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Fredholm Equations and Volterra Equations Arising from Fuzzy Boundary Value Problems

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1 Introduction

There are many fruitful results on representations of fuzzy numbers, differentials and integrals of fuzzy functions. The authors establish fundamental results concerning differentials (e.g., [5, 6, 7, 8, 9, 10, 19, 11, 12, 24]), integrals (e.g., [1, 20]), the existence and uniqueness of solutions for initial value problems of differential equations (e.g., [15, 16, 18, 25, 26]), the asymptotic behaviours of solutions (e.g., [3, 4, 13, 14, 17, 23]). In this study we introduce the couple parametric representation corresponding to the results due to Goetschel-Voxman so that it is easy to analyze fuzzy differential equations. By the couple representation we can discuss differential, integral of fuzzy functions and asymptotic behaviours of solutions for fuzzy differential equations in an analogous way to the theory of ordinary differential equations. In a similar way we treat fuzzy differential equations with fuzzy boundary conditions.

Our aim is to discuss the existence and uniqueness of solutions for the following boundary value problems of fuzzy differential equations:

$$x''(t) = f(t, x, x'), \quad x(a) = A, x(b) = B. \quad (1.1)$$

Here $J = [a, b] \subset \mathbf{R} = (-\infty, +\infty)$, $t \in J$, and fuzzy numbers $A, B \in \mathcal{F}_{\mathbf{b}}^{st}$,which is a set of fuzzy numbers with compact supports and strictly quasi-convexity, and $f : J \times \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st} \to \mathcal{F}_{\mathbf{b}}^{st}$ is an $\mathcal{F}_{\mathbf{b}}^{st}$-valued function.

Let $I = [0, 1]$. In what follows a fuzzy number x is characterized by a membership function μ_x which has four properties. We consider a set of fuzzy numbers with compact supports denoted by $\mathcal{F}_{\mathbf{b}}^{st}$:

Definition 1 *Denote*

$$\mathcal{F}_{\mathbf{b}}^{st} = \{\mu_x : \mathbf{R} \to I \textit{ satisfying } \text{(i)} - \text{(iv) } \textit{below}\}.$$

(i) *There exists a unique* $m \in \mathbf{R}$ *such that* $\mu_x(m) = 1$*;*

(ii) *The support set* $supp(\mu_x) = cl(\{\xi \in \mathbf{R} : \mu(\xi) > 0\})$ *is bounded in* $\mathbf{R}$*;*

(iii) μ_x *is strictly quasi-convex on* $supp(\mu_x)$*;*

(iv) μ_x *is upper semi-continuous on* $\mathbf{R}$.

Function μ_x is strictly quasi-convex , i.e., strictly fuzzy convex, on $supp(\mu_x)$ if

$$\mu_x(\lambda\xi_1 + (1-\lambda)\xi_2) > \min[\mu_x(\xi_1), \mu_x(\xi_2)] \quad (1.2)$$

for $0 < \lambda < 1$ and $\xi_1, \xi_2 \in J$ such that $\xi_1 \neq \xi_2$. In usual case a fuzzy number x satisfies quasi-convex on $\mathbf{R}$, *i.e.*,

$$\mu_x(\lambda\xi_1 + (1-\lambda)\xi_2) \geq \min[\mu_x(\xi_1), \mu_x(\xi_2)]$$

for $0 \leq \lambda \leq 1$ and $\xi_1, \xi_2 \in \mathbf{R}$.

In the similar way as [9, 10] we consider the following parametric representation of $\mu_x \in \mathcal{F}_{\mathbf{b}}^{st}$ such that

$$x_1(\alpha) = \min L_\alpha(\mu_x), \quad x_2(\alpha) = \max L_\alpha(\mu_x)$$

for $0 < \alpha \leq 1$ and that

$$x_1(0) = \min cl(supp(\mu_x)), x_2(0) = \max cl(supp(\mu_x)).$$

We denote a fuzzy numbers x by (x_1, x_2), *i.e.*, $x = (x_1, x_2)$. Condition (iii) in the above definition plays an important role in proving properties of membership function μ_x in Theorem 1, where we show significant properties concerning the end-points of the α-cut set

$$L_\alpha(\mu_x) = \{\xi \in \mathbf{R} : \mu_x(\xi) \geq \alpha\}.$$

Let a metirc in $\mathcal{F}_{\mathbf{b}}^{st}$ be $d(x, y) = \sup_{\alpha \in I}(|x_1(\alpha) - y_1(\alpha)| + |x_2(\alpha) - y_2(\alpha)|)$ for $x = (x_1, x_2), y = (y_1, y_2)$. In [23] it can be shown that the metric space $(\mathcal{F}_{\mathbf{b}}^{st}, d)$ is complete.

We treat fuzzy type of Nagumo's Condition to (1.1) and give the existence and uniqueness theorems to (1.1) by parametric representation of fuzzy numbers. Moreover we show applications of fuzzy type of Nagumo's condition to the Fredholm equation concerning (1.1) by applying the contraction principle and the Schauder's fixed point theorem.

2 Parametric representation of fuzzy functions

In [23] we showed that a fuzzy number $x = (x_1, x_2)$ means a bounded continuous curve in the $\mathbf{R}^2$ space as follows.

Theorem 1 *Denote $x = (x_1, x_2) \in \mathcal{F}_{\mathbf{b}}^{st}$, where $x_1, x_2 : I \to \mathbf{R}$. Then it follows that the following properties* (i)-(iii) *hold:*

(i) *$x_i \in C(I), i = 1, 2$. Here $C(I)$ is the set of all the continuous functions on I;*

(ii) *There exists a unique $m \in \mathbf{R}$ such that*

$$x_1(1) = x_2(1) = m,\ x_1(\alpha) \le m \le x_2(\alpha)$$

for $\alpha \in I$;

(iii) *One of the following statements* (a) *and* (b) *hold;*

(a) *It follows that $x_1(\alpha) < x_2(\alpha)$ for $0 \le \alpha < 1$ and $x_1(\alpha)$, $x_2(\alpha)$ are non-decreasing, non-increasing in $\alpha \in I$, respectively;*

(b) *$x_1(\alpha) = x_2(\alpha) = m$ for $0 < \alpha \le 1$.*

Conversely, under the above conditions (i) -(iii), *if we denote*

$$\mu_x(\xi) = \sup\{\alpha \in I : x_1(\alpha) \le \xi \le x_2(\alpha)\} \quad (2.3)$$

Then μ_x is the membership function of x, i.e., $\mu_x \in \mathcal{F}_{\mathbf{b}}^{st}$.

Denote a fuzzy function $x = (x_1, x_2) : J \to \mathcal{F}_{\mathbf{b}}^{st}$ has a variable $t \in J$ and the parameter $\alpha \in I$ such that x_1, x_2 are functions defined on $J \times I$ to $\mathbf{R}$. Fuzzy function $x = (x_1, x_2)$ is called differentiable at t if there exists an fuzzy number $\eta \in \mathcal{F}_{\mathbf{b}}^{st}$ such that (d1) $x(t + h) = x(t) + h\eta + o(h)$ and (d2) $x(t) = x(t-h)+h\eta+o(h)$ as $h \to +0$. Here $o(h) = (o_1(h), o_2(h))$, *i.e.*, $\lim_{|h| \to 0} \frac{d(o(h),0)}{|h|} = 0$. Denote $x'(t) = \eta$. It's called a differential coefficient of the Hukuhara-differentiation (See [19]). It can be seen that $x = (x_1, x_2)$ is called differentiable at t if and only if $x_1(\cdot, \alpha), x_2(\cdot, \alpha)$ are differentiable at t for any $\alpha \in I$ and there exists $\eta \in \mathcal{F}_{\mathbf{b}}^{st}$ satisfying the above (d1) and (d2).

Fuzzy function $x = (x_1, x_2)$ is called integrable over $[t_1, t_2]$ if x_1, x_2 are integrable over $[t_1, t_2]$ for any $\alpha \in I$. Define

$$\int_{t_1}^{t_2} x(s)ds = \{(\int_{t_1}^{t_2} x_1(s, \alpha)ds, \int_{t_1}^{t_2} x_2(s, \alpha)ds)^T \in \mathbf{R}^2 : \alpha \in I\}.$$

3 Fredholm equation arising from fuzzy boundary problems

1) **Fredholm equations**

Assume that $f : J \times \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st} \to \mathcal{F}_{\mathbf{b}}^{st}$ is continuous. Consider the following Fredholm equation

$$x(t) = w(t) + \int_a^b G(t, s) f(s, x(s), x'(s))ds$$

for $t \in J$. Here a fuzzy function $w \in C(J)$ and an $\mathbf{R}$-valued function $G \in C(\mathbf{R}^2)$ with $G(t, s) \ge 0$ such that

$$w(t) = \frac{A(b-t) + B(t-a)}{b-a}, \quad (3.4)$$

$$G(t, s) = \begin{cases} \frac{(b-t)(s-a)}{b-a} & (a \le t \le s \le b) \\ \frac{(b-s)(t-a)}{b-a} & (a \le s \le t \le b) \end{cases} \quad (3.5)$$

Then we get $w''(t) \equiv 0$ and also $w(a) = A,\ w(b) = B$. It follows that

$$\int_a^b G(t, s)ds \le \frac{(b-a)^2}{8}, \quad \int_a^b \frac{\partial G}{\partial t}(t, s)ds \le \frac{b-a}{2}.$$

In the same way in theory to boundary value problems of ordinary differential equation the following proposition are shown immediately.

Proposition 1 *Fuzzy function x is a continuously differentiable solution of (1.1) if and only if x is a fixed point of $T : C^1(J; \mathcal{F}_{\mathbf{b}}^{st}) \to C^2(J; \mathcal{F}_{\mathbf{b}}^{st})$ such that*

$$[T(x)](t) = w(t) + \int_a^b G(t, s) f(s, x(s), x'(s))ds.$$

Here $C^1(J; \mathcal{F}_{\mathbf{b}}^{st})$ is the set of continuously differentiable functions defined on J to $\mathcal{F}_{\mathbf{b}}^{st}$, etc.

In the same way in applying the contraction principle [17] gets the existence and uniquess theorem of (1.1).

Theorem 2 *Suppose that There exist positive numbers K, L such that*

$$d(f(t, x, y), f(t, u, v)) \le Kd(x, u) + Ld(y, v) \quad (3.6)$$

for $t \in J$ and $x, y, u, v \in \mathcal{F}_{\mathbf{b}}^{st}$ and that

$$\frac{K(b-a)^2}{8} + \frac{L(b-a)}{2} < 1. \quad (3.7)$$

Then (1.1) has one and only one solution in $C^2(J; \mathcal{F}_{\mathbf{b}}^{st})$.

We illustrate the above theorem as follows.

Example 1 *Let fuzzy numbers* $k = (k_1, k_2), \ell = (\ell_1, \ell_2) \in \mathcal{F}_{\mathbf{b}}^{st}$ *with* $k_1(\alpha) \geq 0, \ell_1(\alpha) \geq 0$ *for* $\alpha \in I$ *and* $k_2(0) \leq K, \ell_2(0) \leq L$, *respectively. Assume that positive real numbers* K, L *satisfy the inequality (3.6) and* $p_i \geq 0, q_i \geq 0$ *for* $i = 1, 2$. *We consider fuzzy functions* $f = (f_1, f_2)$ *of* $(t, x, y) \in J \times \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st}$ *with* $x = (x_1, x_2).y = (y_1, y_2)$ *such that*

$$\begin{aligned} &f_i(t, x_1(\alpha), x_2(\alpha), y_1(\alpha), y_2(\alpha), \alpha) \\ &= k_i(\alpha)e^{-p_i t}x_i(\alpha) + \ell_i(\alpha)e^{-q_i t}y_i(\alpha) \end{aligned}$$

for $\alpha \in I, i = 1, 2$. *Then, for any boundary values* $(A, B) \in \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st}$, *there exists a unique solution for (1.1).*

2) **Fuzzy type of Nagumo's condition**
Assume that the following properties (i) -(iii).

(i) Function $f = (f_1, f_2) : J \times \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st} \to \mathcal{F}_{\mathbf{b}}^{st}$ is continuous. Here (f_1, f_2) is the parametric representation of f.

(ii) Let $r_i > 0, i = 1, 2$. There exists a function $h_i : [0, \infty) \to [0, \infty)$ such that

$$|f_i(t, x_1(\alpha), x_2(\alpha), y_1(\alpha), y_2(\alpha), \alpha)| \leq h_i(|y_i(\alpha)|)$$

for $t \in J, \alpha \in I, i = 1, 2$, and $|x_i(\alpha)| \leq r_i$, $y = (y_1, y_2) \in \mathcal{F}_{\mathbf{b}}^{st}$. Here $x = (x_1, x_2), y = (y_1, y_2)$ are parametric representations of x, y, respectively.

(iii) Assume that h_i, $i = 1, 2$, satisfy

$$\int_{+0}^{\infty} \frac{\eta d\eta}{h_i(\eta)} > 2r_i.$$

The above condition is applied to the fuzzy boundary value problem (1.1) in the same way as [2].

Lemma 1 *Assume that* $f = (f_1, f_2)$ *satisfies fuzzy type of Nagumo's condition. Let* $r_i > 0$, $i = 1, 2$, *be in fuzzy type of Nagumo's condition and a solution* $x = (x_1, x_2) \in C^2(J; \mathcal{F}_{\mathbf{b}}^{st})$ *of (1.1) satisfy* $|x_i(t, \alpha)| \leq r_i$ *for* $i = 1, 2, t \in J, \alpha \in I$.

There exist numbers $N_i > 0, i = 1, 2$ *such that* $|x_i'(t, \alpha)| \leq N_i$ *for* $t \in J, \alpha \in I$.

The proof can be proved in the similar to Theorem 1.4.1 in [2]. In case where $f \in C(J \times \mathbf{R}^n \times \mathbf{R}^n)$ [2] gives Nagumo's condition in $\mathbf{R}^n$ and its application to boundary value problems of ordinary differential equations.

3) **Fuzzy type of Nagumo's condition concerning** $x'' = 0$
In what follows we consider fuzzy type of Nagumo's condition concerning $x'' = 0$. Assume that the following properties (i) -(iii).

(i) $f = (f_1, f_2) : J \times \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st} \to \mathcal{F}_{\mathbf{b}}^{st}$ is continuous;

(ii) Let $r_i > 0, i = 1, 2$, and w is the function in (3.4). There exists a function $h_i : [0, \infty) \to [0, \infty)$ such that

$$\begin{aligned} &|f_i(t, x_1(\alpha), x_2(\alpha), y_1(\alpha), y_2(\alpha), \alpha)| \\ &\leq h_i(|y_i(\alpha) - w_i'(t, \alpha)|) \qquad (3.8) \end{aligned}$$

for $t \in J, \alpha \in I, i = 1, 2$, and

$$|x_i(\alpha) - w_i(t, \alpha)| \leq r_i, \; y = (y_1, y_2) \in \mathcal{F}_{\mathbf{b}}^{st}.$$

Here $x = (x_1, x_2), y = (y_1, y_2)$ are parametric representations of x, y, respectively;

(iii) Assume that h_i, $i = 1, 2$, satisfy

$$\int_{+0}^{\infty} \frac{\eta d\eta}{h_i(\eta)} > 2r_i. \qquad (3.9)$$

Lemma 2 *Assume that there exist functions* $h_i, i = 1, 2$, *satisfy (3.8) and (3.9). Let* $r_i > 0$, $i = 1, 2$, *be in (3.9) and a solution* $x = (x_1, x_2) \in C^2(J; \mathcal{F}_{\mathbf{b}}^{st})$ *of (1.1) satisfy*

$$|x_i(t, \alpha) - w_i(t, \alpha)| \leq r_i$$

for $i = 1, 2, t \in J$ *and* $\alpha \in I$.

There exist numbers $N_i > 0, i = 1, 2$, *such that*

$$|x_i'(t, \alpha) - w_i'(t, \alpha)| \leq N_i$$

for $t \in J, \alpha \in I$.

The proof can be done in the similar to Lemma 1.

In cases where $h_i(\eta) = \eta, h_i(\eta) = \eta^2$ for $\eta \geq 0$ it suffices that N_i satisfies $N_i > 2r_i, N_i > 0$, for (3.9), respectively.

4) **Applications of fuzzy type of Nagumo's condition** to $x'' = 0$
In this section we show the existence of solutions for (1.1) by applying Schauder's fixed point

theorem as well as we give the existence and uniqueness of solutions by applying the contraction principle under assumption that Nagumo's condition concerning $x'' = 0$. Let $r = (r_1, r_2)$ and $N = (N_1, N_2)$. Denote

$$S_w(r,N) = \{(x,y) \in \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st} : |x_i(\alpha) - w_i(t,\alpha)| \le r_i, \quad |y_i(\alpha) - w_i'(t,\alpha)| \le N_i, \text{ for } i = 1,2, t \in J, \alpha \in I\}.$$

Theorem 3 *Assume that the same conditions of Lemma 2 hold. Let*

$$|f_i(t, x_1(\alpha), x_2(\alpha), y_1(\alpha), y_2(\alpha), \alpha)| \le \min\left(\frac{2N_i}{b-a}, \frac{8r_i}{(b-a)^2}\right)$$

for $t \in J, (x,y) \in S_w(r,N), i = 1,2, \alpha \in I$.

Then (1.1) has at least one solution x such that $(x(t), x'(t)) \in S_w(r,N)$ for $t \in J$ and any $A, B \in \mathcal{F}_{\mathbf{b}}^{st}$.

[2] show Nagumo's condition of $\mathbf{R}^n$, but they give no theorems of existence of solutions for boundary value problems.

In the following theorem we get the existence and uniqueness of solutions for (1.1).

Theorem 4 *Assume that the same conditions of Theorem 3 hold. Assume that there exist integrable functions $p_1, p_2 : J \to [0,\infty)$ such that for $t \in J, i = 1,2, (x,y), (u,v) \in S_w(r,N)$*

$$\begin{aligned} &|f_i(t, x_1(\alpha), x_2(\alpha), y_1(\alpha), y_2(\alpha), \alpha) \\ &- f_i(t, u_1(\alpha), u_2(\alpha), v_1(\alpha), v_2(\alpha), \alpha)| \\ &\le p_i(t)(d(x,u) + d(y,v)) \end{aligned}$$

and

$$\lambda = \sup_{t\in J} \int_a^b G(t,s)p_1(s)ds + \sup_{t\in J} \int_a^b \frac{\partial G}{\partial t}(t,s)p_2(s)ds < 1. \quad (3.10)$$

Then (1.1) has one and only one solution in $C^2(J; \mathcal{F}_{\mathbf{b}}^{st})$ such that $(x(t), x'(t)) \in S_w(r,N)$ for $t \in J$ and any $A, B \in \mathcal{F}_{\mathbf{b}}^{st}$.

In the following example we illustrate Theorem 4.

Example 2 *Denote fuzzy numbers $k = (k_1, k_2), \ell = (\ell_1, \ell_2), m = (m_1, m_2), n = (n_1, n_2) \in \mathcal{F}_{\mathbf{b}}^{st}$ such that all $k_1(0), \ell_1(0), m_1(0), n_1(0)$ are non-negatige. Denote integrable and non-negative functions a_i, b_i, c_i, d_i defined on $[0,\infty)$ for $i = 1,2$. Assume that $a_1(t) \le a_2(t), b_1(t) \le b_2(t), c_1(t) \le c_2(t), d_1(t) \le d_2(t)$ for $t \in J$. Let*

$$\begin{aligned} & f_i(t, x_1(\alpha), x_2(\alpha), y_1(\alpha), y_2(\alpha), \alpha) \\ = \ & k_i(\alpha)a_i(t)|x_1(\alpha) - w_1(t,\alpha)| \\ + \ & \ell_i(\alpha)b_i(t)|x_2(\alpha) - w_2(t,\alpha)| \\ + \ & m_i(\alpha)c_i(t)[y_1(\alpha) - w_1'(t,\alpha)]^2 \\ + \ & n_i(\alpha)d_i(t)|y_2(\alpha) - w_2'(t,\alpha)| \end{aligned}$$

for $t \in J, x = (x_1, x_2), y = (y_1, y_2) \in S_w(r,N), i = 1,2$.

Assume that r_i, N_i for $i = 1,2$ satisfy the following conditions (i) - (ii).

(i) *There exist p_1, p_2 such that (5.10) holds and that*

$$\begin{aligned} &\max(k_1(1)a_1(t), \ell_1(1)b_1(t), \\ &2N_1 m_1(1)c_1(t), n_1(1)d_1(t)) \\ &\le p_1(t) \\ &\max(k_2(0)a_2(t), \ell_2(0)b_2(t), \\ &2N_2 m_2(0)c_2(t), n_2(0)d_2(t)) \\ &\le p_2(t) \end{aligned}$$

for $t \in J$.

(ii) *Suppose that*

$$\sup_{t\in J} p_i(t)(r_1 + r_2 + N_1^2 + N_2) \le \min\left(\frac{8r_i}{b-a}, \frac{2N_i}{(b-a)^2}\right)$$

for $i = 1,2$ and that $N_1 > 0, N_2 > 2r_2$.

We get $h_1(\eta) = \eta^2, h_2(\eta) = \eta$ for $\eta \ge 0$. It follows that $\int_{+0}^{\infty}(\eta/h_1(\eta))d\eta = \int_{+0}^{\infty}(\eta/h_2(\eta))d\eta = \infty$. Then conditions of Theorem 4 are satisfied. Therefore, by Theorem 4, (1.1) has one and only one solution in $S_w(r,N)$ for any $(A,B) \in \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st}$.

4 Volterra equation arising from fuzzy boundary problems

By putting $y_1 = x_1', y_2 = x_2'$ we have

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}$$

$$+\begin{pmatrix} 0 \\ 0 \\ f_1(t,x_1,x_2,y_1,y_2) \\ f_2(t,x_1,x_2,y_1,y_2) \end{pmatrix}.$$

Then, by denoting $z = (x_1, x_2, y_1, y_2)^T \in \mathbf{R}^4$, we get

$$\frac{dz}{dt}(t) = Mz + F(t,z),\ \mathcal{L}(z) = c \qquad (4.11)$$

Here

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, F(t,z) = \begin{pmatrix} 0 \\ 0 \\ f_1(t,z) \\ f_2(t,z) \end{pmatrix},$$

$$c = \begin{pmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{pmatrix} \qquad (4.12)$$

and $\mathcal{L}$ is a bounded linear operator from $C(J)^4$ to $\mathbf{R}^4$ as follows:

$$\mathcal{L}(z) = (x_1(a), x_2(a), x_1(b), x_2(b))^T.$$

In this case we get the fundamental matrix

$$X(t) = e^{tM} = \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $X(0) = E$, where E is the identity matrix. Let U satisfy

$$\mathcal{L}(X(\cdot)z_0) = \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & a \\ 1 & 0 & b & 0 \\ 0 & 1 & 0 & b \end{pmatrix} z_0 = Uz_0$$

for $z_0 \in \mathbf{R}^4$. It follows that

$$U^{-1} = \frac{1}{b-a}\begin{pmatrix} b & 0 & -a & 0 \\ 0 & b & 0 & -a \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

We denote a norm in $\mathbf{R}^4$ by $\| z \| = |x_1| + |x_2| + |y_1| + |y_2|$ and $\| U \| = \sup_{\|z\|=1} \| Uz \|$.

Then $\| U \| = \max(2, a+b)$ and $\| U^{-1} \| = \dfrac{b+1}{b-a}$.

In what follows we give the existence and uniqueness theorems by applying Schauder's fixed point theorem or the contraction principle as in the similar way as in [21] and [22]. Let $r > 0$. Denote a subset in $C(J \times I)^4$ by

$$S = \{z = (x_1, x_2, y_1, y_2)^T \in C(J \times I)^4 : d_\infty(z, 0) \le r\}.$$

Here

$$d_\infty(z, \bar{z}) = \sup_{t\in J} d(x(t), \bar{x}(t)) + \sup_{t\in J} d(y(t), \bar{y}(t))$$

where

$$z(t,\cdot) = (x(t,\cdot), y(t,\cdot)), \bar{z}(t,\cdot) = (\bar{x}(t,\cdot), \bar{y}(t,\cdot))$$

in $\mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st}$ and

$$\bar{x}(t,\cdot) = (\bar{x}_1(t,\cdot), \bar{x}_2(t,\cdot)), \bar{y}(t,\cdot) = (\bar{y}_1(t,\cdot), \bar{y}_2(t,\cdot))$$

in $\mathcal{F}_{\mathbf{b}}^{st}$ for $t \in J$. Then the following functions

$$\mu_{x(t)}(\xi) = \sup\{\alpha \in I : x_1(t,\alpha) \le \xi \le x_2(t,\alpha)\}$$
$$\mu_{y(t)}(\xi) = \sup\{\alpha \in I : y_1(t,\alpha) \le \xi \le y_2(t,\alpha)\}$$

are membership functions of fuzzy numbers $x(t), y(t)$ in $\mathcal{F}_{\mathbf{b}}^{st}$ for $t \in J$, respectively. Moreover it can be seen that S is a convex and closet subset in $C(I)^4$.

In the similar way of discussion as the theory of ordinary differential equations it follows that $x \in S$ is a continuous solution of (4.2) if and only if

$$z(t) = X(a)U^{-1}(c - \mathcal{L}(q_z)) + \int_a^t Mz(s)ds + \int_a^t F(s, z(s))ds$$

for $t \in J$, where

$$q_z(t) = \int_a^t X(t)X^{-1}(s)F(s, z(s))ds.$$

Putting

$$Q = \int_a^b \max_{\|z\|\le r} (b - s + 1)(|f_1(s,z)| + |f_2(s,z)|)ds,$$

we have $d_\infty(q_z, 0) \le Q$ for $z \in S$. By Schauder's fixed point theorem we get the existence of solutions for (4.1).

Theorem 5 *Assume that positive numbers R, r satisfy $R < e^{-(b-a)}$ and*

$$r > \frac{Q \| \mathcal{L} \| (b+1) \| U^{-1} \|}{e^{-(b-a)} - R}.$$

Let f_1, f_2 satisfy

$$\int_a^b \max_{\|z\|\leq r}(|f_1(s,z)| + |f_2(s,z)|)ds \leq rR.$$

If $c = (A_1, A_2, B_1, B_2)^T \in \mathbf{R}^4$ *with* $(A_1, A_2), (B_1, B_2) \in \mathcal{F}_{\mathbf{b}}^{st}$ *satisfies*

$$\| c \| \leq \frac{r(e^{-(b-a)} - R)}{(b+1) \| U^{-1} \|} - \| \mathcal{L} \| Q,$$

then (4.1) *has at least one solution in S.*

Proof. Let $u \in S$ be fixed. Consider the following boundary linear problem

$$\frac{dz}{dt} = Mz(t) + F(t,u), \mathcal{L}(z) = c.$$

Then there exists a unique solution z_u of the above problem such that

$$\begin{aligned} z_u(t) &= X(t)U^{-1}(c - \mathcal{L}(q_u)) + q_u(t) \\ &= X(a)U^{-1}(c - \mathcal{L}(q_u)) \\ &\quad + \int_a^t Mz_u(s)ds + \int_a^t F(s,u(s))ds \end{aligned}$$

for $t \in J$. Denote a mapping by $[\mathcal{V}(u)](t) = z_u(t)$ for $u \in S, t \in J$. Then the solution z of (4.1) means that z is a fixed point of $\mathcal{V}$ in S.

We shall prove $\mathcal{V}$ is an into mapping. From the definition of $\mathcal{V}$ and $\| X(t) \| \leq b+1$ for $t \in J$ it follows that

$$\begin{aligned} \| z_u(t) \| &\leq (b+1) \| U^{-1} \| (\| c \| \\ &+ \| \mathcal{L} \| d_\infty(q_u, 0)) \\ &+ \int_a^t \| M \| \| z_u(s) \| ds \\ &+ \int_a^t \| F(s,u(s)) \| ds \\ &\leq (b+1) \| U^{-1} \| (\| c \| \\ &+ \| \mathcal{L} \| Q) + \int_a^t \| z_u(s) \| ds + rR \end{aligned}$$

for $t \in J$. By Gronwall's inequality we have

$$\begin{aligned} &\| z_u(t) \| \\ &\leq e^{b-a}((b+1) \| U^{-1} \| (\| c \| + \| \mathcal{L} \| Q) + rR) \leq r. \end{aligned}$$

Thus we have $d_\infty(z_u, 0) \leq r$ for $u \in S$. It's clear that z_u satisfies Conditions (i)- (iii) in Theorem 1. Therefore $z_u \in S$ for $u \in S$. Thus $\mathcal{V}$ is uniformly bounded.

The continuity of q_u on S means that $\mathcal{V}$ is continuous. The uniform continuity of F leads to the equicontinuity of $\mathcal{V}$ and the compactness of $\mathcal{V}$ is proved by Ascoli-Arzela's theorem. By Schauder's fixed point theorem it follows that there exists at least one solution in S. This completes the proof.

Finally we show the existence of uniqueness theorem of solutions of (4.1) by the contraction principle.

Theorem 6 *Assume that there exists an integrable function* $\ell : J \to \mathbf{R}_+$ *such that*

$$((b+1) \| U^{-1} \| + \| \mathcal{L} \|) \int_a^b (b-s+1)\ell(s)ds < 1.$$

Let F be in (4.3) *such that*

$$d(F(t,z), F(t,\bar{z})) \leq \ell(t)d(z,\bar{z})$$

for $z = (x,y), \bar{z} = (\bar{x}, \bar{y})$ *and* $t \in J$, *where* $x, y, \bar{x}, \bar{y} \in \mathcal{F}_{\mathbf{b}}^{st}$. *Then* (4.1) *has one and only one solution for any c in* (4.3).

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