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Uniqueness of solutions with prescribed numbers of zeros for two-point boundary value problems

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We consider the second order ordinary differential equation

$$u'' + a(x)f(u) = 0, \quad x_0 < x < x_1 \tag{1}$$

with the boundary condition

$$u(x_0) = u(x_1) = 0, \tag{2}$$

where $a \in C^2[x_0, x_1]$, $a(x) > 0$ for $x \in [x_0, x_1]$, $f \in C^1(\mathbf{R})$, $f(s) > 0$, $f(-s) = -f(s)$ for $s > 0$.

By a change of variable, it can be shown that the existence of solutions of the problem (1) and (2) is equivalent to the existence of radial solutions of the following Dirichlet problem for elliptic equations in annular domains

$$\begin{cases} \Delta u + K(|x|)f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $K \in C^1[R_1, R_2]$, $\Omega = \{x \in \mathbf{R}^N : R_1 < |x| < R_2\}$, $R_1 > 0$ and $N \geq 2$. (See, for example, [8])

Note that if u is a solution of (1), so is $-u$, because of $f(-s) = -f(s)$. Hence we consider solutions u of the problem (1) and (2) with $u'(x_0) > 0$ only.

In this paper we study the uniqueness of solutions of the problem (1) and (2) having exactly $k-1$ zeros in (x_0, x_1), and hence consider the following problem:

$$(\mathrm{P}_k) \qquad \begin{cases} u'' + a(x)f(u) = 0, \quad x_0 < x < x_1, \\ u(x_0) = u(x_1) = 0, \quad u'(x_0) > 0, \\ u \text{ has exactly } k-1 \text{ zeros in } (x_0, x_1), \end{cases}$$

where k is a positive integer.

For existence of solutions of (P_k), we refer to [1], [2], [3], [6], [7], [8]. In particular we shall describe the result in [7]. We thus assume that there exist limits f_0 and f_∞ such that $0 \leq f_0, f_\infty \leq \infty$,

$$f_0 = \lim_{s \to +0} \frac{f(s)}{s} \quad \text{and} \quad f_\infty = \lim_{s \to \infty} \frac{f(s)}{s}.$$

Let λ_k be the k-th eigenvalue of

$$\begin{cases} \varphi'' + \lambda a(x)\varphi = 0, & x_0 < x < x_1, \\ \varphi(x_0) = \varphi(x_1) = 0. \end{cases}$$

It is known that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty.$$

The following Theorem A has been obtained in [7].

Theorem A. *Let $k \in \mathbf{N} = \{1, 2, \ldots\}$. Then the following* (i) *and* (ii) *holds:*

(i) *if $f_0 < \lambda_k < f_\infty$ or $f_\infty < \lambda_k < f_0$, then* ($\mathrm{P}_k$) *has at least one solution;*

(ii) *if $f(s)/s < \lambda_k$ for $s > 0$ or $f(s)/s > \lambda_k$ for $s > 0$, then* (P_k) *has no solution.*

Now we consider the uniqueness of solutions of (P_k).

Assume moreover that either the following (F1) or (F2) holds:

$$\text{(F1)} \quad \left(\frac{f(s)}{s}\right)' > 0 \quad \text{for } s > 0; \qquad \text{(F2)} \quad \left(\frac{f(s)}{s}\right)' < 0 \quad \text{for } s > 0.$$

The functions

$$f(s) = |s|^{p-1}s \quad (p > 1) \quad \text{and} \quad f(s) = \frac{s}{1 + |s|^q} \quad (q > 1)$$

are typical cases satisfying (F1) and (F2), respectively. From (F1) and (F2) it follows that $f(s)/s$ is monotone function, and hence we note that the limits f_0 and f_∞ exist in $[0, \infty]$.

For the uniqueness of the solutions of (P_k), the following Theorems B–D were obtained.

Theorem B (Coffman [1]). *Let $k \in \mathbf{N}$, $\nu \in \mathbf{R}$ and $p > 1$. Then the solution of the following problem exists and is unique:*

$$\begin{cases} u'' + x^\nu |u|^{p-1}u = 0, \quad 0 < x_0 < x < x_1, \\ u(x_0) = u(x_1) = 0, \quad u'(x_0) > 0, \\ u \textit{ has exactly } k-1 \textit{ zeros in } (x_0, x_1). \end{cases}$$

Theorem C (Coffman–Marcus [2]). *Let $k \in \mathbf{N}$ and $\sigma \in \mathbf{R}$. Suppose that f satisfies* (F1), *$f_0 = 0$ and $f_\infty = \infty$. Then the solution of the following problem exists and is unique:*

$$\begin{cases} u'' + x^{-2-\sigma} f(x^\sigma u) = 0, \quad 0 < x_0 < x < x_1, \\ u(x_0) = u(x_1) = 0, \quad u'(x_0) > 0, \\ u \textit{ has exactly } k-1 \textit{ zeros in } (x_0, x_1). \end{cases}$$

Theorem D (Yanagida [9]). *Let $k \in \mathbf{N}$. Suppose that*

$$q \in C^1[x_0, x_1], \quad q(x) > 0 \quad \text{for} \;\; x_0 \le x \le x_1.$$

Assume moreover that either the following (i) *or* (ii) *holds*:

(i) (F1) *holds and $q'(x)/q(x)$ is nonincreasing in $x \in [x_0, x_1]$*;
(ii) (F2) *holds and $q'(x)/q(x)$ is nondecreasing in $x \in [x_0, x_1]$.*

Then the problem

$$\begin{cases} u'' + h(q(x)u)u = 0, & x_0 < x < x_1, \\ u(x_0) = u(x_1) = 0, & u'(x_0) > 0, \\ u \text{ has exactly } k-1 \text{ zeros in } (x_0, x_1) \end{cases}$$

has at most one solution, where $h(s) = f(s)/s$.

Main results in this paper as follows.

Theorem 1. *Let $k \in \mathbf{N}$. Assume that either the following* (C1) *or* (C2) *holds*:
(C1) (F1) *holds and $([a(x)]^{-\frac{1}{2}})'' \le 0$ for $x_0 \le x \le x_1$;*
(C2) (F2) *holds and $([a(x)]^{-\frac{1}{2}})'' \ge 0$ for $x_0 \le x \le x_1$.*
Then (P_k) *has at most one solution.*

Combining Theorem 1 with Theorem A, we obtain the following result.

Corollary. *Let $k \in \mathbf{N}$. Assume that either* (C1) *or* (C2) *is satisfied. Then the following* (i) *and* (ii) *hold*:

(i) *if $f(s)/s = \lambda_k$ for some $s > 0$, the solution of* (P_k) *exists and is unique*;
(ii) *if $f(s)/s \ne \lambda_k$ for all $s > 0$, then* (P_k) *has no solution.*

Example. Consider the problem

$$\text{(3)} \qquad \begin{cases} u'' + (e^x + \mu)|u|^{p-1}u = 0, & 0 < x < 1, \\ u(0) = u(1) = 0, & u'(0) > 0, \\ u \text{ has exactly } k-1 \text{ zeros in } (0,1), \end{cases}$$

where $p > 1$, $\mu > -1$ and $k \in \mathbf{N}$.

From Theorem A it follows that (3) has at least one solution.

Theorem D implies that if $-1 < \mu \le 0$, then the solution of (3) is unique.

Theorem 1 shows that if $\mu \ge e/2$, then the solution of (3) is unique.

To prove Theorem 1 we use the shooting method. Namely we consider the solution $u(x;\alpha)$ of (1) satisfying the initial condition

$$u(x_0) = 0 \quad \text{and} \quad u'(x_0) = \alpha > 0,$$

and observe the behavior of zeros of $u(x;\alpha)$ in $(0,1]$, where α is a parameter. We note that $u(x;\alpha)$ exists on $[x_0, x_1]$ is unique and satisfies $u \in C^1([x_0, x_1] \times (0,\infty))$, since $a \in C^2[x_0, x_1]$ and $f \in C^1(\mathbf{R})$.

Let $z_k(\alpha)$ be the k-th zero of $u(x;\alpha)$ in $(x_0, x_1]$ (if $z_k(\alpha)$ exists). Note that $u(x;\alpha)$ is a solution of (P_k) if and only if $z_k(\alpha) = x_1$. Since

$$u(z_k(\alpha);\alpha) = 0, \quad u'(z_k(\alpha);\alpha) \neq 0,$$

the implicit function theorem implies that

$$z_k'(\alpha) = -\frac{u_\alpha(z_k(\alpha);\alpha)}{u'(z_k(\alpha);\alpha)}.$$

We can show that if (C1) or (C2) holds, then $z_k'(\alpha) < 0$ or $z_k'(\alpha) > 0$, respectively, by using the similar arguments by Kajikiya [4] and the identity obtained by Korman and Ouyang [5]. Then we conclude that there exists at most one number $\alpha > 0$ such that $z_k(\alpha) = x_1$, so that (P_k) has at most one solution.

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