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Transformations with Improved Chi-Squared Approximations

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1 Introduction

In statistical inference it is basic to obtain the sampling distribution of a statistic. However, we often encounter the situation where the exact distribution cannot be obtained in a closed form, or even if it is obtained, the exact distribution is of no use because of its complexity. To overcome this situation, various approximations of the quantiles as well as the distribution function have been studied. The one to which we restrict attention is that of using asymptotic approximations, especially asymptotic expansions.

In this paper, we consider a nonnegative statistic S whose limiting distribution is a chi-squared distribution χ_f^2 with f degrees of freedom and suppose that a nonnegative statistic S has an asymptotic expansion

$$F(x) \equiv \mathrm{P}(S \leq x) = G_f(x) + \frac{1}{n}\sum_{j=0}^{k} a_j G_{f+2j}(x) + O(n^{-2}), \tag{1.1}$$

where n is a positive number, typically a sample size, k is a positive integer, $G_{f+2j}(\cdot)$ is the distribution function of χ_{f+2j}^2 and coefficients a_j's satisfy the relation $\sum_{j=0}^{k} a_j = 0$. Some examples of the statistic S are as follows: For $k=1$, the likelihood ratio test statistic (see Hayakawa (1977)); for $k=2$, Lawley-Hotelling trace criterion and Bartlett-Nanda-Pillai trace criterion, which are test statistics for multivariate linear hypothesis under normality (see Anderson (1984) and Siotani, Hayakawa and Fujikoshi (1985)); for $k=3$, the score test statistic (see Harris (1985)) and Hotelling's T^2-statistic under nonnormality (see Kano (1995) and Fujikoshi (1997b)).

In order to obtain an approximated quantile of statistic S, we consider a monotone function $T = T(S)$ satisfying

$$\mathrm{P}(T \leq x) = G_f(x) + O(n^{-2}). \tag{1.2}$$

For such a monotone function T, it holds that

$$\mathrm{P}(S \leq b(u_\alpha)) = \mathrm{P}(T(S) \leq u_\alpha) = 1 - \alpha + O(n^{-2}),$$

where u_α is the upper α point of χ_f^2 and $b(\cdot)$ is the inverse function of T. We shall propose methods to use $b(u_\alpha)$ as an approximated upper α point of S.

The transformation $T = T(S)$ satisfying property (1.2) is called the Bartlett correction or a Bartlett type correction and have been investigated by many researchers (e.g., Cordeiro and Ferrari (1991), Kakizawa (1996), Fujikoshi (1997a), Fujisawa (1997), Cordeiro and Ferrari (1998), Cordeiro, Ferrari and Cysneiros (1998), Fujikoshi (2000), and Aoshima, Enoki and Ito (2003)). In this paper, we shall consider new transformations given by a different approach from others. It may be observed that new transformations, proposed in this paper, give a significant improvement to chi-squared approximations. Further, we shall also consider error bounds for the remainder term in (1.2) and for approximated quantiles. These would lead a broad application with a wide class of statistics.

This paper is organized as in the following way. In Section 2, we propose new monotone transformations which are given by a different approach from others. Numerical examples of some test statistics are demonstrated to observe an improvement brought by the proposed transformations beyond the competitors. In Section 3, we give a method to obtain an uniform or non-uniform error bound for an improved χ^2-approximation. Further, we introduce some applications.

2 Transformations with improved chi-squared approximations

For a nonnegative statistic S whose asymptotic distribution is χ_f^2, we assume that the distribution function can be expanded as in (1.1). Then, we consider a monotone transformation $T = T(S)$ based on the Bartlett correction or a Bartlett type correction. That is, we consider a monotone function $T = T(S)$ satisfying property (1.2). The following lemma was given by Cordeiro and Ferrari (1991).

Lemma 2.1 *Suppose that a nonnegative statistic S has an asymptotic expansion (1.1). If the transformation $T = T(S)$ can be expanded as*

$$T = S - \frac{2}{n}\sum_{i=1}^{k}\left(\sum_{j=i}^{k}\frac{a_j}{\prod_{\ell=0}^{i-1}(f+2\ell)}S^i\right) + O_p(n^{-2}), \tag{2.1}$$

then property (1.2) is satisfied.

Some monotone transformations that hold (2.1) have been proposed (e.g., Fujikoshi (2000) for $k = 2$, Cordeiro, Ferrari and Cysneiros (1998) and Aoshima, Enoki and Ito (2003) for $k = 3$ and Kakizawa (1996) for a general k). However, these transformations are not always oriented to a theoretical background except for having (2.1) and the monotoneity. In this section, we propose new transformations that have not only (2.1) and the monotoneity but also a certain theoretical background. Moreover, we introduce some applications with proposed transformations.

2.1 New transformations

Let x_α and u_α be the upper α points of F and G_f, respectively. Then, we note that

$$x_\alpha = F^{-1}(G_f(u_\alpha)). \tag{2.2}$$

Now we define that

$$T^* = G_f^{-1}(F(S)).$$

From (2.2), since it follows for all $\alpha \in (0,1)$ that

$$\begin{aligned} \mathrm{P}(T^* \le u_\alpha) &= \mathrm{P}(S \le F^{-1}(G_f(u_\alpha))) \\ &= \mathrm{P}(S \le x_\alpha) \\ &= 1 - \alpha, \end{aligned}$$

we claim that

$$T^* \sim \chi_f^2.$$

It is easy to see that T^* can be expanded as (2.1). On the other hand, from the property of the distribution function, T^* holds the monotoneity. Further, we note that the upper α point of S is exactly obtained from T^*. Therefore, we can say that T^* is an exact transformation to a chi-squared distribution χ_f^2. If $F(x)$ is completely known, T^* is available. As far as $F(x)$ is available in a form (1.1), we have to replace $F(x)$ with

$$\tilde{F}(x) = G_f(x) + \frac{1}{n}\sum_{j=0}^{k} a_j G_{f+2j}(x). \tag{2.3}$$

However, $\tilde{F}(x)$ does not hold the monotoneity. Now, we modify $\tilde{F}(x)$ so as to hold the monotoneity and construct a transformation with its modified function. Yanagihara and Tonda (2003) proposed an adjustment to $\tilde{F}(x)$ as follows:

$$F_{YT}(x) = d^{-1}\left\{\tilde{F}(x) + \frac{1}{4n^2}\int_0^x G'_f(t)\{a(t)\}^2 dt\right\}, \tag{2.4}$$

where

$$d = \lim_{x\to\infty}\left\{\tilde{F}(x) + \frac{1}{4n^2}\int_0^x G'_f(t)\{a(t)\}^2 dt\right\}, \tag{2.5}$$

$$\begin{aligned} a(x) &= \{G'_f(x)\}^{-1}\left\{\sum_{j=0}^{k} a_j G'_{f+2j}(x)\right\} \\ &= a_0 + \sum_{j=1}^{k} a_j \frac{x^j}{\prod_{\ell=0}^{j-1}(f+\ell)}. \end{aligned} \tag{2.6}$$

Then, $F_{YT}(x)$ is monotone and $\lim_{x\to\infty} F_{YT}(x) = 1$. With a monotone function $F_{YT}(x)$, we propose a new transformation:

$$T_1 = G_f^{-1}(F_{YT}(S)). \tag{2.7}$$

Theorem 2.1 *Suppose that a nonnegative statistic S can be expanded as in (1.1). Then, for a monotone transformation $T_1 = T_1(S)$ defined by (2.7) with (2.4), it holds that*

$$\mathrm{P}(T_1 \leq x) = G_f(x) + O(n^{-2}).$$

Proof. As for a function $\tilde{F}(x)$ defined by (2.3), note that $d = 1 + O(n^{-2})$ in (2.5). Hence, we have that $F_{YT}(x) = F(x) + o(n^{-1})$. Therefore, since T_1 can be expanded as in (2.1), we get the desired result from Lemma 2.1. ■

Next, we consider another adjustment to $\tilde{F}(x)$:

$$F_*(x) = \int_0^x G'_f(t) \exp\left(\frac{1}{n}a(t) - \frac{1}{n^2}p(t)\right) dt, \tag{2.8}$$

where $a(x)$ is given by (2.6) and $p(x)$ is a polynomial such that $k + 1 \leq \deg[p(x)] < \infty$ and $p(x) \to +\infty$ as $x \to +\infty$. Then, $F_*(x)$ is monotone and $F_*(x) = F(x) + o(n^{-1})$. With a monotone function $F_*(x)$, we propose another new transformation:

$$T_2 = G_f^{-1}(F_*(S)). \tag{2.9}$$

Theorem 2.2 *Suppose that a nonnegative statistic S can be expanded as in (1.1). Then, for a monotone transformation $T_2 = T_2(S)$ defined by (2.9) with (2.8), it holds that*

$$\mathrm{P}(T_2 \leq x) = G_f(x) + O(n^{-2}).$$

Proof. The claim is proved similarly to Theorem 2.1. ■

We can see the superiority of T_2 in the following case. Suppose that a nonnegative statistic S can be expanded as

$$F(x) = \mathrm{P}(S \leq x) = G_f(x) + \frac{1}{n}h_1(x)G'_f(x) + \cdots + \frac{1}{n^{r-1}}h_{r-1}(x)G'_f(x) + O(n^{-r}), \tag{2.10}$$

where $h_i(x)$ is a polynomial of degree $i \times k$ without constant term. The form (1.1) is the case when $r = 2$. Then, we consider a monotone transformation $T = T(S)$ satisfying

$$\mathrm{P}(T \leq x) = G_f(x) + O(n^{-r}). \tag{2.11}$$

For such a monotone function T, it holds that

$$\mathrm{P}(S \leq b(u_\alpha)) = \mathrm{P}(T(S) \leq u_\alpha) = 1 - \alpha + O(n^{-r}),$$

where u_α is the upper α point of χ^2_f and $b(\cdot)$ is the inverse function of T. In this situation, Kakizawa (1996) proposed a transformation satisfying (2.11). However, his method requires a quite complex calculation to come to a transformation. Now, let us apply transformation T_2 to this case as follows: Let

$$a_i(x) = \left(\frac{d}{dx}h_i(x)G'_f(x)\right) \Big/ G'_f(x).$$

Note that $a_i(x)$ is a polynomial of degree $i \times k$. We have that

$$F'(x) = G_f'(x)\left\{1 + \frac{1}{n}a_1(x) + \frac{1}{n^2}a_2(x) + \cdots + \frac{1}{n^{r-1}}a_{r-1}(x) + O(n^{-r})\right\}.$$

Define $F_*(x)$ by

$$F_*(x) = \int_0^x G_f'(t)\exp\left\{\frac{1}{n}a(t) - \frac{1}{2n^2}\{a(t)\}^2 + \cdots + \frac{(-1)^{r-1}}{(r-1)n^{r-1}}\{a(t)\}^{r-1} - \frac{1}{n^r}p(t)\right\}dt, \tag{2.12}$$

where

$$a(x) = a_1(x) + \frac{1}{n}a_2(x) + \cdots + \frac{1}{n^{r-2}}a_{r-1}(x),$$

and $p(x)$ is a polynomial such that $\{(r-1)k\}^{r-1} + 1 \leq \deg[p(x)] < \infty$ and $p(x) \to +\infty$ as $x \to +\infty$. Then, $F_*(x)$ is monotone and by using the relation $\exp(x - x^2/2 + \cdots + (-1)^r x^{r-1}/(r-1))$ $= 1 + x + O(x^r)$, we obtain $F_*(x) = F(x) + O(n^{-r})$.

Theorem 2.3 *Suppose that a nonnegative statistic S can be expanded as in (2.10). Then, for a monotone transformation $T_2 = T_2(S)$ defined by (2.9) with (2.12), it holds that*

$$\mathrm{P}(T_2 \leq x) = G_f(x) + O(n^{-r}).$$

In order to prove Theorem 2.3, we give the following lemma.

Lemma 2.2 *Suppose that a nonnegative statistic S has an asymptotic expansion (2.10). Let $b(x)$ be the inverse function of the transformation $T = T(S)$ such that $b(x) = x + O(n^{-1})$. If and only if $b(x)$ is coincident with Cornish-Fisher expansion, $F^{-1}(G_f(x))$, up to the order $O(n^{-r+1})$, then property (2.11) is satisfied.*

Proof. Let $\tilde{b}(x)$ be the one formed by the terms of $F^{-1}(G_f(x))$ up to the order $O(n^{-r+1})$. Then,

$$\begin{aligned} F(\tilde{b}(x)) &= F\left(F^{-1}(G_f(x)) - (F^{-1}(G_f(x)) - \tilde{b}(x))\right) \\ &= F\left(F^{-1}(G_f(x))\right) - F'\left(F^{-1}(G_f(x))\right)\left(F^{-1}(G_f(x)) - \tilde{b}(x)\right) + \cdots \\ &= G_f(x) + O(n^{-r}). \end{aligned} \tag{2.13}$$

On the other hand,

$$F(b(x)) = F(\tilde{b}(x)) + F'(\tilde{b}(x))(b(x) - \tilde{b}(x)) + \frac{1}{2}F''(\tilde{b}(x))(b(x) - \tilde{b}(x))^2 + \cdots. \tag{2.14}$$

It is easy to see that $\tilde{b}(x) = x + O(n^{-1})$. Noting that $b(x) = x + O(n^{-1})$, we get $b(x) - \tilde{b}(x) = O(n^{-1})$. From (2.13) and (2.14), we obtain the desired result. ∎

Proof of Theorem 2.3. Since $F_*(x) = F(x) + O(n^{-r})$, Theorem 2.3 is proved by Lemma 2.2. ∎

2.2 Applications

Here, we shall give the transformations (2.7) and (2.9) for some statistics and examine the accuracy of the approximations to the true percentage point x_α of S. In Example 2.1, we conducted simulation experiments as follows: For parameters given in advance, the approximate percentage point was calculated for each monotone transformation. By using these percentage points, we conducted the Monte Carlo simulation with 100,000 (= R, say) independent trials for a test statistic. Let s_r ($r = 1, ..., R$) be an observed value of S and $p_r = 1$ (or 0) if s_r is (or is not) larger than the approximate percentage point. On the other hand, let $s_{[1]} \le s_{[2]} \le \cdots \le s_{[R]}$ be the ordered values of s_r and let us define $s_{[(1-\alpha)R]}$ as an observed value of x_α. We briefly write it x_α. Let $\bar{p} = 100\sum_{r=1}^{R} p_r/R$ which estimates the test size (100α%) with its estimated standard error $s(\bar{p}) = 100\sqrt{(\bar{p}/100)(1-\bar{p}/100)/R}$. Table 2.1 gives values of the approximate percentage point for each monotone transformation together with the value of x_α. As for the actual test sizes, Table 2.2 gives values of $\bar{p}$ ($s(\bar{p})$), on the first (second) line in each cell, for each monotone transformation. In Example 2.2, we have considered a case that the asymptotic expansion of S is obtained up to the order $O(n^{-2})$.

Example 2.1 Let $S = (n-q)s_h^2/s_e^2$ be a test statistic for testing the equality of means of q nonnormal populations Π_i ($i = 1, ..., q$) with common variance. Here, s_h^2 and s_e^2 are the sums of squares due to the hypothesis and the error, respectively, based on the sample of the size n_i from Π_i. Let $\rho_i = \sqrt{n_i/n}$, where n is the total sample size. Assume that $\rho_i = O(1)$ as n_j's tend to infinity. Let κ_3 and κ_4 be the third and the fourth cumulants of the standardized variate. Then, under a general condition, an asymptotic expansion for the null distribution of S was given by Fujikoshi, Ohmae and Yanagihara (1999) in the form (1.1) with $k = 3$, $f = q - 1$ and the coefficients given by

$$a_0 = \frac{1}{4}(q-1)(q-3) - d_1\kappa_3^2 + d_2\kappa_4, \quad a_1 = -\frac{1}{2}(q-1)^2 + 3d_1\kappa_3^2 - 2d_2\kappa_4,$$
$$a_2 = \frac{1}{4}(q^2-1) - 3d_1\kappa_3^2 + d_2\kappa_4, \qquad a_3 = d_1\kappa_3^2,$$

where

$$d_1 = \frac{5}{24}\left(\sum_{j=1}^{q}\frac{n}{n_j} - q^2\right) + \frac{1}{12}(q-1)(q-2), \quad d_2 = \frac{1}{8}\left(\sum_{j=1}^{q}\frac{n}{n_j} - q^2\right) - \frac{1}{4}(q-1).$$

We examined performance of our new transfomations under the following two nonnormal models:

(i) χ^2 distribution with 4 degrees of freedom;

(ii) Gamma distribution with shape parameter 3 and scale parameter 1/3.

Table 2.1 gives the true percentage point x_α and the approximate percentage points $t_E(u)$, $t_{AEI}(u)$, $t_K(u)$, $t_1(u)$ and $t_2(u)$ for the case $q = 3$. Here, u denotes the upper 5% point of χ_2^2, $t_E(u)$ is computed on the basis of the Cornish-Fisher expansion up to the order $O(n^{-1})$, and $t_{AEI}(u)$, $t_K(u)$, $t_1(u)$ and $t_2(u)$ are computed on the basis of Aoshima, Enoki and Ito (2003), Kakizawa (1996), (2.7) and (2.9) respectively. Note that the Cornish-Fisher expansion yields the percentage point x_α of S in the same form up to the order $O(n^{-1})$. It means that the transformations T aim to find an improvement of approximations to x_α in the terms of $O(n^{-2})$. Table 2.2 gives the actual test sizes denoted by

$$\alpha_1 = \mathrm{P}(T > u), \qquad \alpha_2 = \mathrm{P}(T > t_E(u)), \quad \alpha_3 = \mathrm{P}(T > t_{AEI}(u)),$$
$$\alpha_4 = \mathrm{P}(T > t_K(u)), \quad \alpha_5 = \mathrm{P}(T > t_1(u)), \quad \alpha_6 = \mathrm{P}(T > t_2(u))$$

for the case $q = 3$. For new transformation (2.9), we consider the following case:

$$a(x) = a_0 + \sum_{j=1}^{3} a_j \frac{x^j}{\prod_{\ell=0}^{j-1}(f+2\ell)}, \qquad p(x) = \begin{cases} 2\{a(x)\}^2 & \text{in (i)} \\ -\frac{4}{5}a(x) + \frac{8}{5}\{a(x)\}^2 & \text{in (ii)} \end{cases}.$$

Table 2.1 The percentage points when $q = 3$

	Sample sizes			Upper 5% points ($\chi_2^2(0.05) = 5.9915$)					
	n_1	n_2	n_3	x_α	$t_E(u)$	$t_{AEI}(u)$	$t_K(u)$	$t_1(u)$	$t_2(u)$
(i)	5	5	5	7.455	6.823	7.012	6.986	6.924	7.566
	10	10	10	6.501	6.407	6.449	6.443	6.433	6.528
	3	6	6	7.521	6.815	7.116	7.033	7.025	7.411
	5	5	10	6.916	6.609	6.759	6.720	6.723	6.884
(ii)	5	5	5	7.367	6.945	7.177	7.231	7.143	7.378
	10	10	10	6.375	6.468	6.519	6.528	6.517	6.545
	3	6	6	7.460	6.939	7.228	7.278	7.226	7.330
	5	5	10	6.918	6.702	6.849	6.870	6.859	6.887

100,000 replications

Table 2.2 The actual test sizes when $q = 3$

	Sample sizes			Nominal 5% test					
	n_1	n_2	n_3	α_1	α_2	α_3	α_4	α_5	α_6
(i)	5	5	5	8.077	6.084	5.736	5.782	5.895	4.798
				0.086	0.076	0.074	0.074	0.074	0.068
	10	10	10	6.200	5.191	5.100	5.112	5.131	4.941
				0.076	0.070	0.070	0.070	0.070	0.069
	3	6	6	8.260	6.260	5.690	5.835	5.849	5.172
				0.087	0.077	0.073	0.074	0.074	0.070
	5	5	10	7.151	5.621	5.303	5.389	5.383	5.061
				0.081	0.073	0.071	0.071	0.071	0.069
(ii)	5	5	5	7.958	5.719	5.290	5.199	5.370	4.984
				0.086	0.073	0.071	0.070	0.071	0.069
	10	10	10	5.917	4.814	4.696	4.670	4.703	4.640
				0.075	0.068	0.067	0.067	0.067	0.067
	3	6	6	8.119	5.921	5.379	5.290	5.381	5.196
				0.086	0.075	0.071	0.071	0.071	0.070
	5	5	10	7.067	5.426	5.144	5.100	5.124	5.067
				0.081	0.072	0.070	0.070	0.070	0.069

100,000 replications

From Tables 2.1-2.2, we can see that transformation (2.9) gives most significant improvement for the approximate percentage point among the others. Note that transformation (2.9) is severely affected by the function $p(x)$. As for an optimum choice of $p(x)$, it is under investigation. Note that the value of $t_1(u)$ is close to $t_{AEI}(u)$ or $t_K(u)$. It seems that transformation (2.7) does not make a significant difference from the predecessors.

Example 2.2 Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be independently and identically distributed as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let $\overline{\boldsymbol{X}} = \frac{1}{n}\sum_{j=1}^n \boldsymbol{X}_j$ and $S = \frac{1}{\nu}\sum_{j=1}^n(\boldsymbol{X}_j - \overline{\boldsymbol{X}})(\boldsymbol{X}_j - \overline{\boldsymbol{X}})'$ where $\nu = n - 1 \geq p$. Then, Hotelling's T^2-statistic is defined by

$$T^2 = n(\overline{\boldsymbol{X}} - \boldsymbol{\mu})'S^{-1}(\overline{\boldsymbol{X}} - \boldsymbol{\mu}).$$

The statistic is used for testing hypotheses about the mean vector $\boldsymbol{\mu}$ and for obtaining confidence regions for the unknown $\boldsymbol{\mu}$. Let us put $S = T^2$. Then, an asymptotic expansion for the distribution of S was given by Siotani (1971) as follows:

$$F(x) \equiv \mathrm{P}(S \leq x) = G_p(x) + \frac{1}{\nu}\sum_{j=0}^{2} a_{1j}G_{p+2j}(x) + \frac{1}{\nu^2}\sum_{j=0}^{4} a_{2j}G_{p+2j}(x) + O(\nu^{-3}),$$

where

$$a_{10} = -\frac{p^2}{4}, \quad a_{11} = -\frac{p}{2}, \quad a_{12} = \frac{1}{4}p(p+2), \quad a_{20} = \frac{1}{96}p(3p^3 - 8p^2 + 8), \quad a_{21} = \frac{p^3}{8},$$
$$a_{22} = -\frac{1}{16}p(p+2)(p^2 - 6), \quad a_{23} = -\frac{7}{24}p(p+2)(p+4), \quad a_{24} = \frac{1}{32}p(p+2)(p+4)(p+6).$$

Let

$$a_1(x) = a_{10} + \sum_{j=1}^{2} a_{1j}\frac{x^j}{\prod_{\ell=0}^{j-1}(p+2\ell)} \quad \text{and} \quad a_2(x) = a_{20} + \sum_{j=1}^{4} a_{2j}\frac{x^j}{\prod_{\ell=0}^{j-1}(p+2\ell)}.$$

Let $\tilde{F}_i(x)$ be the one formed by the terms of $F(x)$ up to the order $O(n^{-i})$. We examined performance of our new transformation (2.9) with the following setup:

(1) $\quad a(x) = a_1(x),\ p(x) = \frac{1}{2}\{a(x)\}^2;$

(2) $\quad a(x) = a_1(x),\ p(x) = \frac{1}{2}\{a(x)\}^2 - \frac{1}{3\nu}\{a(x)\}^3 + \frac{1}{4\nu^2}\{a(x)\}^4;$

(3) $\quad a(x) = a_1(x) + \frac{1}{\nu}a_2(x),\ p(x) = \frac{1}{2}\{a(x)\}^2;$

(4) $\quad a(x) = a_1(x) + \frac{1}{\nu}a_2(x),\ p(x) = \frac{1}{2}\{a(x)\}^2 - \frac{1}{3\nu}\{a(x)\}^3 + \frac{1}{4\nu^2}\{a(x)\}^4.$

Under the setup (1), (2), (3) or (4), we note for $F_*(x)$ defined by (2.8) that $F_*(x) = \tilde{F}_1(x) + o(n^{-2})$, $F_*(x) = \tilde{F}_1(x) + o(n^{-4})$, $F_*(x) = \tilde{F}_2(x) + o(n^{-2})$ and $F_*(x) = \tilde{F}_2(x) + o(n^{-4})$, respectively.

Table 2.3 gives the true percentage point x_α, the approximate percentage points t_E, t_1, t_{21}, t_{22}, $\tilde{t}_E$, $\tilde{t}_{21}$ and $\tilde{t}_{22}$ in the case $\alpha = 0.05$, and values of $F(x)$ at each percentage point. Here, t_E ($\tilde{t}_E$) and t_1 are computed on the basis of the Cornish-Fisher expansion up to the order $O(n^{-1})$

$(O(n^{-2}))$ and new transformation (2.7) respectively, and t_{21}, t_{22}, $\tilde{t}_{21}$ and $\tilde{t}_{22}$ are computed on the basis of new transformation (2.9) under the setups (1), (2), (3) and (4), respectively. By using the fact $\{(\nu - p + 1)/(\nu p)\}T^2$ is distributed as F-ratio distribution with the parameter $(p, \nu - p + 1)$, we obtained the exact value of x_α and $F(x)$ at each percentage point.

Table 2.3 The percentage points when $\alpha = 0.05$

p	ν	x_α	t_E	t_1	t_{21}	t_{22}	$\tilde{t}_E$	$\tilde{t}_{21}$	$\tilde{t}_{22}$
2	20	7.4145	7.1885	7.5335	7.2151	7.1984	7.3134	7.4477	7.4325
		0.95	0.9459	0.9518	0.9464	0.9461	0.9482	0.9506	0.9503
	40	6.6423	6.5900	6.6743	6.5994	6.5985	6.6212	6.6450	6.6442
		0.95	0.9489	0.9507	0.9491	0.9491	0.9496	0.9501	0.9501
	60	6.4131	6.3905	6.4278	6.3954	6.3952	6.4044	6.4138	6.4137
		0.95	0.9495	0.9503	0.9496	0.9496	0.9498	0.9500	0.9500
	80	6.3033	6.2907	6.3116	6.2937	6.2937	6.2985	6.3036	6.3035
		0.95	0.9497	0.9502	0.9498	0.9498	0.9499	0.9500	0.9500
	100	6.2389	6.2309	6.2442	6.2329	6.2329	6.2357	6.2390	6.2390
		0.95	0.9498	0.9501	0.9499	0.9499	0.9499	0.9500	0.9500
4	20	13.9516	12.6869	13.3255	14.2262	13.3120	13.2541	-	-
		0.95	0.9339	0.9427	0.9529	0.9425	0.9417	-	-
	40	11.3559	11.0873	11.3218	11.0363	10.9901	11.2291	11.4130	11.3553
		0.95	0.9457	0.9495	0.9449	0.9441	0.9480	0.9509	0.9500
	60	10.6677	10.5541	10.6674	10.5251	10.5156	10.6171	10.6771	10.6674
		0.95	0.9480	0.9500	0.9475	0.9474	0.9491	0.9500	0.9500
	80	10.3499	10.2875	10.3534	10.2711	10.2680	10.3230	10.3530	10.3500
		0.95	0.9489	0.9501	0.9486	0.9485	0.9495	0.9501	0.9500
	100	10.1669	10.1276	10.1704	10.1173	10.1160	10.1503	10.1683	10.1670
		0.95	0.9493	0.9501	0.9491	0.9491	0.9497	0.9500	0.9500
6	20	22.3237	18.4440	18.6314	-	-	19.9083	-	-
		0.95	0.9112	0.9137	-	-	0.9288	-	-
	40	16.2636	15.5178	15.8146	15.6351	15.2400	15.8839	-	-
		0.95	0.9402	0.9443	0.9419	0.9361	0.9452	-	-
	60	14.8498	14.5424	14.7200	14.4394	14.3582	14.7051	14.9598	14.8414
		0.95	0.9455	0.9482	0.9440	0.9427	0.9480	0.9515	0.9499
	80	14.2215	14.0547	14.1672	13.9778	13.9507	14.1462	14.2468	14.2139
		0.95	0.9475	0.9492	0.9462	0.9458	0.9489	0.9504	0.9499
	100	13.8666	13.7621	13.8387	13.7087	13.6970	13.8206	13.8758	13.8629
		0.95	0.9483	0.9496	0.9475	0.9473	0.9493	0.9501	0.9499

In Table 2.3, there are several cases in which the values of t_{21}, t_{22}, $\tilde{t}_{21}$, $\tilde{t}_{22}$ are not available. This is due to the fact that $\lim_{x\to\infty} F_*(x) < 0.95$. From Table 2.3, when the term of $O(n^{-2})$ is obtained, we can observe that the transformaitons give a significant improvement. Comparing t_{21} with t_{22}, it seems that t_{22} does not necessarily improve the approximation of t_{21}. On the other hand, comparing $\tilde{t}_{21}$ with $\tilde{t}_{22}$, we can see an excellent improvement in $\tilde{t}_{22}$. In general, the distance $|F_*(x) - \tilde{F}(x)|$ does not necessarily mean an improvement on the approximation to the percentage point. However, in the present situation that a theoretical determination of $p(x)$ has not been developed, it may be a clue to choose a function $p(x)$ such that $|F_*(x) - \tilde{F}(x)|$ becomes small.

3 Accuracy of improved chi-squared approximations

Suppose that a statistic S has an asymptotic χ^2-approximation as some parameter n tends to infinity. In this case, it is of considerable interest to construct improved χ^2-approximations for the statistic S. A typical approach is to consider a monotone transformation $T = T(S)$ based on the Bartlett correction or a Bartlett type correction. For a Bartlett type correction, we introduced several transformations in the previous section. In this section, our aim is to obtain an error bound for an improved χ^2-approximation.

We write (1.1) as

$$\mathrm{P}(S \le x) = G_f(x) + \frac{1}{n}\sum_{j=0}^{k} a_j G_{f+2j}(x) + R_k(x), \tag{3.1}$$

where $R_k(x) = O(n^{-2})$. For (3.1), Ulyanov and Fujikoshi (2001) proposed an error bound for an improved χ^2-approximation. We shall improve their result and give an extension to a non-uniform bound. Numerical examples of some test statistics are demonstrated to observe an improvement in Theorem 4.1 of Ulyanov and Fujikoshi (2001).

3.1 Uniform error bound

For the remainder term $R_k(x)$ in (3.1), we assume that there exists a positive constant c_k such that

$$|R_k(x)| \le c_k/n^2. \tag{3.2}$$

Theorem 3.1 *Suppose that there exists a positive, increasing function $b(x)$ defined on $[0, +\infty)$ such that, a positive, increasing function $p(x)$ which is bounded above by a polynomial and for some nonnegative constants B_i, $i = 1, 2, 3 : 0 < B_1 \le 1$, $B_2 < B_1/4$, the following conditions are satisfied for all $x > 0$:*

$$b(x) \ge B_1 x, \tag{3.3}$$

$$|b(x) - x| \le p(x)\exp(B_2 x)/n, \tag{3.4}$$

$$G'_f(x)\left| b(x) - x - \frac{2}{n}\sum_{j=1}^{k} a_j \sum_{m=1}^{j} \frac{x^m}{\prod_{\ell=0}^{m-1}(f+2\ell)} \right| \le B_3/n^2, \tag{3.5}$$

where a_j's are the same as in (3.1). If $\mathrm{P}(S \le x)$ can be written in the form (3.1) with (3.2), then

$$|\mathrm{P}(T(S) \le x) - G_f(x)| \le \tilde{c}/n^2, \tag{3.6}$$

where T is the inverse function to $b(x)$ and $\tilde{c}$ is a positive constant depending on f, $p(x)$, B_i, $i = 1, 2, 3$ and c_k in (3.2).

Proof. Since $G_f(x)$ is smooth for all $x > 0$, we can write

$$G_f(b(x)) = G_f(x) + G'_f(x)(b(x) - x) + \frac{1}{2}G''_f(x')(b(x) - x)^2, \tag{3.7}$$

where $x' \in (b(x) \wedge x, b(x) \vee x)$. It is known for $j \geq 1$ that

$$G_{f+2j}(x) = G_f(x) - 2G'_f(x) \sum_{m=1}^{j} \frac{x^m}{\prod_{\ell=0}^{m-1}(f + 2\ell)}. \tag{3.8}$$

Note that in (3.1), it is necessary that $\sum_{j=0}^{k} a_j = 0$. By using (3.1), (3.7) and (3.8), we obtain

$$\begin{aligned} &\mathrm{P}(T(S) \leq x) = \mathrm{P}(S \leq b(x)) \\ &= G_f(x) + G'_f(x)\left\{b(x) - x - \frac{2}{n}\sum_{j=1}^{k} a_j \sum_{m=1}^{j} \frac{x^m}{\prod_{\ell=0}^{m-1}(f+2\ell)}\right\} \\ &+ \frac{1}{2}G''_f(x')(b(x) - x)^2 + \frac{1}{n}\sum_{j=0}^{k} a_j G'_{f+2j}(x'')(b(x) - x) + R_k(b(x)), \end{aligned} \tag{3.9}$$

where $x'' \in (b(x) \wedge x, b(x) \vee x)$.

Now we construct a uniform bound for $G''_f(x')(b(x) - x)^2$. Note that when $f \neq 2$,

$$G''_f(x) = (2^{f/2}\Gamma(f/2))^{-1}\Big(-\frac{x}{2} + \frac{f}{2} - 1\Big)x^{f/2-2}e^{-x/2}. \tag{3.10}$$

We shall employ below the following inequality: For any positive numbers p and q, we have

$$|p - q| \leq \max\{p, q\}.$$

We consider two cases.

Case 1: $b(x) \leq x$.

Then $x' \in (b(x), x)$. When $f \geq 4$, we obtain from (3.3), (3.4) and (3.10) that

$$\begin{aligned} &\sup_{b(x)\leq x} |G''_f(x')|(b(x) - x)^2 \\ &\leq \frac{1}{n^2 2^{f/2}\Gamma(f/2)} \sup_{x>0}\left[\{p(x)\}^2 x^{\frac{f}{2}-2} e^{-\left(\frac{B_1}{2}-2B_2\right)x} \max\{x/2, f/2 - 1\}\right] \\ &\equiv c_{11}/n^2. \end{aligned} \tag{3.11}$$

When $f = 1, 3$, we obtain similarly to (3.11) that

$$\begin{aligned} &\sup_{b(x)\leq x} |G''_f(x')|(b(x) - x)^2 \\ &\leq \frac{B_1^{f/2-2}}{n^2 2^{f/2}\Gamma(f/2)} \sup_{x>0}\left[\{p(x)\}^2 x^{\frac{f}{2}-2} e^{-\left(\frac{B_1}{2}-2B_2\right)x} \max\{x/2, f/2 - 1\}\right] \\ &\equiv c_{12}/n^2. \end{aligned}$$

When $f = 2$, since

$$G''_f(x) = -\frac{1}{2^{f/2+1}\Gamma(f/2)}e^{-x/2}, \tag{3.12}$$

we obtain from (3.3) and (3.4) that

$$\sup_{b(x)\le x} |G_f''(x')|(b(x)-x)^2 \le \frac{1}{n^2 2^{f/2+1}\Gamma(f/2)} \sup_{x>0}\left[\{p(x)\}^2 e^{-\left(\frac{B_1}{2}-2B_2\right)x}\right]$$
$$\equiv c_{13}/n^2. \tag{3.13}$$

By the hypotheses of theorem, we have $c_{11}, c_{12}, c_{13} < \infty$.

Case 2: $b(x) > x$.
Then $x' \in (x, b(x))$. When $f \neq 2$, we obtain from (3.3), (3.4) and (3.10) that

$$\sup_{b(x)>x} |G_f''(x')|(b(x)-x)^2 \le \frac{1}{n^2} \sup_{b(x)>x}\left[|G_f''(x')|\{p(x')\}^2 e^{2B_2 x'}\right]$$
$$\le \frac{1}{n^2 2^{f/2}\Gamma(f/2)} \sup_{x>0}\left[\{p(x)\}^2 x^{\frac{f}{2}-2} e^{-(\frac{1}{2}-2B_2)x}\left|-\frac{x}{2}+\frac{f}{2}-1\right|\right]$$
$$\equiv c_{21}/n^2. \tag{3.14}$$

When $f = 2$, we obtain similarly to (3.13) and (3.14) that

$$\sup_{b(x)>x} |G_f''(x')|(b(x)-x)^2 \le \frac{1}{n^2 2^{f/2+1}\Gamma(f/2)} \sup_{x>0}\left[\{p(x)\}^2 e^{-(\frac{1}{2}-2B_2)x}\right]$$
$$\equiv c_{22}/n^2.$$

By the hypotheses of theorem, we have $c_{21}, c_{22} < \infty$. Since $c_{21} \le c_{11}, c_{12}$ and $c_{22} \le c_{13}$, by defining that

$$c_1 = \begin{cases} c_{11} & f \ge 4 \\ c_{12} & f = 1, 3\,, \\ c_{13} & f = 2 \end{cases}$$

we obtain that

$$\sup_{x>0} |G_f''(x')|(b(x)-x)^2 \le c_1/n^2. \tag{3.15}$$

By using (3.8), we can obtain similarly to (3.15) that when $f \ge 2$,

$$\left|\sum_{j=0}^{k} a_j G_{f+2j}'(x'')(b(x)-x)\right| = |b(x)-x|\left|2\sum_{j=1}^{k} a_j \sum_{m=1}^{j} G_{f+2m}''(x'')\right|$$
$$\le \frac{1}{n 2^{f/2}\Gamma(f/2)} p(x)(x'')^{\frac{f}{2}-1} e^{-\frac{x''}{2}+B_2 x}\left|\sum_{j=1}^{k} a_j \sum_{m=1}^{j} \frac{-x''+f+2m-2}{\prod_{\ell=0}^{m-1}(f+2\ell)}(x'')^{m-1}\right|$$
$$= \frac{1}{n 2^{f/2}\Gamma(f/2)} p(x)(x'')^{\frac{f}{2}-1} e^{-\frac{x''}{2}+B_2 x}\left|a_0 + \sum_{j=1}^{k} a_j \frac{(x'')^j}{\prod_{\ell=0}^{j-1}(f+2\ell)}\right|$$
$$\le \frac{1}{n 2^{f/2}\Gamma(f/2)} \sup_{x>0}\left[p(x) x^{\frac{f}{2}-1} e^{-\left(\frac{B_1}{2}-B_2\right)x}\left\{|a_0| + \sum_{j=1}^{k} |a_j| \frac{x^j}{\prod_{\ell=0}^{j-1}(f+2\ell)}\right\}\right]$$
$$\equiv c_{31}/n. \tag{3.16}$$

When $f = 1$, we obtain similarly to (3.15) and (3.16) that

$$\begin{aligned}&\left|\sum_{j=0}^{k} a_j G'_{f+2j}(x'')(b(x) - x)\right| \\ &\le \frac{B_1^{f/2-1}}{n 2^{f/2}\Gamma(f/2)} \sup_{x>0} \left[p(x) x^{\frac{f}{2}-1} e^{-\left(\frac{B_1}{2}-B_2\right)x} \left\{ |a_0| + \sum_{j=1}^{k} |a_j| \frac{x^j}{\prod_{\ell=0}^{j-1}(f+2\ell)} \right\} \right] \\ &\equiv c_{32}/n.\end{aligned}$$

By the hypotheses of theorem, we have $c_{31}, c_{32} < \infty$. By defining that

$$c_3 = \begin{cases} c_{31} & f \ge 2 \\ c_{32} & f = 1 \end{cases},$$

we obtain that

$$\sup_{x>0} \left| \sum_{j=0}^{k} a_j G'_{f+2j}(x'')(b(x) - x) \right| \le c_3/n. \tag{3.17}$$

Combining (3.2), (3.5), (3.15) and (3.17) with (3.9), we obtain (3.6) with

$$\tilde{c} = B_3 + \frac{1}{2}c_1 + c_3 + c_k.$$

This brings our proof to the end. ■

Remark 3.1 It is clear that a positive function $b(x)$ that satisfies (3.3)-(3.5) may not be increasing. Therefore, we have to require the existence of an increasing function $b(x)$ in Theorem 3.1. Ulyanov and Fujikoshi (2001) showed the existence of the required function $b(x)$.

Remark 3.2 It should be noted that the error bound given by (3.6) is sharper than Ulyanov and Fujikoshi (2001).

Remark 3.3 Note that an increasing function $b(x)$ that satisfies (3.3)-(3.5) is coincident with the Cornish-Fisher expansion for the quantile of S up to the order $O(n^{-1})$.

3.2 Non-uniform error bound

For the remainder term $R_k(x)$ in (3.1), we assume that there exists a positive function $C_k(x)$ such that

$$|R_k(x)| \le C_k(x)/n^2, \qquad C_k(x) \to 0 \quad (x \to \infty). \tag{3.18}$$

Theorem 3.2 *Suppose that there exists a positive and unbounded, increasing function $b(x)$ defined on $[0, +\infty)$ such that, for a positive, increasing function $p(x)$ which is bounded above by a polynomial, for some nonnegative constants B_i, $i = 1, 2, 3$: $0 < B_1 \le 1$, $\max\{2, f/2 + k\}B_2 <$*

$B_1/2$, and for a positive function $B(x)$: $B(x) \to 0 \quad (x \to \infty)$, (3.3), (3.4) and the following condition are satisfied for all $x > 0$:

$$G_f'(x)\left|b(x) - x - \frac{2}{n}\sum_{j=1}^{k} a_j \sum_{m=1}^{j} \frac{x^m}{\prod_{\ell=0}^{m-1}(f+2\ell)}\right| \le B(x)/n^2, \tag{3.19}$$

where a_j's are the same as in (3.1). If $\mathrm{P}(S \le x)$ can be written in the form (3.1) with (3.18), then

$$|\mathrm{P}(T(S) \le x) - G_f(x)| \le \tilde{C}(x)/n^2, \qquad \tilde{C}(x) \to 0 \quad (x \to \infty), \tag{3.20}$$

where T is the inverse function to $b(x)$ and $\tilde{C}(x)$ is a positive function depending on f, $B(x)$ and $C_k(x)$ in (3.18).

Proof. From (3.4), we obtain for all $x > 0$ that

$$\begin{aligned} b(x) &\le x + \frac{1}{n}p(x)\exp(B_2 x) \\ &\le \left\{x + \frac{1}{n}p(x)\right\}\exp(B_2 x). \end{aligned} \tag{3.21}$$

First we construct a non-uniform bound for $G_f''(x')$ in (3.9). Similarly to the proof of Theorem 3.1, we consider two cases.

Case 1: $b(x) \le x$.

Then $x' \in (b(x), x)$. When $f \ge 4$, we obtain from (3.10) that

$$\begin{aligned} |G_f''(x')| &\le \frac{1}{2^{f/2}\Gamma(f/2)} x^{\frac{f}{2}-2} e^{-\frac{1}{2}b(x)} \max\{x/2, f/2-1\} \\ &\equiv C_{11}(x). \end{aligned} \tag{3.22}$$

When $f = 1, 3$, we obtain similarly to (3.22) that

$$\begin{aligned} |G_f''(x')| &\le \frac{1}{2^{f/2}\Gamma(f/2)} \{b(x)\}^{\frac{f}{2}-2} e^{-\frac{1}{2}b(x)} \max\{x/2, f/2-1\} \\ &\equiv C_{12}(x). \end{aligned}$$

When $f = 2$, we obtain from (3.12) that

$$\begin{aligned} |G_f''(x')| &\le \frac{1}{2^{f/2+1}\Gamma(f/2)} e^{-\frac{1}{2}b(x)} \\ &\equiv C_{13}(x). \end{aligned} \tag{3.23}$$

By the hypotheses of theorem, we have $\{C_{11}(x) + C_{12}(x) + C_{13}(x)\}(b(x) - x)^2 \to 0$ as $x \to \infty$.

Case 2: $b(x) > x$.

Then $x' \in (x, b(x))$. When $f \ge 4$, we obtain similarly to (3.22) that

$$\begin{aligned} |G_f''(x')| &\le \frac{1}{2^{f/2}\Gamma(f/2)} \{b(x)\}^{\frac{f}{2}-2} e^{-\frac{x}{2}} \max\{b(x)/2, f/2-1\} \\ &\equiv C_{21}(x). \end{aligned} \tag{3.24}$$

When $f = 1, 3$, we obtain similarly to (3.24) that

$$\begin{aligned}|G''_f(x')| &\le \frac{1}{2^{f/2}\Gamma(f/2)} x^{\frac{f}{2}-2} e^{-\frac{x}{2}} \max\{b(x)/2, f/2-1\} \\ &\equiv C_{22}(x).\end{aligned}$$

When $f = 2$, we obtain similarly to (3.23) that

$$\begin{aligned}|G''_f(x')| &\le \frac{1}{2^{f/2+1}\Gamma(f/2)} e^{-\frac{x}{2}} \\ &\equiv C_{23}(x).\end{aligned}$$

By the hypotheses of theorem and (3.21), we have $\{C_{21}(x) + C_{22}(x) + C_{23}(x)\}(b(x) - x)^2 \to 0$ as $x \to \infty$. Let

$$C_1(x) = \begin{cases} C_{11}(x) & f \ge 4 \\ C_{12}(x) & f = 1, 3 \\ C_{13}(x) & f = 2 \end{cases}, \qquad C_2(x) = \begin{cases} C_{21}(x) & f \ge 4 \\ C_{22}(x) & f = 1, 3 \\ C_{23}(x) & f = 2 \end{cases}.$$

By defining that

$$\tilde{C}_1(x) = \begin{cases} C_1(x) & b(x) \le x \\ C_2(x) & b(x) > x \end{cases},$$

we obtain for all $x > 0$ that

$$|G''_f(x')| \le \tilde{C}_1(x), \qquad \tilde{C}_1(x)(b(x) - x)^2 \to 0 \quad (x \to \infty). \tag{3.25}$$

Next we construct a non-uniform bound for $\sum_{j=0}^{k} a_j G'_{f+2j}(x'')$ in (3.9). Similarly to (3.25), we consider two cases.

Case 1: $b(x) \le x$.

When $f \ge 2$, by refering to (3.16), we obtain

$$\begin{aligned}\left|\sum_{j=0}^{k} a_j G'_{f+2j}(x'')\right| &\le \frac{1}{2^{f/2}\Gamma(f/2)} (x'')^{\frac{f}{2}-1} e^{-\frac{x''}{2}} \left\{|a_0| + \sum_{j=1}^{k} |a_j| \frac{(x'')^j}{\prod_{\ell=0}^{j-1}(f+2\ell)}\right\} \\ &\le \frac{1}{2^{f/2}\Gamma(f/2)} x^{\frac{f}{2}-1} e^{-\frac{b(x)}{2}} \left\{|a_0| + \sum_{j=1}^{k} |a_j| \frac{x^j}{\prod_{\ell=0}^{j-1}(f+2\ell)}\right\} \\ &\equiv C_{31}(x).\end{aligned} \tag{3.26}$$

When $f = 1$, we obtain

$$\begin{aligned}\left|\sum_{j=0}^{k} a_j G'_{f+2j}(x'')\right| &\le \frac{1}{2^{f/2}\Gamma(f/2)} \{b(x)\}^{\frac{f}{2}-1} e^{-\frac{b(x)}{2}} \left\{|a_0| + \sum_{j=1}^{k} |a_j| \frac{x^j}{\prod_{\ell=0}^{j-1}(f+2\ell)}\right\} \\ &\equiv C_{32}(x).\end{aligned} \tag{3.27}$$

Case 2: $b(x) > x$.
When $f \geq 2$, we obtain similarly to (3.26) that

$$\begin{aligned}\left|\sum_{j=0}^{k} a_j G'_{f+2j}(x'')\right| &\leq \frac{1}{2^{f/2}\Gamma(f/2)}\{b(x)\}^{\frac{f}{2}-1}e^{-\frac{x}{2}}\left\{|a_0| + \sum_{j=1}^{k}|a_j|\frac{\{b(x)\}^j}{\prod_{\ell=0}^{j-1}(f+2\ell)}\right\} \\ &\equiv C_{41}(x).\end{aligned}$$

When $f = 1$, we obtain similarly to (3.27) that

$$\begin{aligned}\left|\sum_{j=0}^{k} a_j G'_{f+2j}(x'')\right| &\leq \frac{1}{2^{f/2}\Gamma(f/2)}x^{\frac{f}{2}-1}e^{-\frac{x}{2}}\left\{|a_0| + \sum_{j=1}^{k}|a_j|\frac{\{b(x)\}^j}{\prod_{\ell=0}^{j-1}(f+2\ell)}\right\} \\ &\equiv C_{42}(x).\end{aligned}$$

By the hypotheses of theorem and (3.21), we have $\{C_{31}(x)+C_{32}(x)+C_{41}(x)+C_{42}\}|b(x)-x| \to 0$ as $x \to \infty$. Let

$$C_3(x) = \begin{cases} C_{31}(x) & f \geq 2 \\ C_{32}(x) & f = 1 \end{cases}, \qquad C_4(x) = \begin{cases} C_{41}(x) & f \geq 2 \\ C_{42}(x) & f = 1 \end{cases}.$$

By defining that

$$\tilde{C}_2(x) = \begin{cases} C_3(x) & b(x) \leq x \\ C_4(x) & b(x) > x \end{cases},$$

we obtain for all $x > 0$ that

$$\left|\sum_{j=0}^{k} a_j G'_{f+2j}(x'')\right| \leq \tilde{C}_2(x), \qquad \tilde{C}_2(x)|b(x)-x| \to 0 \quad (x \to \infty). \tag{3.28}$$

We define

$$\tilde{C}(x) = B(x) + \frac{n^2}{2}\tilde{C}_1(x)(b(x)-x)^2 + n\tilde{C}_2(x)|b(x)-x| + C_k(b(x)). \tag{3.29}$$

Then, by the hypotheses of theorem, (3.25) and (3.28), we have $\tilde{C}(x) \to 0 \quad (x \to \infty)$. Therefore, by combining (3.18), (3.19), (3.25) and (3.28) with (3.9), we obtain (3.20) with (3.29). ■

Remark 3.4 It should be noted that the error bound given by (3.20) is sharper than (3.6).

Remark 3.5 Note that an increasing function $b(x)$ that satisfies (3.3), (3.4) and (3.19) is coincident with the Cornish-Fisher expansion for the quantile of S up to the order $O(n^{-1})$.

Remark 3.6 For a constant c_k and a function $C_k(x)$ defined by (3.2) and (3.18), respectively, if $\sup_{x>0} C_k(x) < c_k$, then $\sup_{x>0} \tilde{C}_k(x) < \tilde{c}_k$, where $\tilde{c}_k$ and $\tilde{C}_k(x)$ are defined by (3.6) and (3.20), respectively. Therefore, we can obtain a smaller uniform bound.

3.3 Applications

Here, we examine Theorem 3.1 and Theorem 3.2 numerically. For Theorem 3.1, we compare with Theorem 4.1 in Ulyanov and Fujikoshi (2001). We conducted simulation experiments as follows: For parameters given in advance, error bounds for the remainder term of type (3.2) or (3.18) were calculated. We refered to Fujikohi (1993) for a calculation of (3.2). For a calculation of (3.18), we refered to Fujikoshi (1988, 1993) and obtained non-uniform bounds of type $\frac{c}{1+x^\ell}$ with a constant c. Here, we set $\ell = 1$. By using these error bounds and giving some constants and functions, we calculated error bounds of type (3.6) or (3.20). Tables 3.1 and 3.3 present values of the uniform bound (3.2) on the first line in each cell and values of the uniform bound (3.6) given by Theorem 3.1 (Theorem 4.1 in Ulyanov and Fujikoshi (2001)) on the third (second) line in each cell. As for non-uniform bounds, Tables 3.2 and 3.4 give values of the error bound (3.20) given by Theorem 3.2 ((3.18)) on the second (first) line in each cell. As for non-uniform bounds, we consider a case $x = u_\alpha$, the upper α point of χ_f^2.

Example 3.1 We consider the case when $S = \chi_f^2/Y$ with $Y = \chi_n^2/n$ where χ_f^2 and Y are independent. An asymptotic expansion for the distribution of S was given by Siotani (1956) in the form (3.1) with

$$k = 2, \quad a_0 = \frac{1}{4}f(f-2), \quad a_1 = -\frac{1}{2}f^2, \quad a_2 = \frac{1}{4}f(f+2).$$

Then, a uniform bound for the remainder term, of type (3.2), can be obtained (see, e.g., Fujikoshi (1993) and Shimizu and Fujikoshi (1997)). Therefore, we can apply Theorem 3.1 to this case.

Let

$$\begin{aligned} b(x) &= x + \frac{2}{n}\sum_{j=1}^{k} a_j \sum_{m=1}^{j} \frac{x^m}{\prod_{\ell=0}^{m-1}(f+2\ell)} \\ &= x - \frac{1}{n}\left\{\left(\frac{f}{2}-1\right)x - \frac{1}{2}x^2\right\}. \end{aligned}$$

Note that for $n \geq (f/2-1)$, $b(x)$ is monotone. We examine Theorem 3.1 and compare with Theorem 4.1 in Ulyanov and Fujikoshi (2001). On Theorem 4.1 in Ulyanov and Fujikoshi (2001), we set constants D_i as follows:

$$D_1 = 1 - \frac{1}{n}\left(\frac{f}{2}-1\right), \quad D_2 = \begin{cases} 1 & f=2 \\ \frac{f}{2}-1 & f>2 \end{cases}, \quad D_3 = \begin{cases} \frac{1}{2e} & f=2 \\ a & f>2 \end{cases}, \quad D_4 = 0,$$

where a is a constant that satisfies $\frac{1}{a(f-2)} - \frac{1}{a(f-2)}\log\frac{1}{a(f-2)} + 1 = 0$. On the other hand, we set constants and a function in Theorem 3.1 as follows:

$$B_1 = 1 - \frac{1}{n}\left(\frac{f}{2}-1\right), \qquad B_2 = B_3 = 0, \qquad p(x) = \begin{cases} \frac{x^2}{2} & f=2 \\ \max\{\frac{x}{2}-\frac{f}{2}+1, \frac{f}{2}-1\} & f>2 \end{cases}.$$

Table 3.1 Uniform error bounds

$f \backslash n$	10	20	30	40	50	70	100	150
	1.3645	0.0311	0.0081	0.0036	0.0020	0.0009	0.0004	0.0002
2	1.4928	0.0632	0.0224	0.0116	0.0072	0.0035	0.0017	0.0007
	1.4593	0.0548	0.0187	0.0095	0.0058	0.0028	0.0013	0.0006
	1.4011	0.0316	0.0082	0.0036	0.0020	0.0009	0.0004	0.0002
4	2.2471	0.1921	0.0737	0.0390	0.0241	0.0119	0.0057	0.0025
	2.0690	0.1600	0.0607	0.0319	0.0197	0.0097	0.0046	0.0020
	0.3794	0.0084	0.0021	0.0009	0.0005	0.0002	0.0001	0.00004
6	5.3692	0.7058	0.2628	0.1358	0.0827	0.0399	0.0188	0.0081
	4.1610	0.4760	0.1693	0.0855	0.0513	0.0244	0.0113	0.0048
	1.0496	0.0266	0.0075	0.0035	0.0020	0.0009	0.0004	0.0002
8	35.6148	2.8350	0.9156	0.4429	0.2596	0.1200	0.0547	0.0230
	25.7372	1.5949	0.4718	0.2183	0.1246	0.0559	0.0249	0.0103
	0.1141	0.0038	0.0012	0.0006	0.0003	0.0002	0.00008	0.00004
10	299.30	10.5607	2.7890	1.2314	0.6851	0.2991	0.1308	0.0533
	220.26	5.3625	1.2374	0.5116	0.2835	0.1232	0.0538	0.0220

Table 3.2 Non-uniform error bounds at $x = u_\alpha$

α	$f \backslash n$	10	20	30	40	50	70	100	150
	2	-	0.1244	0.0212	0.0078	0.0039	0.0015	0.0006	0.0002
		-	0.1364	0.0305	0.0133	0.0075	0.0034	0.0015	0.0006
	4	-	0.0530	0.0090	0.0033	0.0016	0.0006	0.0003	0.0001
		-	0.1757	0.0600	0.0308	0.0188	0.0091	0.0043	0.0019
0.05	6	-	0.0079	0.0012	0.0004	0.0002	0.00008	0.00003	0.00001
		-	0.4211	0.1538	0.0787	0.0476	0.0228	0.0107	0.0046
	8	-	0.0265	0.0051	0.0021	0.0011	0.0005	0.0002	0.00009
		-	1.0922	0.3652	0.1783	0.1049	0.0486	0.0222	0.0093
	10	-	0.0497	0.0083	0.0030	0.0015	0.0006	0.0002	0.00009
		-	2.5054	0.7635	0.3545	0.2021	0.0904	0.0402	0.0165
	2	-	0.0852	0.0145	0.0053	0.0027	0.0011	0.0004	0.0002
		-	0.0992	0.0244	0.0111	0.0064	0.0029	0.0013	0.0006
	4	-	0.0389	0.0066	0.0024	0.0012	0.0005	0.0002	0.00007
		-	0.1469	0.0490	0.0246	0,0148	0.0071	0.0033	0.0014
0.01	6	-	0.0053	0.0009	0.0003	0.0002	0.00006	0.00003	0.00001
		-	0.3703	0.1230	0.0599	0.0352	0.0164	0.0075	0.0032
	8	-	0.0207	0.0040	0.0016	0.0009	0.0004	0.0002	0.00007
		-	0.9976	0.2947	0.1344	0.0759	0.0336	0.0149	0.0061
	10	-	0.0397	0.0066	0.0024	0.0012	0.0005	0.0002	0.00007
		-	2.4427	0.6385	0.2719	0.1468	0.0617	0.0262	0.0105

Further we consider non-uniform error bounds at $x = u_\alpha$. An error bound for the remainder term of type (3.18) can be obtained (see, e.g., Fujikoshi (1988, 1993) and Ulyanov, Fujikoshi and Shimizu (1999)). Therefore, Theorem 3.2 is also applicable to this case.

From Table 3.1, we can see a significant improvement to Theorem 4.1 in Ulyanov and Fujikoshi (2001). Non-uniform bounds in this example are defined in the case $n > 10$. Therefore, when $n = 10$ in Table 3.2, the error bound is not available. We can see that error bounds $\tilde{c}$ and $\tilde{C}(x)$ tend to enlarge as f becomes large. Especially, non-uniform bounds $\tilde{C}(x)$ at $x = u_\alpha$ have a tendency to be small as α is small. That is, Theorem 3.2 improves uniform bounds more successfully in the tail part of the distribution of S. A non-uniform bound of type (3.18) proposed by Fujikoshi (1988) does not necessarily improve the uniform bound of type (3.2) (see Fujikoshi (1988)). In fact, from Table 3.1, we can see the phenomenon. On the other hand, we observe in Table 3.2 that Theorem 3.2 improves uniform bounds in most cases.

Example 3.2 Suppose that, for a p-variate normal population $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are unknown, we wish to construct a set of simultaneous confidence intervals on $\boldsymbol{a}'\boldsymbol{\mu}$ with a given length 2ℓ for all $\boldsymbol{a}$, $\boldsymbol{a}'\boldsymbol{a} = 1$. A solution to this problem, given by Hyakutake and Siotani (1987), is as follows: First, take a pilot sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ of a given size m and compute

$$\overline{\boldsymbol{X}} = \frac{1}{m}\sum_{j=1}^{m}\boldsymbol{X}_j, \qquad \boldsymbol{S} = \frac{1}{\nu}\sum_{j=1}^{m}(\boldsymbol{X}_j - \overline{\boldsymbol{X}})(\boldsymbol{X}_j - \overline{\boldsymbol{X}})',$$

where $\nu = m - 1 \geq p$. Then, define the total sample size as

$$N = \max\{m + p^2, [c \cdot \mathrm{tr}(\boldsymbol{T}\boldsymbol{S})] + 1\},$$

where c is a positive constant, $[a]$ stands for the greatest integer less than a real number a, and $\boldsymbol{T}$ is a given positive difinite matrix which is assumed to be symmetric. Next, take an additional sample $\boldsymbol{X}_{m+1}, \ldots, \boldsymbol{X}_N$ of size $N - m$ and construct the basic random variate $\boldsymbol{Z}$ in the following way:

Choosing p matrices $\boldsymbol{A}_j : p \times N = [\boldsymbol{a}_1^{(j)}, \ldots, \boldsymbol{a}_m^{(j)}, \boldsymbol{a}_{m+1}^{(j)}, \ldots, \boldsymbol{a}_N^{(j)}]$, $j = 1, \ldots, p$, satisfying that

(1) $\boldsymbol{a}_1^{(j)} = \cdots = \boldsymbol{a}_m^{(j)} \equiv \boldsymbol{a}_0^{(j)}$ (say) $(j = 1, \ldots, p)$;

(2) $\boldsymbol{A}_j \boldsymbol{1}_N = \boldsymbol{e}_j$, where $\boldsymbol{1}_N : N \times 1 = (1, 1, \ldots, 1)'$ and $\boldsymbol{e}_j : p \times 1 = (0, \ldots, 0, \overset{(j)}{1}, 0, \ldots, 0)'$;

(3) $\boldsymbol{A}\boldsymbol{A}' = \frac{1}{c}\boldsymbol{T}^{-1} \otimes \boldsymbol{S}^{-1}$, where $\boldsymbol{A} : p^2 \times N = [\boldsymbol{A}_1', \boldsymbol{A}_2', \ldots, \boldsymbol{A}_p']'$ and $\otimes$ denotes the direct product.

Define a new random vector $\boldsymbol{Z}$ by

$$\boldsymbol{Z} : p \times 1 = [\mathrm{tr}(A_1\boldsymbol{X}'), \mathrm{tr}(A_2\boldsymbol{X}'), \ldots, \mathrm{tr}(A_p\boldsymbol{X}')]'$$

where $\boldsymbol{X} : p \times N = [\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m, \boldsymbol{X}_{m+1}, \ldots, \boldsymbol{X}_N]$. Now, by using the statistic

$$S = \frac{c}{2p}(\boldsymbol{Z} - \boldsymbol{\mu})'\boldsymbol{T}(\boldsymbol{Z} - \boldsymbol{\mu}),$$

taking $\boldsymbol{T} = \boldsymbol{I}_p$ and choosing c as $c = 2px_\alpha/\ell^2$ with x_α the upper α point of S, the solution is obtained as follows:

$$\boldsymbol{a}'\boldsymbol{\mu} \in [\boldsymbol{a}'\boldsymbol{Z} \pm \ell\,] \text{ for all } \boldsymbol{a} \text{ such that } \boldsymbol{a}'\boldsymbol{a} = 1.$$

When $p = 1, 2$, we can evaluate the distribution of S exactly. For $p \geq 3$, the exact treatment of the distribution of S becomes complicated. Hyakutake and Siotani (1987) gave an asymptotic expansion of S in the form (1.1) with $k = 2$, $f = p$, $n = \nu$ and the coefficients given by

$$a_0 = -\frac{1}{4}(2p^2 + p - 2), \qquad a_1 = \frac{1}{2}(p^2 - 2), \qquad a_2 = \frac{1}{4}(p + 2).$$

Then, a uniform bound for the remainder term, of type (3.2), can be obtained (see, e.g., Fujikoshi (1993), Mukaihata and Fujikoshi (1993) and Shimizu and Fujikoshi (1997)). Therefore, we can apply Theorem 3.1 to this case. Let

$$\begin{aligned} b(x) &= x + \frac{2}{\nu}\sum_{j=1}^{2} a_j \sum_{m=1}^{j} \frac{x^m}{\prod_{\ell=0}^{m-1}(p+2\ell)} \\ &= x + \frac{1}{2p\nu}\{(2p^2+p-2)x + x^2\}. \end{aligned}$$

Note that $b(x)$ is monotone. On Theorem 4.1 in Ulyanov and Fujikoshi (2001), we set constants D_i as follows:

$$D_1 = 1, \quad D_2 = \frac{2p^2+p-2}{2p}, \quad D_3 = \frac{1}{2p^2+p-2}, \qquad D_4 = 0.$$

On the other hand, we set constants and a function in Theorem 3.1 as follows:

$$B_1 = 1, \qquad B_2 = B_3 = 0, \qquad p(x) = \frac{1}{2p}\{(2p^2+p-2)x + x^2\}.$$

Similarly to Example 3.1, we also consider non-uniform error bounds at $x = u_\alpha$.

Let

$$\sigma = \frac{1}{p}\mathrm{tr}(\Sigma S^{-1}).$$

In order to obtain error bounds for the remainder term, of type (3.2) or (3.18), by using a method by Fujikoshi (1993), it is necessary to evaluate the exact moments of $\sigma^{\pm 1}$. It is difficult to obtain those in general p, however, Mukaihata and Fujikoshi (1993) gave the ones in the case $p = 2$. Here, we examine Theorem 3.1 and Theorem 3.2 in the case $p = 2$.

Table 3.3 Uniform error bounds when $p = 2$

$\nu = 10$	20	30	40	50	70	100	150
1.5955	0.0132	0.0032	0.0014	0.0008	0.0004	0.0002	0.00007
1.7590	0.0541	0.0214	0.0116	0.0073	0.0037	0.0018	0.0008
1.7015	0.0397	0.0150	0.0080	0.0050	0.0025	0.0012	0.0005

Table 3.4 Non-uniform error bounds at $x = u_\alpha$ when $p = 2$

$\alpha \backslash \nu$	10	20	30	40	50	70	100	150
0.05	-	0.0410	0.0067	0.0025	0.0013	0.0005	0.0002	0.00009
	-	0.0578	0.0154	0.0075	0.0045	0.0021	0.0010	0.0004
0.01	-	0.0281	0.0046	0.0017	0.0009	0.0004	0.0001	0.00006
	-	0.0399	0.0109	0.0053	0.0031	0.0015	0.0007	0.0003

From Table 3.3, we can see a significant improvement to Theorem 4.1 in Ulyanov and Fujikoshi (2001). Non-uniform bounds in this example are defined in the case $n > 11$. Therefore, when $n = 10$ in Table 3.4, the error bound is not available. From Tables 3.3-3.4, we can see a similar tendency to Example 3.1 for error bounds $\tilde{c}$ and $\tilde{C}(x)$.

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