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On the density of the set of primes which are related to decimal expansion of rational numbers

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We give several conjectures on the set of prime numbers which are closely related to 10-adic decimal expansion of rational numbers. The starting point is the following theorem.

Theorem 1 *Let* $p\,(\neq 2,5)$ *be a prime number.* $1/p$ *has a purely periodic decimal expansion*

$$1/p = 0.\dot{c}_1\cdots\dot{c}_e = 0.c_1\cdots c_e c_1\cdots c_e\cdots, \quad (0 \le c_i \le 9)$$

where we assume that e *is the minimal length of periods, i.e.* $e =$ *the order of* $10 \mod p$. *Suppose* $e = nk$ *for natural numbers* $n\ (>1), k$. *We divide the period to* n *parts of equal length and add them. Then we have*

$$\begin{aligned} & c_1\cdots c_k + c_{k+1}\cdots c_{2k} + \cdots + c_{(n-1)k+1}\cdots c_{nk} \\ = \quad & 9\cdots 9 \times \begin{cases} n/2 \text{ if } n \text{ is even,} \\ s(p) \text{ if } n \text{ is odd,} \end{cases} \end{aligned}$$

where $9\cdots 9 = 10^k - 1$ *and* $s(p)$ *is an integer such that* $1 \le s(p) \le n-2$.

We are concerned with the density of the set of primes for given n and $s = s(p)$. Hereafter we assume that $n\ (\ge 3)$ is an odd natural number and $1 \le s \le n-2$. Put

$$P(n,s,x) = \frac{\#\{p \mid p \le x, n|e, s(p) = s\}}{\#\{p \mid p \le x, n|e\}},$$

where $p\ (\neq 2,5)$ stands for a prime number and $e =$ the order of 10 mod p.

The following table of $P(n,s,10^9)$ is made by computer.

s	n = 5	n = 9	n = 11
1	0.1666	0	0.0000
2	0.6667	0	0.0014
3	0.1667	0.2499	0.0403
4		0.5001	0.2432
5		0.2500	0.4301
6		0	0.2433
7		0	0.0403
8			0.0014
9			0.0000

As a matter of fact, the graph of $P(n, s, x)$ in x is almost straight line. The ratios are symmetric at $(n-1)/2$. In the table, 0.0000 means that primes which take the values $s = 1, 9$ are very rare in the case of $n = 11$, and 0 for $n = 9$ means that the set is empty, which can be proven. The first conjecture is

Conjecture 1 $\lim_{x\to\infty} P(n, s, x)$ *exists, and by denoting it by* $P(n, s)$

$$P(n, s) = P(n, n-1-s) \text{ for } 1 \le s \le n-2.$$

Moreover $P(n, s) > 0$ *holds if* n *is an odd prime number.*

Moreover the table above looks like normal distribution. Let us recall notations of statistics. For the table of frequency distribution

value	x_1	x_2	$\cdots$	x_m	sum
relative frequency	r_1	r_2	$\cdots$	r_m	1

define the average μ and the standard deviation σ by

$$\mu = \sum_{i=1}^{m} x_i r_i, \quad \sigma = \sqrt{\sum_{i=1}^{m} x_i^2 r_i - \mu^2}.$$

Then we get

n	μ	σ
5	2.0001	0.5774
9	4.0002	0.7070
11	5.0002	0.9132
37	18.0010	1.7325

This table suggests

Conjecture 2

$$\lim_{x\to\infty} \mu = (n-1)/2.$$

To formulate being normal distribution, we denote the density function of normal distribution of average μ and standard deviation σ by

$$f_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

and compare the ratio with it. The table is

n	$\max_{1\le s\le n-2} \lvert P(n,s,x) - f_{\mu,\sigma}(s)\rvert$
5	0.0243
9	0.0641
11	0.0067
37	0.0006

This table and more general table for odd $n \le 101$ suggest

Conjecture 3

$$\lim_{n\to\infty}\lim_{x\to\infty}\max_{1\le s\le n-2} \lvert P(n,s,x) - f_{\mu,\sigma}(s)\rvert = 0.$$

We considered 10-adic expansion. But in the proof of Theorem 1, the number 10 is not important. It is generalized as follows:

Theorem 2 *Let a $(\neq 0, \pm 1)$ be an integer and p a prime number. Put $e =$ the order of $a \mod p$ and suppose $e = nk$, where $n \ge 3$ and $(a^k - 1, p) = 1$. Define an integer r_i by*

$$r_i \equiv a^{ki} \mod p, \quad 0 \le r_i < p.$$

Then $s(p) = (\sum_{i=0}^{n-1} r_i)/p$ is an integer such that $1 \le s(p) \le n-2$.

The former part is the case of $a = 10$. Similarly as above, we put

$$P_a(n,s,x) = \frac{\#\{p \mid p \le x, n|e, s(p) = s\}}{\#\{p \mid p \le x, n|e\}}.$$

The numerical data suggest the final

Conjecture 4

$$\lim_{x\to\infty} P_a(n,s,x) = \lim_{x\to\infty} P_{10}(n,s,x) (= P(n,s)).$$

The proof of theorems are easy and other probably new observations will be included in 本格的に代数を学ぶ前に.