

provided that $h_+, h_- > K$.

Let N_U be the number of the unstable eigenvalues and also let N_N be the number of the zero eigenvalues (counting the multiplicity). Then, by virtue of Lemmas 4.1, 4.2, 4.3 and 5.1, we are led to the following theorem.

Theorem 5.2 *Case A: If $\mathcal{D}(h_-, h_+, \kappa_*) > 0$ and if $h_- > -b/c$, then*

$$N_U = N_N = 0\,.$$

Case B: If $\mathcal{D}(h_-, h_+, \kappa_*) = 0$ and if $h_- > -b/c$, then

$$N_U = 0\,,\ N_N = 1\,.$$

Case C: If $\mathcal{D}(h_-, h_+, \kappa_*) < 0$, then

$$N_U = 1\,,\ N_N = 0\,.$$

Case D: If $\mathcal{D}(h_-, h_+, \kappa_*) = 0$ and if $h_- < -b/c$, then

$$N_U = 1\,,\ N_N = 1\,.$$

Case E: If $\mathcal{D}(h_-, h_+, \kappa_*) > 0$ and if $h_- < -b/c$, then

$$N_U = 2\,,\ N_N = 0\,.$$

Remark 5.3 *(a) In the cases A,B,D and E the condition, $h_- > -b/c$ ($h_- < -b/c$ respectively) can be replaced by $h_+ > -b/c$ ($h_+ < -b/c$ respectively).*
(b) Theorem 5.2 says that above the upper arc of the hyperbola (see Figures 1-5) we have only negative eigenvalues, which imply the stability of stationary solutions. Underneath of it and above the lower arc of the hyperbola, we have one positive eigenvalue, which means that the number of unstable modes is one. Furthermore, underneath of it, we have two positive eigenvalues, which mean that the number of unstable modes is two.

Proof of Theorem 5.2. The proof is a simple consequence of the Lemmas 4.2, 4.3 and 5.1. For large h_+ and h_- we have stability. If we decrease h_+ or h_-, the stability behaviour only changes on the curves defined by $\mathcal{D}(h_-, h_+, \kappa_*) = 0$. By virtue of Lemma 4.3(iv), only one eigenvalue can pass through zero when crossing the curves $\mathcal{D}(h_-, h_+, \kappa_*) = 0$. The monotonicity of the eigenvalues with respect to h_+ and h_- implies that the number of unstable modes can only increase if we further decrease h_+ or h_-. This proves the theorem. □

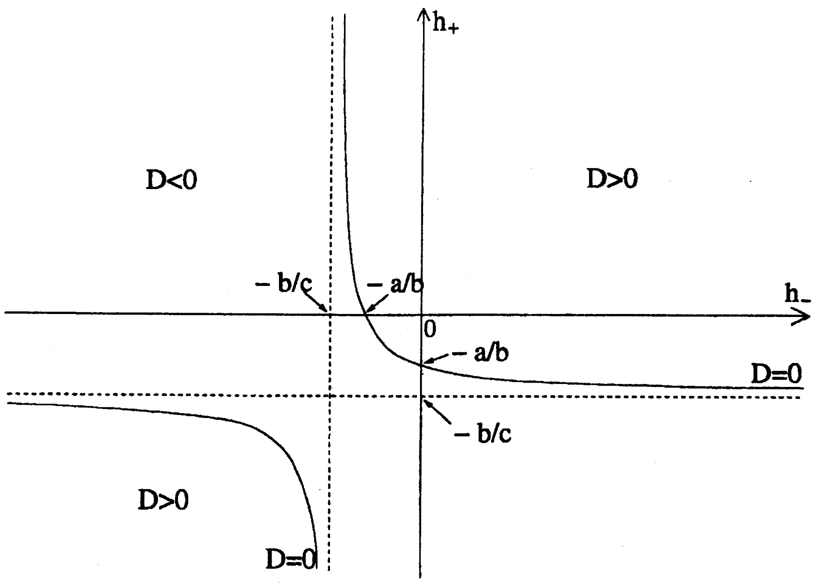

Figure 1: $\kappa_* l < \pi/2, a < 0, b < 0, c < 0$

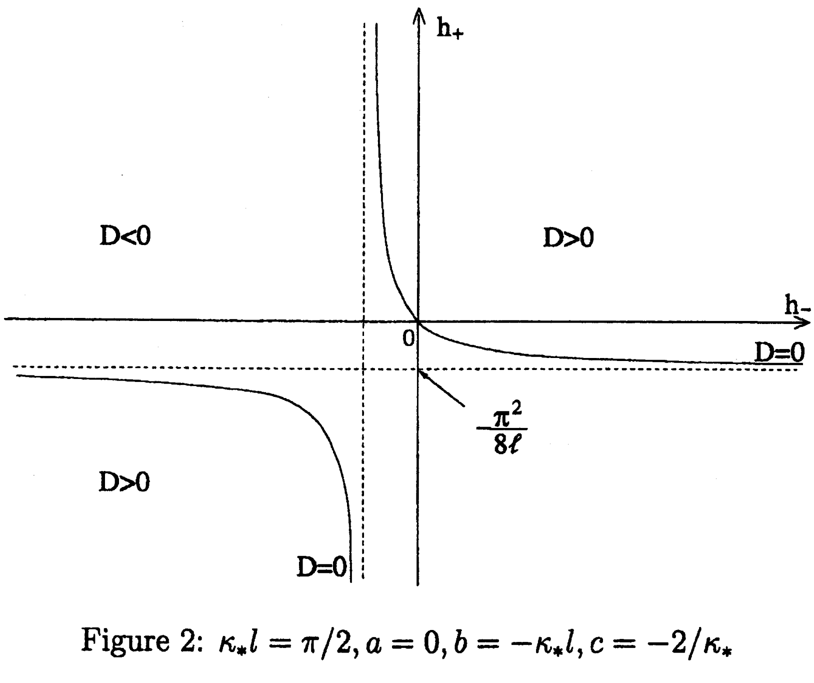

Figure 2: $\kappa_* l = \pi/2, a = 0, b = -\kappa_* l, c = -2/\kappa_*$

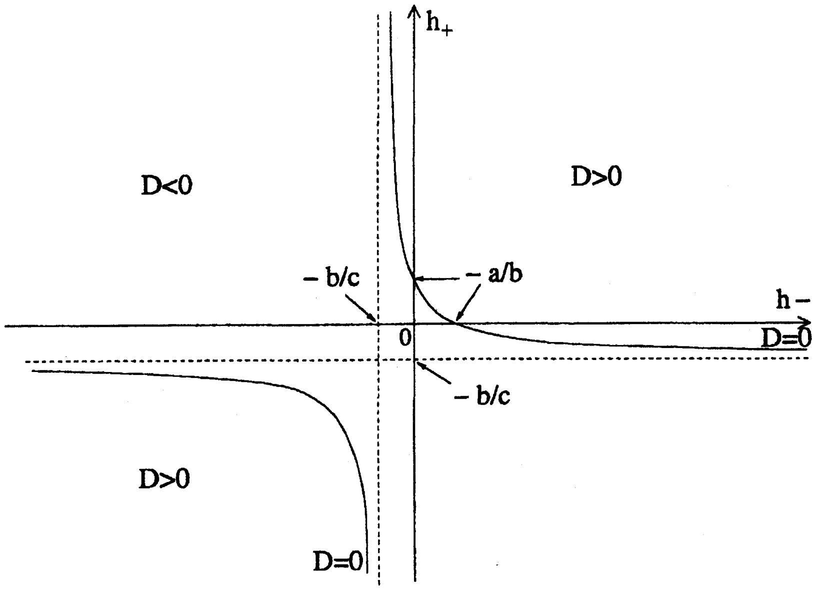

Figure 3: $\kappa_* l > \pi/2, a > 0, b < 0, c < 0$

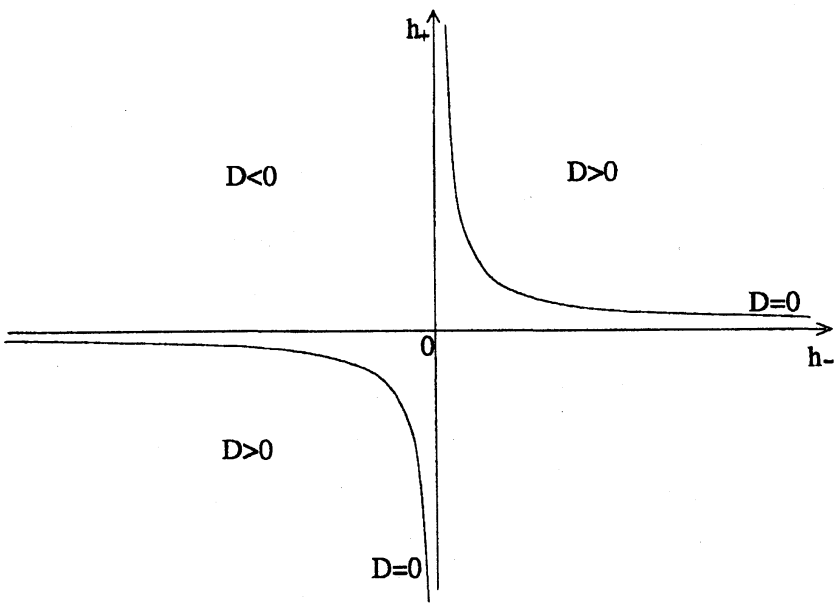

Figure 4: $\kappa_* l > \pi/2, a > 0, b = 0, c < 0$

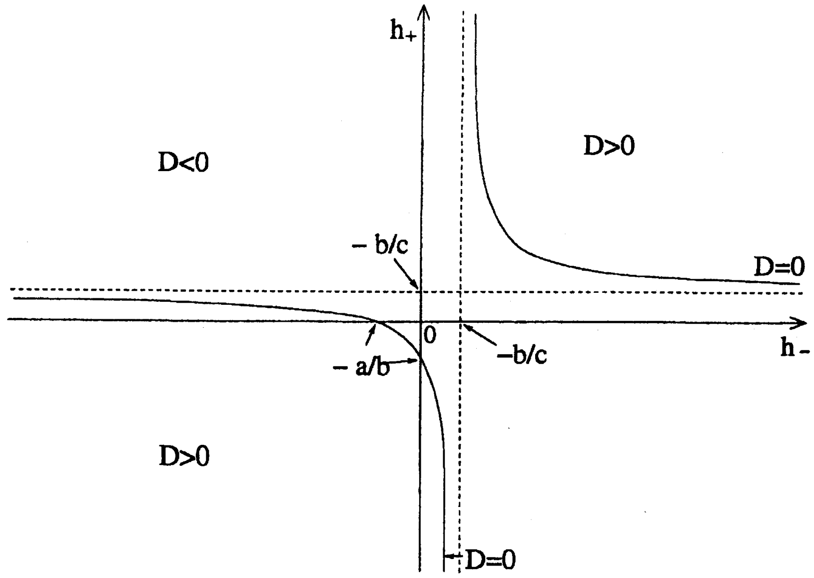

Figure 5: $\kappa_* l > \pi/2, a > 0, b > 0, c < 0$

A Linearization of the area functional

In this section we show that the linearization of the area-preserving property implies the mean value zero, i.e. (2.10).

Let A_Γ be the area of a domain enclosed by Γ and $\partial\Omega$. Then A_Γ is represented as

$$A_\Gamma(\rho) = \int_{-l}^{l} (\Psi(\cdot, \rho), N(\rho))_{\mathbb{R}^2} J(\rho)\, d\sigma + \int_{\partial\Omega: S^+(\rho) \to S^-(\rho)} (Q(s), N_{\partial\Omega}(s))_{\mathbb{R}^2}\, ds,$$

where $Q(s)$ is the parameterization of $\partial\Omega$ with respect to the arc-length parameter s and also satisfies

$$Q(S^\pm(\rho)) = \Psi(\cdot, \rho)\big|_{\sigma = \pm l}. \tag{A.1}$$

In addition, let A_{Γ_*} be the area of a domain enclosed by Γ_* and $\partial\Omega$. Then A_{Γ_*} is represented as

$$A_{\Gamma_*} = \int_{-l}^{l} (\Phi_*, N_*)_{\mathbb{R}^2}\, d\sigma + \int_{\partial\Omega: s_*^+ \to s_*^-} (Q(s), N_{\partial\Omega}(s))_{\mathbb{R}^2}\, ds,$$

where it holds at $s = s_*^\pm$

$$Q(s_*^\pm) = \Phi_*(\pm l).$$

Thus the area-preserving property is denoted by

$$\Xi(\rho) := A_\Gamma(\rho) - A_{\Gamma_*} = 0. \tag{A.2}$$

Set

$$F(\rho) := \int_{-l}^{l} (\Psi(\cdot,\rho), N(\rho))_{\mathbb{R}^2} J(\rho)\, d\sigma$$

$$G(\rho) := \int_{\partial\Omega: S^+(\rho)\to S^-(\rho)} (Q(s), N_{\partial\Omega}(s))_{\mathbb{R}^2}\, ds$$

Then we have the following lemmas.

Lemma A.1 *It holds for a smooth function* ρ

$$\partial F(0)\rho = 2\int_{-l}^{l} \rho\, d\sigma - \left[(\Phi_*, T_*)_{\mathbb{R}^2}\rho\right]_{\sigma=-l}^{\sigma=l},$$

where $\partial F(0)$ *is the Fréchet derivative of* F.

Proof. Note that

$$J(0) = 1, \qquad \Psi_q(\cdot,0) = N_*,$$
$$\left.\frac{d}{d\varepsilon}J(\varepsilon\rho)\right|_{\varepsilon=0} = -\kappa_*\rho, \qquad \left.\frac{d}{d\varepsilon}N(\varepsilon\rho)\right|_{\varepsilon=0} = -\rho_\sigma T_*.$$

Then it follows that

$$\left.\frac{d}{d\varepsilon}F(\varepsilon\rho)\right|_{\varepsilon=0} = \int_{-l}^{l} \rho\, d\sigma - \int_{-l}^{l} (\Phi_*, T_*)_{\mathbb{R}^2}\rho_\sigma\, d\sigma - \kappa_*\int_{-l}^{l} (\Phi_*, N_*)_{\mathbb{R}^2}\rho\, d\sigma.$$

Integrating by parts in the second term with $\Phi_{*,\sigma} = T_*$ and $T_{*,\sigma} = \kappa_* N_*$, we are led to the assertion. □

Lemma A.2 *It holds for a smooth function* ρ

$$\partial G(0)\rho = \left[(\Phi_*, T_*)_{\mathbb{R}^2}\rho\right]_{\sigma=-l}^{\sigma=l},$$

where $\partial G(0)$ *is the Fréchet derivative of* G.

Proof. Note that the identity (A.1) implies

$$Q(S^\pm(0)) = \Psi(\cdot,0)\big|_{\sigma=\pm l} = \Phi_*(\pm l). \tag{A.3}$$

Since $\dot{Q}(S^\pm(0)) = T_{\partial\Omega}(s_*^\pm) = \mp N_*(\pm l)$, we also have

$$(S^\pm)'(0)\rho = \mp\rho(\pm l). \tag{A.4}$$

Then it follows that

$$\left.\frac{d}{d\varepsilon}G(\varepsilon\rho)\right|_{\varepsilon=0} = (Q(S^-(0)), N_{\partial\Omega}(S^-(0)))_{\mathbb{R}^2}(S^-)'(0)\rho$$
$$-(Q(S^+(0)), N_{\partial\Omega}(S^+(0)))_{\mathbb{R}^2}(S^+)'(0)\rho.$$

By means of (A.3), (A.4) and $N_{\partial\Omega}(S^{\pm}(0)) = N_{\partial\Omega}(s_*^{\pm}) = \pm T_*(\pm l)$, we derive

$$\begin{aligned} \frac{d}{d\varepsilon}G(\varepsilon\rho)\bigg|_{\varepsilon=0} &= -(\Phi_*(-l), T_*(-l))_{\mathbb{R}^2}\rho(-l) + (\Phi_*(l), T_*(l))_{\mathbb{R}^2}\rho(l) \\ &= \big[(\Phi_*, T_*)_{\mathbb{R}^2}\rho\big]_{\sigma=-l}^{\sigma=l}. \end{aligned}$$

This completes the proof. □

These lemmas imply the following proposition.

Proposition A.3 *(The linearization of Ξ) It holds for a smooth function ρ*

$$\partial\Xi(0)\rho = 2\int_{-l}^{l} \rho \, d\sigma,$$

where $\partial\Xi(0)$ is the Fréchet derivative of Ξ.

Proof. Since $\Xi(\rho) = A_\Gamma(\rho) - A_{\Gamma_*}$, we have

$$\frac{d}{d\varepsilon}\Xi(\varepsilon\rho)\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}A_\Gamma(\varepsilon\rho)\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}F(\varepsilon\rho)\bigg|_{\varepsilon=0} + \frac{d}{d\varepsilon}G(\varepsilon\rho)\bigg|_{\varepsilon=0}.$$

The assertion follows from Lemma A.1 and Lemma A.2. □

Thus it follows from (A.2) and Proposition A.3 that the area-preserving property gives

$$\int_{-l}^{l} \rho \, d\sigma = 0.$$

B Second variation of length under area constraint

In this section we show that the second variation of the length functional under the area constraint gives the bilinear form I defined by (3.3).

Let $L_\Gamma(\rho)$ be the length of Γ. Then the length functional $L_\Gamma(\rho)$ is represented as

$$L_\Gamma(\rho) = \int_{-l}^{l} J(\rho) \, d\sigma$$

where $J(\rho)$ is defined by (2.2). Using (2.8), we derive

$$\frac{d}{d\varepsilon}J(\varepsilon\rho)\bigg|_{\varepsilon=0} = -\kappa_*\rho,$$

so that the first variation of L_Γ is

$$\frac{d}{d\varepsilon}L_\Gamma(\varepsilon\rho)\bigg|_{\varepsilon=0} = -\kappa_*\int_{-l}^{l} \rho \, d\sigma.$$

According to Section A, the area constraint is denoted by $\Xi(\rho) := A_\Gamma(\rho) - A_{\Gamma_*} = 0$. Note that the first variation of the functional $\Xi(\rho)$ is

$$\frac{d}{d\varepsilon}\Xi(\varepsilon\rho)\Big|_{\varepsilon=0} = 2\int_{-l}^{l} \rho \, d\sigma.$$

If $\rho \equiv 0$ is the extremal value of the length functional $L_\Gamma(\rho)$ under the area constraint $\Xi(\rho) = 0$, we have

$$\frac{d}{d\varepsilon}L_\Gamma(\varepsilon\rho)\Big|_{\varepsilon=0} + \gamma\frac{d}{d\varepsilon}\Xi(\varepsilon\rho)\Big|_{\varepsilon=0} = -\kappa_*\int_{-l}^{l} \rho \, d\sigma + 2\gamma\int_{-l}^{l} \rho \, d\sigma = 0$$

where $\gamma \in \mathbb{R}$ is Lagrange multiplier. Since ρ is arbitrary, we see $\gamma = \kappa_*/2$.

Let us derive the second variation of $L_\Gamma(\rho)$ and $\Xi(\rho)$. We first observe

$$\begin{cases} \Psi_{qq}(\cdot,0) = \xi_{qq}(\cdot,0)T_*, \quad \Psi_{\sigma qq}(\cdot,0) = \xi_{\sigma qq}(\cdot,0)T_* + \xi_{qq}(\cdot,0)\kappa_* N_*, \\ \xi_{qq}(\sigma,0) = -h_- + \dfrac{\sigma+l}{2l}(h_+ + h_-). \end{cases} \tag{B.1}$$

Then (though we omit the details of the calculation) it follows from (2.8) and (B.1) that

$$\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}L_\Gamma(\varepsilon_1\rho_1+\varepsilon_2\rho_2)\Big|_{\varepsilon_1=\varepsilon_2=0} = \int_{-l}^{l} \partial_\sigma\rho_1\partial_\sigma\rho_2 \, d\sigma + h_+\rho_1(l)\rho_2(l) + h_-\rho_1(-l)\rho_2(-l),$$

$$\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\Xi(\varepsilon_1\rho_1+\varepsilon_2\rho_2)\Big|_{\varepsilon_1=\varepsilon_2=0} = -2\kappa_*\int_{-l}^{l} \rho_1\rho_2 \, d\sigma.$$

Thus the second variation of $L_\Gamma(\rho)$ under the constraint $\Xi(\rho) = 0$ is

$$\begin{aligned} &\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}L_\Gamma(\varepsilon_1\rho_1+\varepsilon_2\rho_2)\Big|_{\varepsilon_1=\varepsilon_2=0} + \frac{\kappa_*}{2}\left\{\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\Xi(\varepsilon_1\rho_1+\varepsilon_2\rho_2)\Big|_{\varepsilon_1=\varepsilon_2=0}\right\} \\ &= \int_{-l}^{l} \partial_\sigma\rho_1\partial_\sigma\rho_2 \, d\sigma + h_+\rho_1(l)\rho_2(l) + h_-\rho_1(-l)\rho_2(-l) + \frac{\kappa_*}{2}\left\{-2\kappa_*\int_{-l}^{l} \rho_1\rho_2 \, d\sigma\right\} \\ &= I(\rho_1,\rho_2). \end{aligned}$$

This is the desired assertion.

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