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STRONG CONVERGENCE OF ISHIKAWA ITERATIONS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract—Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space. We prove that if $T : C \to C$ is both compact iterates and asymptotically nonexpansive, the Ishikawa iteration process with errors defined by $x_1 \in C$, $x_{n+1} = \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n$, and $y_n = \alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n$ converges strongly to some fixed point of T. This generalizes the recent theorems due to Rhoades [5], Schu [6] and Schu [7].

Keywords—strong convergence, fixed point, Mann and Ishikawa iteration process, asymptotically nonexpansive mapping.

1. Introduction

Let C be a nonempty bounded closed convex subset of a Banach space E and let T be a mapping of C into itself. Then T is said to be *asymptotically nonexpansive* [1] if there exists a sequence $\{k_n\}$, $k_n \geq 1$, with $\lim_{n\to\infty} k_n = 1$, such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and $n \geq 1$. In particular, if $k_n = 1$ for all $n \geq 1$, T is said to be *nonexpansive*. T is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$, such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x, y \in C$ and $n \geq 1$. T is said to be compact if it maps bounded sets into relatively compact ones. We denote by $F(T)$ the set of all fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$. We also denote by $\mathbb{N}$ the set of all positive integers. A Banach space E is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ will denote strong convergence of the sequence $\{x_n\}$ to x. For a mappings T of C into itself, Rhoades [5] considered the following modified Ishikawa iteration process (cf. Ishikawa [3]) in C defined by

$$\begin{aligned} & x_1 \in C, \\ (1) \qquad & x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \\ & y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$. If $\beta_n = 0$ for all $n \geq 1$, then the iteration process (1) becomes the following modified Mann iteration process (cf. Mann [4], Schu [6]):

$$(2) \qquad \begin{aligned} &x_1 \in C, \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \end{aligned}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$.

Recently, Schu [7] proved that if E is a uniformly convex Banach space, C is a nonempty bounded closed and convex subset of E, and $T : C \to C$ is an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and T^m is compact for some $m \in \mathbb{N}$, then for any $x_1 \in C$, the sequence $\{x_n\}$ defined by (2), where $\{\alpha_n\}$ is chosen so that $0 < a \leq \alpha_n \leq b < 1$, for all $n \geq 1$ and some $a, b \in (0,1)$, converges strongly to some fixed point of T. This extended a result of Schu [6] to uniformly convex Banach spaces. On the other hand, Rhoades [5] proved that if E is a uniformly convex Banach space, C is a nonempty bounded closed convex subset of E, and $T : C \to C$ is a completely continuous asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$, $r = \max\{2,p\}$, then for any $x_1 \in C$, the sequence $\{x_n\}$ defined by (1), where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy $a \leq (1-\alpha_n), (1-\beta_n) \leq 1-a$ for all $n \geq 1$ and some $a > 0$, converges strongly to some fixed point of T. We consider a more general iterative process of the type (cf. Xu [10]) emphasizing the randomness of errors as follows:

$$(3) \qquad \begin{aligned} &x_1 \in C, \\ &x_{n+1} = \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \\ &y_n = \alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are real sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$ are two sequences in C such that

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$ for all $n \geq 1$,
(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$.

If $\gamma_n = \gamma'_n = 0$ for all $n \geq 1$, then the iteration process (3) reduces to the Ishikawa iteration process [3], while setting $\beta'_n = 0$ and $\gamma'_n = 0$ for all $n \geq 1$, (3) reduces to the Mann iteration process with errors, which is a generalized case of the Mann iteration process [4]

In this paper, we prove strong convergence theorems of the Ishikawa (and Mann) iteration process with errors defined by (3) for a compact iterates and asymptotically nonexpansive mapping in a uniformly convex Banach space, which generalize the recent theorems due to Rhoades [5], Schu [6] and Schu [7].

2. Strong convergence theorems

We first begin with the following:

Lemma 1 [9]. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that* $\sum_{n=1}^{\infty} b_n < \infty$ *and*

$$a_{n+1} \le a_n + b_n$$

for all $n \ge 1$. Then $\lim_{n\to\infty} a_n$ exists.

Lemma 2 [2]. *Let E be a uniformly convex Banach space. Let $x, y \in E$. If $\|x\| \le 1$, $\|y\| \le 1$, and $\|x-y\| \ge \epsilon > 0$, then $\|\lambda x+(1-\lambda)y\| \le 1-2\lambda(1-\lambda)\delta(\epsilon)$ for λ with $0 \le \lambda \le 1$.*

Lemma 3 (cf. [6]). *Let E be a normed space and let C be a nonempty bounded convex subset of E. Let $T : C \to C$ be a uniformly L-Lipschitzian mapping. Define the sequence $\{x_n\}$ defined by (3). Set $w_n = \|T^n x_n - x_n\|$, for all $n \ge 1$. Then*

$$\|x_n - Tx_n\| \le w_n + L(2+2L+L^2)w_{n-1} + L^2(1+L)M^*\gamma'_{n-1} + L(1+L)M^*\gamma_{n-1},$$

for all $n \ge 1$, where $M^* := \sup_{n\ge1} \|x_n - u_n\| \vee \sup_{n\ge1} \|x_n - v_n\| < \infty$.

Proof. Since

$$\begin{aligned}\|y_n - x_n\| &= \|\alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n - x_n\| \\ &\le \beta'_n \|T^n x_n - x_n\| + \gamma'_n \|v_n - x_n\| \\ &\le w_n + \gamma'_n M^*,\end{aligned}$$

$$\begin{aligned}\|T^n y_n - x_n\| &\le \|T^n y_n - T^n x_n\| + \|T^n x_n - x_n\| \\ &\le L\|y_n - x_n\| + w_n \\ &\le L\{w_n + \gamma'_n M^*\} + w_n \\ &= (1+L)w_n + LM^*\gamma'_n\end{aligned}$$

and thus

$$\begin{aligned}\|x_n - x_{n-1}\| &= \|\alpha_{n-1}x_{n-1} + \beta_{n-1}T^{n-1}y_{n-1} + \gamma_{n-1}u_{n-1} - x_{n-1}\| \\ &\le \beta_{n-1}\|T^{n-1}y_{n-1} - x_{n-1}\| + \gamma_{n-1}\|u_{n-1} - x_{n-1}\| \\ &\le (1+L)w_{n-1} + LM^*\gamma'_{n-1} + M^*\gamma_{n-1},\end{aligned}$$

$$\begin{aligned}\|T^{n-1}x_n - x_n\| &\le \|T^{n-1}x_n - T^{n-1}x_{n-1}\| + \|T^{n-1}x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\| \\ &\le w_{n-1} + (1+L)\|x_n - x_{n-1}\| \\ &\le w_{n-1} + (1+L)\{(1+L)w_{n-1} + LM^*\gamma'_{n-1} + M^*\gamma_{n-1}\}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - T^n x_n\| + \|T^n x_n - Tx_n\| \\ &\le w_n + L\|T^{n-1}x_n - x_n\| \\ &\le w_n + L[w_{n-1} + (1+L)\{(1+L)w_{n-1} + LM^*\gamma'_{n-1} + M^*\gamma_{n-1}\}] \\ &= w_n + L(2+2L+L^2)w_{n-1} + L^2(1+L)M^*\gamma'_{n-1} + L(1+L)M^*\gamma_{n-1}.\end{aligned}$$

□

Using Lemma 1, we have the following:

Lemma 4. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that the sequence $\{x_n\}$ defined by (3). Then $\lim_{n\to\infty} \|x_n - z\|$ exists, for any $z \in F(T)$.*

Proof. The existence of a fixed point of T follows from Goebel-Kirk [1]. For a fixed $z \in F(T)$, since $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ are bounded, let

$$M := \sup_{n\geq 1} \|x_n - z\| \vee \sup_{n\geq 1} \|u_n - z\| \vee \sup_{n\geq 1} \|v_n - z\| < \infty.$$

Put $c_n = k_n - 1$. Since

$$\begin{aligned}
\|T^n y_n - z\| &\leq k_n \|y_n - z\| \\
&= (1 + c_n)\|\alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n - z\| \\
&\leq (1 + c_n)\{\alpha'_n \|x_n - z\| + \beta'_n \|T^n x_n - z\| + \gamma'_n \|v_n - z\|\} \\
&\leq (1 + c_n)\{\alpha'_n \|x_n - z\| + \beta'_n (1 + c_n)\|x_n - z\| + \gamma'_n \|v_n - z\|\} \\
&\leq (1 + c_n)\{\alpha'_n \|x_n - z\| + \beta'_n \|x_n - z\| + c_n \|x_n - z\| + \gamma'_n \|v_n - z\|\} \\
&= \alpha'_n \|x_n - z\| + \beta'_n \|x_n - z\| + c_n \|x_n - z\| + \gamma'_n \|v_n - z\| \\
&\quad + c_n\{\alpha'_n \|x_n - z\| + \beta'_n \|x_n - z\| + c_n \|x_n - z\| + \gamma'_n \|v_n - z\|\} \\
&\leq (1 - \gamma'_n)\|x_n - z\| + 4Mc_n + M\gamma'_n,
\end{aligned}$$

we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n - z\| \\
&\leq \alpha_n \|x_n - z\| + \beta_n \|T^n y_n - z\| + \gamma_n \|u_n - z\| \\
&\leq \alpha_n \|x_n - z\| + \beta_n\{(1 - \gamma'_n)\|x_n - z\| + 4Mc_n + M\gamma'_n\} + \gamma_n M \\
&= (1 - (\gamma_n + \beta_n \gamma'_n))\|x_n - z\| + 4M\beta_n c_n + M(\gamma_n + \beta_n \gamma'_n) \\
&\leq \|x_n - z\| + 4Mc_n + M(\gamma_n + \gamma'_n).
\end{aligned}$$

By Lemma 1, we readily see that $\lim_{n\to\infty} \|x_n - z\|$ exists. □

By using Lemma 1–Lemma 4, we have the following:

Theorem 1. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose $x_1 \in C$, and the sequence $\{x_n\}$ defined by (3) satisfies $0 < a \leq \alpha_n \leq b < 1$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $0 \leq \beta'_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0, 1)$ or $0 < a \leq \beta_n \leq 1$, $0 < a \leq \alpha'_n \leq b < 1$, $\sum_{n=1}^{\infty} \beta'_n = \infty$ for all $n \geq 1$ and some $a, b \in (0, 1)$. Then $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Proof. The existence of a fixed point of T follows from Goebel-Kirk [1]. For a fixed $z \in F(T)$, since $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ are bounded, let

$$M := \sup_{n\geq 1}\|x_n - z\| \vee \sup_{n\geq 1}\|u_n - z\| \vee \sup_{n\geq 1}\|v_n - z\| < \infty.$$

By Lemma 4, we see that $\lim_{n\to\infty}\|x_n - z\|(\equiv r)$ exists. If $r = 0$, then the conclusion is obvious. So, we assume $r > 0$. Note that $d_n := \max\{\gamma_n', \gamma_n/a, \gamma_n'/a\} \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$. Put $c_n = k_n - 1$. Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, we have

$$\lim_{n\to\infty} c_n = 0. \tag{4}$$

Since $\|T^n y_n - z\| \leq \|x_n - z\| + 4Mc_n + Md_n$ and

$$\left\|\frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z\right\| \leq \|x_n - z\| + 4Mc_n + Md_n,$$

by using Lemma 2 and Takahashi [8], we obtain

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n - z\| \\
&= \left\|\beta_n(T^n y_n - z) + (1 - \beta_n)\Big(\frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z\Big)\right\| \\
&\leq (\|x_n - z\| + 4Mc_n + Md_n)\Big[1 - 2\beta_n(1 - \beta_n) \\
&\quad \times \delta_E\Big(\frac{1}{\alpha_n + \gamma_n} \cdot \frac{\|\alpha_n(T^n y_n - x_n) + \gamma_n(T^n y_n - u_n)\|}{\|x_n - z\| + 4Mc_n + Md_n}\Big)\Big].
\end{aligned}$$

Thus, by using $0 < a \leq \alpha_n \leq b < 1$, we obtain

$$\begin{aligned}
&2\beta_n a(\|x_n - z\| + 4Mc_n + Md_n)\delta_E\Big(\frac{1}{\alpha_n + \gamma_n} \cdot \frac{\|\alpha_n(T^n y_n - x_n) + \gamma_n(T^n y_n - u_n)\|}{\|x_n - z\| + 4Mc_n + Md_n}\Big) \\
&\leq 2\beta_n(1 - \beta_n)(\|x_n - z\| + 4Mc_n + Md_n)\delta_E\Big(\frac{1}{\alpha_n + \gamma_n} \cdot \frac{\|\alpha_n(T^n y_n - x_n) + \gamma_n(T^n y_n - u_n)\|}{\|x_n - z\| + 4Mc_n + Md_n}\Big) \\
&\leq \|x_n - z\| - \|x_{n+1} - z\| + 4Mc_n + Md_n.
\end{aligned}$$

Since

$$2a\sum_{n=1}^{\infty}\beta_n(\|x_n - z\| + 4Mc_n + Md_n)\delta_E\Big(\frac{1}{\alpha_n + \gamma_n} \cdot \frac{\|\alpha_n(T^n y_n - x_n) + \gamma_n(T^n y_n - u_n)\|}{\|x_n - z\| + 4Mc_n + Md_n}\Big) < \infty,$$

$\sup_{n\geq 1}\|T^n y_n - u_n\| < \infty$, and δ_E is strictly increasing and continuous, we obtain

$$\liminf_{n\to\infty}\|T^n y_n - x_n\| = 0. \tag{5}$$

Since

$$\begin{aligned}
\|T^n x_n - x_n\| &\le \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \\
&\le (1+c_n)\|x_n - y_n\| + \|T^n y_n - x_n\| \\
&= (1+c_n)\|x_n - \alpha'_n x_n - \beta'_n T^n x_n - \gamma'_n v_n\| + \|T^n y_n - x_n\| \\
&\le (1+c_n)\beta'_n \|T^n x_n - x_n\| + (1+c_n)\gamma'_n \|x_n - v_n\| + \|T^n y_n - x_n\| \\
&\le (1+c_n) b \|T^n x_n - x_n\| + (1+c_n)\gamma'_n \|x_n - v_n\| + \|T^n y_n - x_n\| \\
&= b\|T^n x_n - x_n\| + c_n b \|T^n x_n - x_n\| + (1+c_n)\gamma'_n \|x_n - v_n\| + \|T^n y_n - x_n\| \\
&\le b\|T^n x_n - x_n\| + c_n(2+c_n) b \|x_n - z\| + (1+c_n)\gamma'_n \|x_n - v_n\| + \|T^n y_n - x_n\|,
\end{aligned}$$

we obtain

$$\begin{aligned}
(1-b)\|T^n x_n - x_n\| &\le c_n(2+c_n) b \|x_n - z\| + (1+c_n)\gamma'_n \|x_n - v_n\| + \|T^n y_n - x_n\| \\
&\le c_n(2+c_n) bM + 2(1+c_n)\gamma'_n M + \|T^n y_n - x_n\|.
\end{aligned}$$

By using (4) and (5), we obtain

$$\liminf_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{6}$$

On the other hand, if $0 < a \le \beta_n \le 1$, $0 < a \le \alpha'_n \le b < 1$, $\sum_{n=1}^{\infty} \beta'_n = \infty$ for all $n \ge 1$ and some $a, b \in (0,1)$, then we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n - z\| \\
&\le \alpha_n \|x_n - z\| + \beta_n \|T^n y_n - z\| + \gamma_n \|u_n - z\| \\
&\le \alpha_n \|x_n - z\| + \beta_n (1+c_n)\|y_n - z\| + \gamma_n \|u_n - z\| \\
&\le \alpha_n \|x_n - z\| + \beta_n \|y_n - z\| + \beta_n c_n \|y_n - z\| + M\gamma_n \\
&= (1 - \beta_n - \gamma_n)\|x_n - z\| + \beta_n \|y_n - z\| + \beta_n c_n \|y_n - z\| + M\gamma_n \\
&\le (1 - \beta_n)\|x_n - z\| + \beta_n \|y_n - z\| + \beta_n c_n \|y_n - z\| + M\gamma_n
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{\|x_{n+1} - z\| - \|x_n - z\|}{\beta_n} &\le \|y_n - z\| - \|x_n - z\| + c_n \|y_n - z\| + M\frac{\gamma_n}{a} \\
&\le \|y_n - z\| - \|x_n - z\| + c_n\{\|x_n - z\| + Mc_n + M\gamma'_n\} + Md_n.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|x_n - z\| - \|y_n - z\| &\le \frac{\|x_n - z\| - \|x_{n+1} - z\|}{\beta_n} + c_n\{\|x_n - z\| + Mc_n + M\gamma'_n\} + Md_n \\
&\le \frac{\|x_n - z\| - \|x_{n+1} - z\|}{a} + c_n\{M(1+c_n) + M\gamma'_n\} + Md_n. \qquad (7)
\end{aligned}$$

Since

$$\begin{aligned}\|T^n x_n - z\| &\le (1+c_n)\|x_n - z\| \\ &\le \|x_n - z\| + Mc_n + Md_n\end{aligned}$$

and

$$\left\|\frac{\alpha'_n x_n}{\alpha'_n+\gamma'_n} + \frac{\gamma'_n v_n}{\alpha'_n+\gamma'_n} - z\right\| \le \|x_n - z\| + Mc_n + Md_n,$$

we obtain

$$\begin{aligned}\|y_n - z\| &= \|\alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n - z\| \\ &= \left\|\beta'_n(T^n x_n - z) + (1-\beta'_n)\Big(\frac{\alpha'_n x_n}{\alpha'_n+\gamma'_n} + \frac{\gamma'_n v_n}{\alpha'_n+\gamma'_n} - z\Big)\right\| \\ &\le (\|x_n - z\| + Mc_n + Md_n)\Big[1 - 2\beta'_n(1-\beta'_n) \\ &\quad \times \delta_E\Big(\frac{1}{\alpha'_n+\gamma'_n}\cdot\frac{\|\alpha'_n(T^n x_n - x_n) + \gamma'_n(T^n x_n - v_n)\|}{\|x_n - z\| + Mc_n + Md_n}\Big)\Big]. \qquad (8)\end{aligned}$$

By using (7), (8) and $0 < a \le \alpha'_n \le b < 1$, we obtain

$$\begin{aligned}&2\beta'_n a(\|x_n - z\| + Mc_n + Md_n)\delta_E\Big(\frac{1}{\alpha'_n+\gamma'_n}\cdot\frac{\|\alpha'_n(T^n x_n - x_n) + \gamma'_n(T^n x_n - v_n)\|}{\|x_n - z\| + Mc_n + Md_n}\Big) \\ &\le 2\beta'_n(1-\beta'_n)(\|x_n - z\| + Mc_n + Md_n)\delta_E\Big(\frac{1}{\alpha'_n+\gamma'_n}\cdot\frac{\|\alpha'_n(T^n x_n - x_n) + \gamma'_n(T^n x_n - v_n)\|}{\|x_n - z\| + Mc_n + Md_n}\Big) \\ &\le \|x_n - z\| - \|y_n - z\| + Mc_n + Md_n \\ &\le \frac{\|x_n - z\| - \|x_{n+1} - z\|}{a} + c_n\{M(1+c_n) + M\gamma'_n\} + Md_n + Mc_n + Md_n \\ &= \frac{\|x_n - z\| - \|x_{n+1} - z\|}{a} + c_n\{M(2+c_n) + M\gamma'_n\} + 2Md_n.\end{aligned}$$

Hence

$$2a\sum_{n=1}^{\infty}\beta'_n(\|x_n - z\| + Mc_n + Md_n)\delta_E\Big(\frac{1}{\alpha'_n+\gamma'_n}\cdot\frac{\|\alpha'_n(T^n x_n - x_n) + \gamma'_n(T^n x_n - v_n)\|}{\|x_n - z\| + Mc_n + Md_n}\Big) < \infty.$$

We also obtain (6) similarly to the arguement above. By using Lemma 3, we obtain $\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$. □

Our Theorem 2 improves Theorem 1.5 of Schu [6], Theorem 2.2 of Schu [7] and Theorem 3 of Rhoades [5] to a more general Ishikawa type scheme under much less restrictions on the iterative parameters $\{\alpha_n\}$ and $\{\beta_n\}$.

Theorem 2. *Let E be a uniformly convex Banach space, and let C be a nonempty bounded closed convex subset of E, and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \ge 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and let T^m be compact for some $m \in \mathbf{N}$. If*

$x_1 \in C$, *and the sequence* $\{x_n\}$ *defined by* (3) *satisfies* $0 < a \le \alpha_n \le b < 1$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $0 \le \beta'_n \le b < 1$ *for all* $n \ge 1$ *and some* $a, b \in (0,1)$ *or* $0 < a \le \beta_n \le 1$, $0 < a \le \alpha'_n \le b < 1$, $\sum_{n=1}^{\infty} \beta'_n = \infty$ *for all* $n \ge 1$ *and some* $a, b \in (0,1)$, *then* $\{x_n\}$ *converges strongly to some fixed point of* T.

Proof. From Theorem 1, there exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that

$$\lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0. \tag{9}$$

Since

$$\begin{aligned}\|T^m x_{n_k} - x_{n_k}\| &\le \|T^m x_{n_k} - T^{m-1}x_{n_k}\| + \|T^{m-1}x_{n_k} - T^{m-2}x_{n_k}\| + \cdots + \|Tx_{n_k} - x_{n_k}\| \\ &\le \|Tx_{n_k} - x_{n_k}\| \sum_{j=1}^{m-1} k_j + \|Tx_{n_k} - x_{n_k}\|,\end{aligned}$$

we obtain

$$\lim_{k\to\infty} \|x_{n_k} - T^m x_{n_k}\| = 0.$$

Since T^m is compact, there exist a subsequence $\{x_{n_{k_i}}\}$ of the sequence $\{x_{n_k}\}$ and a point $p \in C$ such that $x_{n_{k_i}} \to p$. Thus we obtain $p \in F(T)$ by the continuity of T and (9). Hence we obtain $\lim_{n\to\infty} \|x_n - p\| = 0$ by Lemma 4. □

Our Theorem 3 improves Theorem 1.5 of Schu [6], Theorem 2.2 of Schu [7] and Theorem 3 of Rhoades [5] under much less restrictions on the iterative parameters $\{\alpha_n\}$ and $\{\beta_n\}$.

Theorem 3. *Let* E *be a uniformly convex Banach space, and let* C *be a nonempty bounded closed convex subset of* E, *and let* $T : C \to C$ *be an asymptotically nonexpansive mapping with* $\{k_n\}$ *satisfying* $k_n \ge 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, *and let* T^m *be compact for some* $m \in \mathbb{N}$. *If* $x_1 \in C$, *and the sequence* $\{x_n\}$ *defined by* (1) *satisfies* $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, $0 \le \beta_n \le b < 1$ *for all* $n \ge 1$ *and some* $b \in (0,1)$ *or* $0 < a \le \alpha_n \le 1$, $\sum_{n=1}^{\infty} \beta_n(1-\beta_n) = \infty$ *for all* $n \ge 1$ *and some* $a \in (0,1)$, *then* $\{x_n\}$ *converges strongly to some fixed point of* T.

As a direct consequence, taking $\beta'_n = 0$ and $\gamma'_n = 0$ for $n \in \mathbb{N}$ in Theorem 2, we obtain the following result, which improves Theorem 2.2 of Schu [7] and Theorem 2 of Rhoades [5] under much less restrictions on the iterative parameter $\{\alpha_n\}$.

Theorem 4. *Let* E *be a uniformly convex Banach space, and let* C *be a nonempty bounded closed convex subset of* E, *and let* $T : C \to C$ *be an asymptotically nonexpansive mapping with* $\{k_n\}$ *satisfying* $k_n \ge 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, *and let* T^m *be compact for some* $m \in \mathbb{N}$. *Suppose that* $x_1 \in C$, *and the sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = \alpha_n x_n + \beta_n T^n x_n + \gamma_n u_n,$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying* $0 < a \le \alpha_n \le b < 1$ *for some* $a, b \in (0,1)$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \ge 1$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\{u_n\}$ *is a sequence in* C *Then* $\{x_n\}$ *converges strongly to some fixed point of* T.

Remark. *If* $\{\alpha_n\}$ *is bounded away from both* 0 *and* 1, *i.e.,* $a \le \alpha_n \le b$ *for all* $n \ge 1$ *and some* $a, b \in (0,1)$, *then* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ *hold. However, the converse is not true.*

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