数理解析研究所講究録 1365 巻 2004 年 41-47

DEFINITIONS OF ORDER COMPLETENESS IN ORDERED LINEAR SPACES

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ABSTRACT

In this paper we define a notion of weakly order completeness in ordered linear spaces. This plays important roles in dealing with the generalized supremum. We will consider the relation between the weakly order completeness and some other definitions of order completeness. Also we will give some examples in sequence spaces with order.

§1 INTRODUCTION AND THE GENERALIZED SUPREMUM

Let E be a linear space over $\mathbb{R}$, and P be a convex cone in E satisfying

(P1) $E = P - P$,

(P2) $P \cap (-P) = \{0\}$.

By (E, P), we denote an ordered linear space with the order $x \leq y \Longleftrightarrow y - x \in P$. For a subset A of E, we denote the set of upper bounds and lower bounds by $U(A) = \{x \in E \mid y \leq x,\ \forall y \in A\}, L(A) = \{x \in E \mid y \geq x,\ \forall y \in A\}$ respectively. (E, P) is said to be order complete if $U(A) \neq \emptyset$ implies the existence of the least upper bound of A (lub A). In this note we consider some weaker conditions which can be regarded as the definitions of order completeness in wider sense.

Let $\mathfrak{B}$ ($\mathfrak{B}'$) be the family of all upper bounded subset (lower bounded subset) in E, i.e. $\mathfrak{B} = \{A \subset E \mid A \neq \emptyset,\ U(A) \neq \emptyset\}$, $\mathfrak{B}' = \{B \subset E \mid B \neq \emptyset,\ L(B) \neq \emptyset\}$.
For $A \in \mathfrak{B}$, and $A' \in \mathfrak{B}'$ the generalized supremum and the generalized infimum are defined by

$$\operatorname{Sup} A = \{a \in U(A) | \, b \leq a,\ \ b \in U(A) \Longrightarrow a = b\} \quad (A \in \mathfrak{B}),$$
$$\operatorname{Inf} A' = \{a \in L(A') | \, b \geq a,\ \ b \in L(A') \Longrightarrow a = b\} \quad (A' \in \mathfrak{B}').$$

The properties of generalized supremum has been investigated in [1],[2],[4]. The main result among them is the following. We denote $\tilde{E} = \{\operatorname{Sup} A \mid A \in \mathfrak{B}\}$, and define an order relation '$\leq$', and a vector operation '$\oplus$' and '$*$' on $\tilde{E}$ as follows. For $\operatorname{Sup} A$, $\operatorname{Sup} B \in \tilde{E}$ and $\lambda \in \mathbb{R}$,

$$\operatorname{Sup} A \leq \operatorname{Sup} B \Leftrightarrow \operatorname{Sup} B \subset \operatorname{Sup} A + P$$
$$\operatorname{Sup} A \oplus \operatorname{Sup} B = \operatorname{Sup}(A + B)$$
$$\lambda * \operatorname{Sup} A = \begin{cases} \operatorname{Sup}(\lambda A) & (\lambda > 0) \\ \{0\} & (\lambda = 0) \\ \operatorname{Sup}(\lambda U(A)) & (\lambda < 0). \end{cases}$$
$$\tilde{P} = \{\operatorname{Sup} A \in \tilde{E} \mid \operatorname{Sup} A \subset P\}$$
$$\tilde{E}_1 = \{\operatorname{Sup} A \in \tilde{E} \mid \operatorname{Sup} A = \{a_0\} \text{ for some } a_0 \in E\}$$

We consider the case when the space (E,P) has the property that for each element x there exsists $y \in \operatorname{Sup} A$ such that $y \leq x$, i.e.

$$(1.1) \qquad U(A) = (\operatorname{Sup} A) + P \qquad (\forall A \in \mathfrak{B}).$$

Proposition 1. ([2]) *Let E be a Banach space with a closed positive cone P. If (E,P) has the property* (1.1), *then $\tilde{E}$, forms an order complete vector lattice. Moreover,*

(a) *$\tilde{P}$ is a convex cone in $\tilde{E}$ and satisfies* (P1), (P2), *and* $\operatorname{Sup} A \leq \operatorname{Sup} B \Longleftrightarrow \operatorname{Sup} B \oplus (-1) * \operatorname{Sup} A \in \tilde{P}$.

(b) *$\tilde{E}_1$ is a subspace which is order isomorphic to (E,P) by* $E \ni a \longleftrightarrow \operatorname{Sup} A = \{a\} \in \tilde{E}_1$

Corollary 1. *For* $\operatorname{Sup} A$, $\operatorname{Sup} B \in \tilde{E}$,

(a) $\operatorname{Sup} A \vee \operatorname{Sup} B = \operatorname{Sup}(L(U(A) \cap U(B)))$,

(b) $\operatorname{Sup} A \wedge \operatorname{Sup} B = \operatorname{Sup}(L(U(A)) \cap L(U(B)))$.

Not only this result, but also many good properties of the generalized supremum holds under the condition (1.1) on the space (E,P). If the generalized supremum $\operatorname{Sup} A$ permits only what satisfies $U(A) = (\operatorname{Sup} A) + P$, it is natural to consider that the condition (1.1) is one of the conditions for order completeness of (E,P) of wide sense.

In §2, we introduce some types of definitions of order completeness, and state the relations among these definitions including the condition (1.1). In §3, we consider some examples in sequence spaces which are not order complete but satisfy the condition (1.1).

§2 Definitions of order completeness

We say that an ordered linear space (E,P) is **weakly order complete** (w.o.c.) if every subset A of E with $U(A) \neq \emptyset$ has the generalized supremum $\operatorname{Sup} A$ satisfying $U(A) = (\operatorname{Sup} A) + P$. As mentioned in §1, the collection of all the generalized supremum forms the order completion of (E,P) if it is w.o.c. Moreover, in such a space we have the following.

Proposition 2. ([1]) *If an ordered linear space (E,P) is weakly order complete, then*
(1) $\operatorname{Sup} A = \{a\}$ *if and only if* $\operatorname{lub} A = a$,
(2) $\operatorname{Sup} \operatorname{Inf} \operatorname{Sup} A = \operatorname{Sup} A$,
(3) $\operatorname{Sup}(A+B) + P \supset \operatorname{Sup} A + \operatorname{Sup} B$,
(4) $L(\operatorname{Sup} A + \operatorname{Sup} B) = L(\operatorname{Sup}(A+B))$.

One of the sufficient conditions for the weakly order completeness is given in terms of the facial structure of the positive cone P. We suppose that P is algebraically closed, that is, every straight line in E meets P by a closed interval. A point x of a convex subset $A \subset E$ is called an algebraic interior point of A if for every $z \in E$, there exists $\lambda > 0$ such that $x + \lambda z \in A$. A convex set C of P is called an exposed face of P if there exists a supporting hyperplane H of P such that $C = P \cap H$. By $\mathfrak{F}(P)$, we denote the set of all exposed faces of P. For $C \in \mathfrak{F}(P)$, $\dim C$ is defined as the dimension of aff$C$ where aff$C$ denotes the affine hull of C. Let (E,P) be an ordered linear space with algebraically closed positive cone P, and suppose that P has at least an algebraic interior point. It has been proved in [4] that if $\dim C < \infty$ for every $C \in \mathfrak{F}(P)$, then (E,P) is weakly order complete. In particular, in finite dimensional cases, (E,P) is w.o.c. if and only if P is closed. Some other sufficient conditions for weakly order completeness in ordered Banach spaces are given in [2].

An ordered linear space (E, P) is said to be **monotone order complete** (m.o.c.) if every upper bounded totally ordered subset of E has the least upper bound in E. In finite dimensional cases $(\mathbb{R}^d, P)$ is m.o.c. if and only if P is closed ([4]). Moreover, in the case when (E, P) is a Banach space and the dual cone $P^* = \{x^* \mid \ < x^*, x > \geq 0 \text{ for } x \in P\}$ in E^* satisfies $P^* - P^* = E^*$, (E^*, P^*) is m.o.c. It is also known that the algebraic closedness of P in an ordered linear space (E, P) is a necessary condition for the monotone order completeness.

Proposition 3. ([4]) *Suppose that an ordered linear space (E, P) is monotone order complete. Then it is weakly order complete.*

As an example, the space of $n \times n$ symmetric matrices with the positive cone P consisting of all positive semidefinite matrices is m.o.c., because P is closed. Hence it is also w.o.c. by Proposition 3.

By $(E, P, \| \ \|)$ we denote an ordered linear space which is also a normed space. $(E, P, \| \ \|)$ is said to be **boundedly order complete** (b.o.c.) if every $\| \ \|$-bounded increasing net A in E has a least upper bound $\operatorname{lub} A$. The positive cone P is said to be **normal** if there is a neighborhood basis of the origin consisting of neighborhoods V satisfying $(V + P) \cap (V - P) = V$. Let $(E, P, \| \ \|)$ be a normed space with a normal positive cone P, and let A be a totally ordered subset in E such that $U(A) \neq \emptyset$. For $a_0 \in A$ the set $A' = \{a \in A \mid a_0 \leq a\}$ has the same upper bounds as $U(A)$, and $A' \subset [a_0, u] = \{x \in E \mid a_0 \leq x \leq u\}$ for some $u \in U(A)$. Hence A' is $\| \ \|$-bounded since P is normal. If $(E, P, \| \ \|)$ is b.o.c., there exists $\operatorname{lub} A' = \operatorname{lub} A$. Thus we have

Proposition 4. *If $(E, P, \| \ \|)$ is a normed space with a normal positive cone P, then the boundedly order completeness implies the monotone order completeness. Particularly, the weakly order completeness also follows.*

§3 Examples in sequence spaces

In the case $E = \mathbb{R}^3$, the two positive cone $P_1 = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq |x| + |y|\}$ and $P_2 = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 \geq x^2 + y^2\}$ are fundamental in considering the generalized supremum. They are not order complete but weakly order complete. $(\mathbb{R}^3, P_2)$ is order isomorphic to the space of 2×2 symmetric matrices with the positive cone P consisting of all positive semidefinite matrices. Since P_2 is a circular cone, every nontrivial face of P_2 is one dimensional. Moreover, the order completion of $(\mathbb{R}^3, P_2)$ is infinite dimensional while that of $(\mathbb{R}^3, P_1)$ is four dimensional. In this section we consider some examples in sequence spaces with the positive cones which are considered to be the natural extension of P_1 and P_2 in $\mathbb{R}^3$.

Let $l_1 = \{x = (x_0, x_1, x_2, \cdots) \mid \Sigma_{n=0}^{\infty} |x_n| < \infty\}$ and $l_2 = \{x = (x_0, x_1, x_2, \cdots) \mid \Sigma_{n=0}^{\infty} x_n^2 < \infty\}$. We define two cones;

$$P_1 = \{x = (x_0, x_1, x_2, \cdots) \in l_1 \mid x_0 \geq \sum_{n=1}^{\infty} |x_n|\},$$

$$P_2 = \{x = (x_0, x_1, x_2, \cdots) \in l_2 \mid x_0 \geq (\sum_{n=1}^{\infty} x_n^2)^{\frac{1}{2}}\}.$$

There is a face of P_1 which is infinite dimensional. Indeed, $H = \{(x_0, x_1, x_2, \cdots) \in l_1 \mid x_0 = \Sigma_{n=1}^{\infty} x_n\}$ is a supporting hyperplane of P_1 and the face $F = H \cap P_1$ is infinite dimensional. In contrast, every nontrivial face of (l_1, P_2) is one dimensional, and hence (l_2, P_2) and (l_1, P_2) are weakly order complete. Here in (l_1, P_2) we consider the positive cone to be $P_2 \cap l_1$.

Proposition 5. *(l_1, P_1) is m.o.c., and it is weakly order complete in particular.*

An ordered linear space (E, P) is said to be sequentially monotone order complete (s.m.o.c.) if every totally ordered countable subset A of E with $U(A) \neq \emptyset$ has the least upper bound $\operatorname{lub} A$ in E. This condition is slightly weaker than the monotone order completeness in general.

Proposition 6. *For every upper bounded totally ordered subset A in (l_1, P_1), there exsits a countable subset $\{a_n\}_{n=1}^{\infty}$ of A such that $U(A) = U(\{a_n\})$.*

proof. We write $A = \{a_\lambda = (a_{\lambda 0}, a_{\lambda 1}, a_{\lambda 2}, \cdots) \mid \lambda \in \Lambda\}$, and let $(b_0, b_1, b_2, \cdots)$ be an upper bound of A. Since $a_{\lambda 0} \leq b_0 \quad (\lambda \in \Lambda)$, there exists $a_0 = \sup a_{\lambda 0}$. If there exists $a_\lambda = (a_{\lambda 0}, a_{\lambda 1}, a_{\lambda 2}, \cdots) \in A$ such that $a_{\lambda 0} = a_0$, then a_λ is the maximum of A and the lemma is trivial. Hence we assume that $a_{\lambda 0} < a_0 \quad (\lambda \in \Lambda)$. We can choose a sequence $\lambda_1, \lambda_2, \cdots$ such that $\{a_{\lambda_n}\}_{n=1}^{\infty}$ is nondecreasing and $a_{\lambda_n} \longrightarrow a_0$. For arbitrary $a_\lambda = (a_{\lambda 0}, a_{\lambda 1}, a_{\lambda 2}, \cdots) \in A$, there exists $n \in \mathbb{N}$ such that $a_\lambda \leq a_{\lambda_n}$, and this means that $U(A) = U(\{a_{\lambda_n}\})$.

proof of Proposition 5. By Proposition 6, it suffices to show that (l_1, P_1) is s.m.o.c. Let $a_m = (a_{m0}, a_{m1}, a_{m2}, \cdots)$ $(m = 1, 2, 3, \cdots)$ be an upper bounded increasing sequence in (l_1, P_1), and let $(b_0, b_1, b_2, \cdots)$ be an upper bound of $\{a_m\}$. Since $\{a_{m0}\}_{m=1}^{\infty}$ is nondecreasing and $a_{m0} \leq b_0$ $(m = 1, 2, \cdots)$, it is a convergent sequence. Moreover, $a_m \leq a_n$ $(1 \leq m \leq n)$ implies

$$a_{n0} - a_{m0} \geq \sum_{i=1}^{\infty} |a_{ni} - a_{mi}| \quad (1 \leq m \leq n). \tag{3.1}$$

Hence, for each $i = 1, 2, \cdots$, $\{a_{ni}\}_{n=1}^{\infty}$ is a convergent sequence. Thus we can define $a_0 = (a_{00}, a_{01}, a_{02}, \cdots)$ by $a_{0i} = \lim_{n \to \infty} a_{ni} \quad (i = 0, 1, 2 \cdots)$. By (3.1), we have for each $N = 1, 2, \cdots$,
$a_{n0} - a_{m0} \geq \Sigma_{i=1}^{N} |a_{ni} - a_{mi}| \quad (1 \leq m \leq n)$. Hence we obtain by letting $n \longrightarrow \infty$ that $a_{00} - a_{m0} \geq \Sigma_{i=1}^{N} |a_{0i} - a_{mi}| \quad (m, N \in \mathbb{N})$. Since $N \in \mathbb{N}$ is arbitrary and $a_m \in l_1$, this inequality yields that $a_0 \in l_1$ and

$$a_{00} - a_{m0} \geq \sum_{i=1}^{\infty} |a_{0i} - a_{mi}| \quad (m \in \mathbb{N}).$$

This means $a_0 \geq a_m \quad (m \in \mathbb{N})$, and $a_0 \in U(\{a_m\})$. It remains to prove that a_0 is the minimum of $U(\{a_m\})$. For $b = (b_0, b_1, b_2, \cdots) \in U(\{a_m\})$, we have

$$b_0 - a_{m0} \geq \sum_{i=1}^{N} |b_i - a_{mi}| \quad (m, N \in \mathbb{N}).$$

Letting $m \longrightarrow \infty$, we obtain $b_0 - a_{00} \geq \sum_{i=1}^{N} |b_i - a_{0i}| \quad (N \in \mathbb{N})$. Since $N \in \mathbb{N}$ is arbitrary we also have $b_0 - a_{00} \geq \sum_{i=1}^{\infty} |b_i - a_{0i}|$. This means $b \geq a_0$ and the proof is complete.

Proposition 7. (l_1, P_2) *is not m.o.c.*

Remark As mentioned at the beginning of §3, (l_1, P_2) is weakly order complete. Moreover, the proof of this proposition answers a natural question; $\operatorname{Sup} A$ consists of at most a single element if A is tatally ordered subset in E ? For a cirtain totally ordered subset A in the following proof, one can see that for every $b \in U(A)$ there is $b' \in U(A)$ such that $b \not\leq b'$ and $b \not\geq b'$. This fact means that the generalized supremum $\operatorname{Sup} A$ has at least two elements.

proof. Let us consider the convergent series $\Sigma_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. First we show that there is a subsequence $\{n_k\}_{k=0}^{\infty}$ of the sequence $1, 2, 3, \cdots$ such that $n_0 = 1$ and

$$\begin{aligned} S &= \sqrt{A_1} + \sqrt{A_2} + \sqrt{A_3} + \cdots \\ &< +\infty, \end{aligned}$$

$$\text{where } A_k = \frac{1}{(n_{k-1}+1)^2} + \frac{1}{(n_{k-1}+2)^2} + \cdots + \frac{1}{{n_k}^2} \qquad (\, k = 1,\ 2,\ 3,\ \cdots).$$

Indeed, if we choose the subsequence $\{n_k\}_{k=0}^{\infty}$ by $n_k = 2^k$ $(k = 0, 1, 2, 3, \cdots)$, then

$$\begin{aligned} A_k &= \frac{1}{(2^{k-1}+1)^2} + \frac{1}{(2^{k-1}+2)^2} + \cdots + \frac{1}{2^{2k}} \\ &\leq \frac{1}{2^{2(k-1)}} + \frac{1}{2^{2(k-1)}} + \cdots + \frac{1}{2^{2(k-1)}} = \frac{1}{2^{k-1}}. \end{aligned}$$

Hence we have

$$\sum_{k=1}^{\infty} \sqrt{A_k} \leqq \sum_{k=1}^{\infty} \sqrt{\frac{1}{2^{k-1}}} < +\infty.$$

Now we define a sequence $\{a_n\}_{n=0}^{\infty}$ in l_1 by

$$\begin{aligned} a_0 &= (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \cdots\cdots\cdots\cdots\cdots), \\ a_1 &= (S_1,\ \frac{1}{2},\ \cdots,\ \frac{1}{n_1},\ 0,\ 0,\ 0,\ 0,\ \cdots\cdots\cdots\cdots\cdots\cdots), \\ a_2 &= (S_2,\ \frac{1}{2},\ \cdots,\ \frac{1}{n_1},\ \frac{1}{n_1+1},\ \cdots,\ \frac{1}{n_2},\ 0,\ 0,\ 0,\ \cdots\cdots\cdots\cdots), \\ a_3 &= (S_3,\ \frac{1}{2},\ \cdots,\ \frac{1}{n_1},\ \frac{1}{n_1+1},\ \cdots,\ \frac{1}{n_2},\ \frac{1}{n_2+1},\ \cdots,\ \frac{1}{n_3},\ 0,\ 0,\ \cdots\cdots\cdots), \\ &\ \vdots \\ b_0 &= (2S,\ 0,\ 0,\ 0,\ 0\ \cdots\cdots\cdots), \end{aligned}$$

where $S_n = \sum_{k=1}^{n} \sqrt{A_k}$ $(n = 1, 2, \cdots)$. Since $\sum_{k=1}^{n} A_k \leq S_n^2$ for every n, we see that

$$\frac{\pi^2}{6} - 1 < S^2. \tag{3.2}$$

By the definition of A_k, we have $\sqrt{A_k} = (\frac{1}{(n_{k-1}+1)^2} + \frac{1}{(n_{k-1}+2)^2} + \cdots + \frac{1}{{n_k}^2})^{\frac{1}{2}}$ $(\, k = 1,\ 2,\ 3,\ \cdots)$. Therefore,

$$\begin{aligned} a_k - a_{k-1} &= (\sqrt{A_k},\ 0,\ \cdots\ 0,\ \frac{1}{n_{k-1}+1},\ \cdots,\ \frac{1}{n_k},\ 0,\ \cdots) \\ &\in P_2 \quad (\, k = 1,\ 2,\ 3,\ \cdots). \end{aligned}$$

Moreover, by (3.2), $(2S-S_k)^2-(\frac{1}{2})^2-\cdots-(\frac{1}{n_k})^2=(2S-S_k)^2-A_1-A_2-\cdots-A_k \geqq S^2-A_1-A_2-\cdots-A_k > \frac{\pi^2}{6}-1-A_1-A_2-\cdots-A_k>0$, it follows that

$$b_0-a_k=(2S-S_k,\ -\frac{1}{2},\ \cdots,-\frac{1}{n_k},\ 0,\ \cdots)\in P_2,$$

for every $k\in\mathbb{N}$. Hence the sequence $\{a_k\}$ is increasing and upper bounded in (l_1,P_2). Let $b=(b_1,b_2,b_3,\cdots)$ be an arbitrary element in $U(\{a_k\})$. Since $b\in l_1$, there is at least a number $n\in\mathbb{N}$ such that $b_n\neq\frac{1}{n}$. We define

$$b'=(b_1,\ b_2,\ b_3,\ \cdots,\ b_{n-1},\ \frac{1}{n},\ b_{n+1},\ \cdots),$$

then $b-b'=(0,\ 0,\ \cdots,\ b_n-\frac{1}{n},\ 0,\ 0,\ \cdots)\notin P_2\cup(-P_2)$. This means that b and b' are not comparable with respect to the order of P_2. Moreover, it follows from the relation $b\geq a_k$ $(k=0,1,2,\cdots)$ that

$$\begin{aligned}0 &\leqq (b_1-S_k)^2-(b_2-\frac{1}{2})^2-(b_3-\frac{1}{3})^2-\cdots\\&\quad -(b_{n-1}-\frac{1}{n-1})^2-(b_n-\frac{1}{n})^2-(b_{n+1}-\frac{1}{n+1})^2-\cdots\\&\leqq (b_1-S_k)^2-(b_2-\frac{1}{2})^2-(b_3-\frac{1}{3})^2-\cdots\\&\quad -(b_{n-1}-\frac{1}{n-1})^2-(b_{n+1}-\frac{1}{n+1})^2-\cdots,\end{aligned}$$

for sufficiently large k. This means $b'\geq a_k$ $(k=0,1,2,\cdots)$. Thus we find that b is not the minimum of $U(\{a_k\})$, and since b is arbitrary it follows that $\mathrm{lub}\{a_k\}$ does not exist.

Let $(E,P,\|\ \|)$ is a normed space with a positive cone P. P is said to be a **strict b-cone** if there is a constant $M>0$ such that each $x\in E$ has a decomposition $x=y-z$ where $y,z\in P$ and $\|y\|,\|z\|\leq M\|x\|$.

Proposition 8. *Let $(E,P,\|\ \|)$ is a normed space with a strict b-cone P, and suppose that every order interval $[x,y]=\{z\in E\mid x\leq z\leq y\}$ is $\|\ \|$ bounded. If $(E,P,\|\ \|)$ is boundedly order complete then E is complete with respect to the norm $\|\ \|$.*

Proposition 9. *$(l_1,P_2,\|\ \|_2)$ is not b.o.c. where $\|x\|_2=\{\sum_{n=0}^{\infty}x_n^2\}^{\frac{1}{2}}$.*

proof. For $x=(x_0,x_1,x_2,\cdots)\in l_2$, we take $y=(\alpha,x_1,x_2,\cdots)$ and $z=(\alpha-x_0,0,0,\cdots)$ where $\alpha=\{\sum_{n=1}^{\infty}x_n^2\}^{\frac{1}{2}}$. Clearly, $x,y\in P_2$ and $x=y-z$. It is easy to see that $\|y\|_2,\|z\|_2\leq 2\|x\|_2$. Hence P_2 is a strict b-cone in $(l_1,P_2,\|\ \|_2)$. Next for $y=(y_0,y_1,y_2,\cdots)\in P_2$, we take $x=(x_0,x_1,x_2,\cdots)\in[0,y]$ arbitrarily, and put $x'=(x_1,x_2,x_3,\cdots)$, $y'=(y_1,y_2,y_3,\cdots)$. Since $0\leq x_0\leq y_0\leq\|y\|_2$, we have $\|x'\|_2-\|y'\|_2\leq\|x'-y'\|_2\leq y_0-x_0\leq y_0\leq\|y\|_2$. Hence $\|x'\|_2\leq\|y\|_2+\|y'\|_2\leq 2\|y\|_2$, and $\|x\|_2\leq x_0+\|x'\|_2\leq 3\|y\|_2$. This means that $[0,y]$ is $\|\ \|_2$ bounded, and so is every order interval. If $(l_1,P_2,\|\ \|_2)$ is b.o.c., it follows from Proposition 8 that it must be complete with respect to the norm $\|\ \|_2$. But it is not, and a contradiction yields.

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