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PARTIAL SUMS OF CERTAIN MEROMORPHIC FUNCTIONS

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The purpose of the present paper is to establish some results concerning the partial sums of meromorphic starlike and meromorphic convex functions analogous to the results due to H. Silverman (J. Math. Anal. Appl. 209(1997), 221-227). Furthermore, we consider the partial sums of certain integral operator.

KEY WORDS: partial sum, meromorphic starlike, meromorphic convex, meromorphic close-to-convex, integral operator.

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1. Introduction

Let Σ be the class consisting of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the punctured open unit disk $\mathcal{D} = \{z : 0 < |z| < 1\}$. Let $\Sigma^*(\alpha)$ and $\Sigma_k(\alpha)$ be the subclasses of Σ consisting of all functions which are, respectively, meromorphic starlike and meromorphic convex of order $\alpha (0 \leq \alpha < 1)$ in $\mathcal{D}$. We also denote by $\Sigma_c(\alpha)$ the subclass of Σ which satisfy

$$-\mathrm{Re}\{z^2 f'(z)\} > \alpha \quad (0 \leq \alpha < 1;\ z \in \mathcal{U} = \mathcal{D} \cup \{0\}).$$

We note that every function belonging to the class $\Sigma_c(\alpha)$ is meromorphic close-to-convex of order α in $\mathcal{D}$ (see, [2]).

If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ are analytic in $\mathcal{U}$, then their Hadamard product (or convolution), denote by $f * g$, is the function defined by the power series

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k \quad (z \in \mathcal{U}).$$

A sufficient condition for a function f of the form (1.1) to be in $\Sigma^*(\alpha)$ is that

$$\sum_{k=1}^{\infty} (k+\alpha)|a_k| \leq 1-\alpha \tag{1.2}$$

and to be in $\Sigma_k(\alpha)$ is that

$$\sum_{k=1}^{\infty} k(k+\alpha)|a_k| \leq 1-\alpha. \tag{1.3}$$

Further, we note that these sufficient conditions are also necessary for functions of the form (1.1) with positive or negative coefficients([6,13], also see [7]).

Recently, Silverman [10] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. Also, Li and Owa [4] obtained the sharp radius which for the normalized univalent functions in $\mathcal{U}$, the partial sums of the well-known Libera integral operator [5] imply starlikeness. Further, for various other interesting developments concerning partial sums of analytic univalent functions, the reader may be (for examples) refered to the works of Brickman et al. [1], Sheil-Small [9], Silvia [11], Singh and Singh [12] and Yang and Owa [14].

Since to a certain extent the work in the meromorphic univalent case has paralleled that of analytic univalent case, one is tempted to search results analogous to Silverman [10] for meromorphic univalent functions in $\mathcal{D}$.

In the present paper, motivated essentially by the work of Silverman [10], we will investigate the ratio of a function of the form (1.1) to its sequence of partial sums $f_n(z) = 1/z + \sum_{k=1}^{n} a_k z^k$ when the coefficients are sufficiently small to satisfy either condition (1.2) or (1.3). More precisely, we will determine sharp lower bounds for $\mathrm{Re}\{f(z)/f_n(z)\}$, $\mathrm{Re}\{f_n(z)/f(z)\}$, $\mathrm{Re}\{f'(z)/f_n'(z)\}$, and $\mathrm{Re}\{f_n'(z)/f'(z)\}$. Further, we give a property for the partial sums of certain integral opreator in connection with meromorphic close-to-convex functions.

In the sequel, we will make use of the well-known result that $\mathrm{Re}\{(1+w(z))/(1-w(z))\} > 0$ $(z \in \mathcal{U})$ if and only if $w(z) = \sum_{k=1}^{\infty} c_k z^k$ satisfies the inequality $|w(z)| < |z|$. Unless otherwise stated, we will assume that f is of the form (1.1) and its sequence of partial sums is denoted by $f_n(z) = 1/z + \sum_{k=1}^{n} a_k z^k$.

2. Main Results

Theorem 2.1. *If f of the form* (1.1) *satisfies condition* (1.2), *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{n+2\alpha}{n+1+\alpha} \quad (z \in \mathcal{U}).$$

The result is sharp for every n, with extremal function

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{n+1+\alpha} z^{n+1} \quad (n \geq 0). \tag{2.1}$$

Proof. We may write

$$\begin{aligned}
&\frac{n+1+\alpha}{1-\alpha}\left[\frac{f(z)}{f_n(z)} - \frac{n+2\alpha}{n+1+\alpha}\right] \\
&= \frac{1+\sum_{k=1}^{n} a_k z^{k+1} + \frac{n+1+\alpha}{1-\alpha}\sum_{k=n+1}^{\infty} a_k z^{k+1}}{1+\sum_{k=1}^{n} a_k z^{k+1}} \\
&:= \frac{1+A(z)}{1+B(z)}.
\end{aligned}$$

Set $(1+A(z))/(1+B(z)) = (1+w(z))/(1-w(z))$, so that $w(z) = (A(z)-B(z))/(2+A(z)+B(z))$. Then

$$w(z) = \frac{\frac{n+1+\alpha}{1-\alpha}\sum_{k=n+1}^{\infty} a_k z^{k+1}}{2+2\sum_{k=1}^{n} a_k z^{k+1} + \frac{n+1+\alpha}{1-\alpha}\sum_{k=n+1}^{\infty} a_k z^{k+1}}$$

and

$$|w(z)| \leq \frac{\frac{n+1+\alpha}{1-\alpha}\sum_{k=n+1}^{\infty} |a_k|}{2-2\sum_{k=1}^{n} |a_k| - \frac{n+1+\alpha}{1-\alpha}\sum_{k=n+1}^{\infty} |a_k|}.$$

Now $|w(z)| \leq 1$ if and only if

$$2\left(\frac{n+1+\alpha}{1-\alpha}\right)\sum_{k=n+1}^{\infty} |a_k| \leq 2 - 2\sum_{k=1}^{n} |a_k|,$$

which is equivalent to

$$\sum_{k=1}^{n} |a_k| + \frac{n+1+\alpha}{1-\alpha}\sum_{k=n+1}^{\infty} |a_k| \leq 1. \tag{2.2}$$

It suffices to show that the left hand side of (2.2) is bounded above by $\sum_{k=1}^{\infty}((k+\alpha)/(1-\alpha))|a_k|$, which is equivalent to

$$\sum_{k=1}^{n}\left(\frac{k+2\alpha-1}{1-\alpha}\right)|a_k|+\sum_{k=n+1}^{\infty}\left(\frac{k-n-1}{1-\alpha}\right)|a_k|\geq 0.$$

To see that the function f given by (2.1) gives the sharp result, we observe for $z=re^{\pi i/(n+2)}$ that

$$\begin{aligned}\frac{f(z)}{f_n(z)}&=1+\frac{1-\alpha}{n+1+\alpha}z^{n+2}\longrightarrow 1-\frac{1-\alpha}{n+1+\alpha}\\&=\frac{n+2\alpha}{n+1+\alpha}\ \text{when } r\to 1^-.\end{aligned}$$

Therefore we complete the proof of Theorem 2.1.

Theorem 2.2. *If f of the form* (1.1) *satisfies condition* (1.3), *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\}\geq\frac{(n+2)(n+\alpha)}{(n+1)(n+1+\alpha)}\quad(z\in\mathcal{U}).$$

The result is sharp for every n, with extremal function

$$f(z)=\frac{1}{z}+\frac{1-\alpha}{(n+1)(n+1+\alpha)}z^{n+1}\quad(n\geq 0).\tag{2.3}$$

Proof. We write

$$\begin{aligned}&\frac{(n+1)(n+1+\alpha)}{1-\alpha}\left[\frac{f(z)}{f_n(z)}-\frac{(n+2)(n+\alpha)}{(n+1)(n+1+\alpha)}\right]\\&=\frac{1+\sum_{k=1}^{n}a_kz^{k+1}+\frac{(n+1)(n+1+\alpha)}{1-\alpha}\sum_{k=n+1}^{\infty}a_kz^{k+1}}{1+\sum_{k=1}^{n}a_kz^{k+1}}\\&:=\frac{1+w(z)}{1-w(z)},\end{aligned}$$

where

$$w(z)=\frac{\frac{(n+1)(n+1+\alpha)}{1-\alpha}\sum_{k=n+1}^{\infty}a_kz^{k+1}}{2+2\sum_{k=1}^{n}a_kz^{k+1}+\frac{(n+1)(n+1+\alpha)}{1-\alpha}\sum_{k=n+1}^{\infty}a_kz^{k+1}}.$$

Now

$$|w(z)|\leq\frac{\frac{(n+1)(n+1+\alpha)}{1-\alpha}\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=1}^{n}|a_k|-\frac{(n+1)(n+1+\alpha)}{1-\alpha}\sum_{k=n+1}^{\infty}|a_k|}\leq 1,$$

if

$$\sum_{k=1}^{n}|a_k| + \frac{(n+1)(n+1+\alpha)}{1-\alpha}\sum_{k=n+1}^{\infty}|a_k| \leq 1. \tag{2.4}$$

The left hand side of (2.4) is bounded above by $\sum_{k=1}^{\infty}(k(k+\alpha)/(1-\alpha))|a_k|$ if

$$\frac{1}{1-\alpha}\left\{\sum_{k=1}^{n}(k(k+\alpha)-(1-\alpha)|a_k| + \sum_{k=n+1}^{\infty}(k(k+\alpha)-(n+1)(n+1+\alpha))|a_k|\right\} \geq 0,$$

and the proof is completed.

We next determine bounds for $\text{Re}\{f_n(z)/f(z)\}$.

Theorem 2.3. (a) *If f of the form* (1.1) *satisfies condition* (1.2), *then*

$$\text{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{n+1+\alpha}{n+2} \quad (z \in \mathcal{U}).$$

(b) *If f of the form* (1.1) *satisfies condition* (1.3), *then*

$$\text{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{(n+1)(n+1+\alpha)}{(n+1)(n+2)-n(1-\alpha)} \quad (z \in \mathcal{U}).$$

Equalities hold in (a) *and* (b) *for the functions given by* (2.1) *and* (2.3), *respectively.*

Proof. We prove (a). The proof of (b) is similar to (a) and will be omitted. We write

$$\begin{aligned}
&\frac{n+2}{1-\alpha}\left[\frac{f_n(z)}{f(z)} - \frac{n+1+\alpha}{n+2}\right] \\
&= \frac{1+\sum_{k=1}^{n} a_k z^{k+1} + \frac{n+1+\alpha}{1-\alpha}\sum_{k=n+1}^{\infty} a_k z^{k+1}}{1+\sum_{k=1}^{n} a_k z^{k+1}} \\
&:= \frac{1+w(z)}{1-w(z)},
\end{aligned}$$

where

$$|w(z)| \leq \frac{\frac{n+2}{1-\alpha}\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=1}^{n}|a_k| - \frac{n+2\alpha}{1-\alpha}\sum_{k=n+1}^{\infty}|a_k|} \leq 1.$$

This last inequality is equivalent to

$$\sum_{k=1}^{n}|a_k| + \frac{n+1+\alpha}{1-\alpha}\sum_{k=n+1}^{\infty}|a_k| \leq 1. \tag{2.5}$$

Since the left hand side of (2.5) is bounded above by $\sum_{k=1}^{\infty}((k+\alpha)/(1-\alpha))|a_k|$, the proof is completed.

We turn to ratios involving derivatives. The proofs of Theorem 2.4 below follows the pattern of those in Theorem 2.1 and (a) of Theorem 2.3 and so the details may be omitted.

Theorem 2.4. *If f of the form* (1.1) *satisfies condition* (1.2) *with $\alpha = 0$, then*

(a) $$\mathrm{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq 0 \quad (z \in \mathcal{U}),$$

(b) $$\mathrm{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{1}{2} \quad (z \in \mathcal{U}).$$

In both cases, the extremal function is given by (2.1) *with $\alpha = 0$.*

Theorem 2.5. *If f of the form* (1.1) *satisfies condition* (1.3), *then*

(a) $$\mathrm{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{n+2\alpha}{n+1+\alpha} \quad (z \in \mathcal{U}),$$

(b) $$\mathrm{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{n+1+\alpha}{n+2} \quad (z \in \mathcal{U}).$$

In both cases, the extremal function is given by (2.3).

Proof. It is well known that $f \in \Sigma_k(\alpha) \Leftrightarrow -zf' \in \Sigma^*(\alpha)$. In particular, f satisfies condition (1.3) if and only if $-zf'$ satisfies condition (1.2). Thus, (a) is an immediate consequence of Theorem 2.1 and (b) follows directly from (a) of Theorem 2.3.

For a function $f \in \Sigma$, we define the integral operator F as follows:

$$\begin{aligned} F(z) &= \frac{1}{z^2}\int_0^z tf(t)dt \\ &= \frac{1}{z} + \sum_{k=1}^{\infty}\frac{1}{k+2}a_k z^k \quad (z \in \mathcal{D}). \end{aligned}$$

The n-th partial sums F_n of the integral operator F is given by

$$F_n(z) = \frac{1}{z} + \sum_{k=1}^{n} \frac{1}{k+2} a_k z^k \quad (z \in \mathcal{D}).$$

The following lemmas will be required for the proof of Theorem 2.6 below.

Lemma 2.1. *For* $0 \le \theta \le \pi$,

$$\frac{1}{2} + \sum_{k=1}^{m} \frac{\cos(k\theta)}{k+1} \ge 0.$$

Lemma 2.2. *Let P be analytic in $\mathcal{U}$ with $P(0) = 1$ and* $\operatorname{Re}\{P(z)\} > 1/2$ *in $\mathcal{U}$. For any function Q analytic in $\mathcal{U}$, the function $P * Q$ takes values in the convex hull of the image on $\mathcal{U}$ under Q.*

Lemma 2.1 is due to Rogosinski and Szegö [8] and Lemma 2.2 is a well-known result (c.f. [3, 12]) that can be derived from the Herglotz' representation for P.

Finally, we derive

Theorem 2.6. *If $f \in \Sigma_c(\alpha)$, then $F_n \in \Sigma_c(\alpha)$*

Proof. Let f be of the form (1.1) and belong to the class $\Sigma_c(\alpha)$ for $0 \le \alpha < 1$. Since $-\operatorname{Re}\{z^2 f'(z)\} > \alpha$, we have

$$\operatorname{Re}\left\{1 - \frac{1}{2(1-\alpha)} \sum_{k=1}^{\infty} k a_k z^{k+1}\right\} > \frac{1}{2} \quad (z \in \mathcal{U}). \tag{2.6}$$

Applying the convolution properties of power series to F_n', we may write

$$\begin{aligned} -z^2 F_n'(z) &= 1 - \sum_{k=1}^{n} \frac{k}{k+2} a_k z^{k+1} \\ &= \left(1 - \frac{1}{2(1-\alpha)} \sum_{k=1}^{\infty} k a_k z^{k+1}\right) * \left(1 + 2(1-\alpha) \sum_{k=1}^{n+1} \frac{1}{k+1} z^k\right). \end{aligned} \tag{2.7}$$

Putting $z = re^{i\theta} (0 \le r < 1,\ 0 \le |\theta| \le \pi)$, and making use of the minimum principle for harmonic functions along with Lemma 2.1, we obtain

$$\begin{aligned}
&\operatorname{Re}\left\{1+2(1-\alpha)\sum_{k=1}^{n+1}\frac{1}{k+1}z^k\right\} \\
&=1+2(1-\alpha)\sum_{k=1}^{n+1}\frac{r^k\cos k\theta}{k+1} \\
&>1+2(1-\alpha)\sum_{k=1}^{n+1}\frac{\cos k\theta}{k+1} \\
&\geq \alpha.
\end{aligned} \tag{2.8}$$

In view of (2.6), (2.7), (2.8) and Lemma 2.2, we deduce that

$$-\operatorname{Re}\{z^2F_n'(z)\} \;>\; \alpha \quad (0\leq\alpha<1;\ z\in\mathcal{U}).$$

Therefore we complete the proof of Theorem 2.6.

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