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On spectra of q-deformed operators

九州大学 大学院芸術工学研究院 太田 昇一 (Schôichi Ôta)
Faculty of Design, Department of Art and Information Design
Kyushu University

1.

The formal algebraic relation $xx^* = qx^*x$ ($q > 0$, $q \neq 1$) appears in several different situations related to the theory of quantum groups. This leads us to the study of an operator obeying this relation in a Hilbert space. Let q be a positive real number with $q \neq 1$. Let T be a closed densely defined operator in $\mathcal{H}$. If T satisfies

$$TT^* = qT^*T,$$

then T is called a *deformed normal operator* with deformation parameter q. Let T be a closed densely defined operator in $\mathcal{H}$ with polar decomposition $T = U|T|$. If T satisfies the relation

$$U|T| = \sqrt{q}\,|T|U,$$

then T is called a *deformed quasinormal operator* with deformation parameter q. For a deformed normal (resp. deformed quasinormal) operator T with deformation parameter q, we will simply say T is *q-normal.* (resp. *q-quasinormal*)

If T is q-normal then T is q-quasinormal. A closed densely defined operator T is q-normal if and only if

$$\mathcal{D}(T) = \mathcal{D}(T^*) \quad \text{and} \quad \|T^*\eta\| = \sqrt{q}\|T\eta\| \quad (\eta \in \mathcal{D}(T)).$$

A densely defined operator T is called a *q-hyponormal* operator (or a *deformed hyponormal* operator with deformation parameter q) if it satisfies

$$\mathcal{D}(T) \subseteqq \mathcal{D}(T^*) \quad \text{and} \quad \|T^*\eta\| \leqq \sqrt{q}\,\|T\eta\|$$

for all $\eta \in \mathcal{D}(T)$. If T is q-quasinormal, then T is q-deformed hyponormal.

Let T be a q-deformed hyponormal operator in $\mathcal{H}$. Then there exists uniquely a contraction K_T such that

$$T^* \supseteqq \sqrt{q}K_T T \quad \text{and} \quad \ker K_T \supseteqq \ker T^*.$$

K_T is called the attached contraction to T. If, in addition, T is closed and $T = U|T|$ is the polar decomposition, then T is q-quasinormal if and only if $K_T = (U^*)^2$.

2. Unbounded weighted shifts

Let S_b be a closed densely defined operator in a separable Hilbert space $\mathcal{H}$. If there are an orthonormal basis $\{e_n\}$ $(n \in \mathbb{Z})$ and a sequence $\{w_n\}(w_n \neq 0\ , n \in \mathbb{Z})$ of complex numbers such that

$$\mathcal{D}(S_b) = \left\{ \sum_{-\infty}^{\infty} \alpha_n e_n \in \mathcal{H} : \sum_{-\infty}^{\infty} |\alpha_n|^2 |w_n|^2 < \infty \right\}$$

and

$$S_b e_n = w_n e_{n+1}$$

for all $n \in \mathbb{Z}$, then S_b is called a *bilateral (injective) weighted shift* with weight sequence $\{w_n\}$ (with respect to $\{e_n\}$). A unilateral weighted shift S_u is defined analogously.

Proposition. The following statements hold:

1. A unilateral weighted shift S_u in $\mathcal{H}$ with weights $\{w_n\}$ is q-quasinormal if and only if
$$|w_n| = \left(\frac{1}{\sqrt{q}}\right)^n |w_0|$$
for all $n \geqq 0$. In particular, a unilateral weighted shift cannot be q-normal.
2. A bilateral weighted shift S_b in $\mathcal{H}$ with weights $\{w_n\}$ is q-normal if and only if the above equation is valid for all $n \in \mathbb{Z}$
3. A weighted shift S_u (resp. S_b) is q-hyponormal if and only if
$$|w_{n+1}| \geqq \frac{1}{\sqrt{q}} |w_n|$$
for all $n \geqq 0$ (resp. $n \in \mathbb{Z}$).

The spectrum of a q-normal weighted shift S_b:

	σ_p	σ_c	σ_r	σ
$S_b\,(0 < q < 1)$	$\emptyset$	$\{0\}$	$\mathbb{C} \setminus \{0\}$	$\mathbb{C}$
$S_b\,(q > 1)$	$\mathbb{C} \setminus \{0\}$	$\{0\}$	$\emptyset$	$\mathbb{C}$

The spectrum of a q-quasinormal weighted shift S_u:

	σ_p	σ_c	σ_r	σ
$S_u\,(0 < q < 1)$	$\emptyset$	$\emptyset$	$\mathbb{C}$	$\mathbb{C}$
$S_u\,(q > 1)$	$\emptyset$	$\emptyset$	$\{0\}$	$\{0\}$

3. Spectra of a q-hyponormal operator

Theorem. Let T_1 and T_2 be q-hyponormal operators in a Hilbert space $\mathcal{H}$. Then $T_1 \oplus T_2$ is also q-hyponormal in $\mathcal{H} \oplus \mathcal{H}$ and

$$K_{T_1 \oplus T_2} = K_{T_1} \oplus K_{T_2}.$$

Moreover, $T_1 \oplus T_2$ is q-normal (resp. q-quasinormal) if and only if both T_1 and T_2 are q-normal (resp. q-quasinormal).

In case that $0 < q < 1$, a non-trivial q-hyponormal operator is always unbounded and the planar Lebesgue measure of its spectrum is positive.

Let $q > 1$. Then, there are various kinds of q-deformed operators, bounded or unbounded:

- A q-quasinormal unilateral weighted shift is always bounded.
- There exist q-quasinormal operators which are unbounded; they are q-normal ones.
- Using Theorem, one can construct an unbounded q-quasinormal operator which is not q-normal. (For this take T_1 to be any q-normal operator (which must be unbounded) and T_2 to be a bounded q-quasinormal unilateral weighted shift.)
- There exists a q-hyponormal operator which has empty spectrum, which is given in the following section; this is in contrast to the fact that every closed densely defined hyponormal operator ($q = 1$) has to have non-empty spectrum.

4. A q-deformed operator with empty spectrum

Let T be a closed densely defined operator in a Hilbert space $\mathcal{H}$. Recall that the resolvent set $\rho(T)$ of T is defined as the set of all $\lambda \in \mathbb{C}$ for which $\ker(\lambda - T) = \{0\}$, $\mathcal{R}(\lambda - T) = \mathcal{H}$ and the inverse $(\lambda - T)^{-1}$ is bounded on $\mathcal{H}$. Especially,

$$0 \in \rho(T)$$

if and only if there is a bounded operator S on $\mathcal{H}$ such that

$$ST \subseteqq 1 \quad \textit{and} \quad TS = 1.$$

Lemma. Let T be a closed densely defined operator in $\mathcal{H}$. Suppose that

$$\rho(T) \ni 0 \ .$$

If $\sigma(T^{-1}) = \{0\}$, then

$$\sigma(T) = \phi \ .$$

Let $q > 1$. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis$\{e_n\}_{n \in \mathbb{Z}}$. Take numbers r and ℓ such that

$$\ell > 1 > r \geqq \frac{1}{\sqrt{q}}.$$

Put

$$w_n = \begin{cases} r^n & \text{if} \quad n \geqq 0 \ , \\ \ell^n & \text{if} \quad n \leqq -1 \ . \end{cases}$$

Let us consider the weighted shift S_0 with the weight sequence $\{w_n\}$. Then, clearly S_0 is bounded with $\mathcal{D}(S_0) = \mathcal{H}$. Since the sequence $\{w_n\}$ tends to zero as $|n| \to \infty$, S_0 is compact and so $\sigma(S_0)$ is countable. On the other hand,

$$\sigma(S_0) = c\,\sigma(S_0)$$

for all $c \in \mathbb{C}$ with $|c| = 1$. It follows that $\sigma(S_0) = \{0\}$.

Since $\ker(S_0) = \ker(S_0{}^*) = \{0\}$, S_0 is injective and has dense range. This means that the inverse $S_0{}^{-1}$ is closed and densely defined. Hence, it follows from Lemma that $S_0{}^{-1}$ has empty spectrum. On the other hand, we have

$$\frac{w_{n+1}}{w_n} = r \geqq \frac{1}{\sqrt{q}} \qquad \textit{for} \quad n \geqq 0 \ ,$$

and

$$\frac{w_{n+1}}{w_n} = \ell > 1 > \frac{1}{\sqrt{q}} \qquad \text{for} \quad n \leqq -1 .$$

These inequalities imply that S_0 is q-hyponormal. Therefore, ${S_0}^{-1}$ is also q-hyponormal. Thus we have:

Theorem. Let $q > 1$. Then, there exists a q-hyponormal operator with empty spectrum.

5. Order relations for q-deformed operators

Let us recall some inequalities by Kato and Rellich ([1] and [5]) :

$$S \ll T \text{ means } \mathcal{D}(T) \subseteqq \mathcal{D}(S), \text{ and } \|S\eta\| \leqq \|T\eta\| \text{ for } \eta \in \mathcal{D}(T)$$

and

$$S \preceq T \text{ means } \mathcal{D}(T^{\frac{1}{2}}) \subseteqq \mathcal{D}(S^{\frac{1}{2}}) \text{ and } \|S^{\frac{1}{2}}\eta\| \leqq \|T^{\frac{1}{2}}\eta\| \text{ for } \eta \in \mathcal{D}(T^{\frac{1}{2}})$$

provided S and T are selfadjoint and nonegative.

Definition. Let S and T be symmetric (densely defined) operators in $\mathcal{H}$. If

$$\mathcal{D}(T) \subseteqq \mathcal{D}(S) \quad \text{and} \quad \langle S\eta , \eta \rangle \leqq \langle T\eta , \eta \rangle$$

for all $\eta \in \mathcal{D}(T)$, then we write

$$S \leqq T .$$

Theorem. Let T be a closed densely defined operator in $\mathcal{H}$. We consider the following statements:

(1) T is q-hyponormal.

(2) T satisfies the condition $|T^*| \ll \sqrt{q}\,|T|$.

(3) T satisfies the condition $|T^*| \leqq \sqrt{q}\,|T|$.

(4) T satisfies the condition $|T^*| \preceq \sqrt{q}\,|T|$.

Then, (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4).

Especially, if T is a weighted shift, unilateral or bilateral, then all these statements are equivalent.

Theorem. If a closed densely defined operator T in $\mathcal{H}$ satisfies condition

$$TT^* \leqq qT^*T,$$

then T is q-hyponormal.

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