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Blow-up solutions for quasilinear degenerate elliptic equation

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Abstract

We treat the equations with a positive nonlinearity in the right hand side. Namely

$$\begin{cases} L_p(u) = \lambda f(u), & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where

$$L_p(u) = -\operatorname{div}(|\nabla u|^{p-2}\nabla u) \tag{0.2}$$

Here $\lambda \geq 0$, and the nonlinearity f is, roughly speaking, positive, increasing and strictly convex on $[0, +\infty)$. In connection with combustion theory and other applications, we are interested in the study of positive minimal solutions. This is a résumé of the preprint [9].

1 Introduction.

In connection with combustion theory and other applications, we are interested in the study of positive solutions of the following:

$$\begin{cases} L_p(u) = \lambda f(u), & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$L_p(u) = -\operatorname{div}(|\nabla u|^{p-2}\nabla u) \tag{1.2}$$

Here $\lambda \geq 0$, and the nonlinearity f is, roughly speaking, positive, increasing and strictly convex on $[0, +\infty)$.

When $p = 2$, it is known that there is a finite number λ^* such that (1.2) has a classical positive solution $u \in C^2(\overline{\Omega})$ if $0 < \lambda < \lambda^*$. On the other hand no solution exists, even in the weak sense, for $\lambda > \lambda^*$. This value λ^* is often called the extremal value and solutions for this extremal value are called extremal solutions. It has been a very interesting problem to study the properties of these extremal solutions.

As for a nonlinearity $f(t)$ we adopt the following.

Definition 1.1 $f(t) \in C^1([0, +\infty))$, *increasing, strictly convex and*

$$f(0) > 0, \qquad \liminf_{t\to\infty} \frac{f'(t)t}{f(t)} > p - 1.$$

Definition 1.2 (*Weak solution*)

A function $u \in W_0^{1,p}(\Omega)$ *is called a weak solution if* $f(u)$ *satisfy*

$$dist(x, \partial\Omega) \cdot f(u) \in L^1(\Omega)$$

and u *satisfies*

$$\int_\Omega (|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - \lambda f(u)\varphi)\, dx = 0$$

for all $\varphi \in C_0^1(\Omega)$.

Lemma 1.1 *Let* $u \in W_0^{1,p}(\Omega) \cap L^\infty$ *be a weak solution. Then* $\exists C > 0$ *and* $\exists \sigma \in (0, 1)$ *such that*

$$\begin{cases} |\nabla u| \le C, \\ |\nabla u(x) - \nabla u(y)| \le C|x - y|^\sigma. \end{cases} \tag{1.3}$$

Then we have

Lemma 1.2

$^\exists u$; *a classical solution for a sufficiently small* $\lambda > 0$.

2 Minimal solution and extremal solution

Definition 2.1 (Minimal solution)

The mimimal solution $u_\lambda \in C^1(\overline{\Omega})$ *is the smallest solution among all possible solutions.*

Then we have

Lemma 2.1 $\exists_1\, u_\lambda \in C^1(\overline{\Omega})$; *the minimal solution for a sufficiently small* $\lambda \geq 0$.

Lemma 2.2 u_λ *satisfies:*

1. $u_\lambda \in C^{1,\sigma}(\overline{\Omega})$ *for some* $\sigma \in (0,1)$,
2. *For* $\lambda > 0$, $u_\lambda > 0$ *in* Ω *and* $u_\lambda = 0$ *on* $\partial\Omega$.
3. *monotone increasing and left-continuous on* λ.

Definition 2.2 (Extremal value λ^*)
The extremal value λ^* *is the supremum of* μ *such that:*

(a) *For* $\forall\lambda \in (0,\mu]$, $^\exists u_\lambda$ *(minimal solution).*
(b) *The following Hardy type inequality is valid:*

$$\int_\Omega |\nabla u_\lambda|^{p-2}\Big(|\nabla\varphi|^2 + (p-2)\frac{(\nabla u_\lambda, \nabla\varphi)^2}{|\nabla u_\lambda|^2}\Big)\, dx \geq \lambda \int_\Omega f'(u_\lambda)\varphi^2\, dx$$

for any $\varphi \in V_{\lambda,p}(\Omega)$.

$$V_{\lambda,p}(\Omega) = \{\varphi : ||\varphi||_{V_{\lambda,p}} < +\infty, \varphi = 0 \ on\ \partial\Omega\},$$
$$||\varphi||_{V_{\lambda,p}} = \Big(\int_\Omega |\nabla u_\lambda(x)|^{p-2}|\nabla\varphi|^2\, dx\Big)^{\frac{1}{2}}.$$

Under these preparations, we see

Proposition 2.1

$$u_{\lambda^*}(x) = \lim_{\lambda\to\lambda^*} u_\lambda(x) \quad a.e..$$

Moreover $u_{\lambda^*} \in W_0^{1,p}(\Omega)$ *is a weak solution.*

Proof: From the definition of $V_{\lambda,p}(\Omega)$, we see $u_\lambda \in V_{\lambda,p}(\Omega)$. By the assumption we have

$$(p-1)\int_\Omega |\nabla u_\lambda|^p\, dx \geq \lambda \int_\Omega f'(u_\lambda)u_\lambda^2\, dx$$

Since u_λ is a solution of (2.3), we have

$$\int_\Omega |\nabla u_\lambda|^p\,dx = \int_\Omega f(u_\lambda)u_\lambda\,dx$$

Then for any $\varepsilon > 0$ there is a positive number $C_\varepsilon > 0$ such that

$$(p-1+\varepsilon)f(t)t \leq f'(t)t^2 + C_\varepsilon$$

Hence

$$\int_\Omega f'(u_\lambda)u_\lambda^2\,dx \leq \frac{p-1}{p-1+\varepsilon}\int_\Omega f'(u_\lambda)u_\lambda^2\,dx + C'_\varepsilon.$$

Here C'_ε is a positive number independent of each $\lambda < \lambda^*$. Then,for some positive number C

$$\int_\Omega |\nabla u_\lambda|^p\,dx = \lambda\int_\Omega f(u_\lambda)u_\lambda\,dx \leq C$$
$$\int_\Omega f'(u_\lambda)u_\lambda^2\,dx \leq C,$$

and so u_λ is uniformly bounded in $W_0^{1,p}(\Omega)$ for $\lambda < \lambda^*$. Therefore $\{u_\lambda\}$ contains a weakly convergent subsequence in $W_0^{1,p}(\Omega)$. Since u_λ is increasing in λ, the limit $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ uniquely exists a.e. and clearly $u^* \in W_0^{1,p}(\Omega)$ becomes a weak solution. □

Definition 2.3 (Singular solution)
A unbounded solution is called singular.

3 The linearized operator of $L_p(u)$ at u_λ

Recall the linearized operator and $V_{\lambda,p}$:

$$L'_p(u)\varphi = -\operatorname{div}\left(|\nabla u|^{p-2}(\nabla\varphi + (p-2)\frac{(\nabla u, \nabla\varphi)}{|\nabla u|^2}\nabla u)\right).$$

When $p \geq 2$, $L_p(u)$ is Frechet differentiable in $W_0^{1,p}(\Omega)$. But if $1 < p < 2$, it is not differentiable. Therefore we have to prepare proper space for the linearized operator $L'_p(u_\lambda)$ with u_λ being the minimal solution.

Definition 3.1 *Let us set*

$$||\varphi||_{V_{\lambda,p}} = \left(\int_\Omega |\nabla u_\lambda(x)|^{p-2} |\nabla \varphi|^2 \, dx \right)^{\frac{1}{2}},$$

$$V_{\lambda,p}(\Omega) = \{\varphi : ||\varphi||_{V_{\lambda,p}} < +\infty, \varphi = 0 \text{ on } \partial\Omega\}.$$

Lemma 3.1 (Coercivity) *For* $\forall \varphi \in V_{\lambda,p}(\Omega)$,

$$\varphi \in V_{\lambda,p}(\Omega) \Longrightarrow L_p'(u_\lambda)\varphi \in [V_{\lambda,p}(\Omega)]'$$

$$|\langle L_p'(u_\lambda)\varphi, \varphi \rangle_{V_{\lambda,p}' \times V_{\lambda,p}}| \geq C ||\nabla \varphi||^2_{V_{\lambda,p}}$$

We need more notations.

Definition 3.2

$$F_{\lambda,p} = \{x \in \Omega : |\nabla u_\lambda(x)| = 0\}.$$

Definition 3.3

$$\tilde{V}_{\lambda,p}(\Omega) = \begin{cases} \psi \in C_0^\infty(\Omega) \\ |\nabla \psi| \equiv 0 \text{ on some nbd of } F_{\lambda,p} \end{cases}$$

Lemma 3.2 *Assume that* $0 < \lambda < \lambda^*$.
If $p \geq 2$, *then*

$$\tilde{V}_{\lambda,p}(\Omega) \subset W_0^{1,p}(\Omega) \subset V_{\lambda,p}(\Omega),$$

If $1 < p < 2$, *then*

$$\tilde{V}_{\lambda,p}(\Omega) \subset V_{\lambda,p}(\Omega) \subset W_0^{1,p}(\Omega)$$

Definition 3.4 (Differentiability in $V_{\lambda,p}(\Omega)$)
$L_p(\cdot)$ is said to be differentiable at u_λ in the direction to φ in $V_{\lambda,p}(\Omega)$, if

$$\frac{1}{t}\left(L_p(u_\lambda + t\varphi) - L_p(u_\lambda) - L_p'(u_\lambda)\varphi\right) = o(1), \qquad \text{in } [V_{\lambda,p}(\Omega)]'.$$

In addition if S is dense in $V_{\lambda,p}(\Omega)$, then $L_p(\cdot)$ is said to be differentiable at u_λ in $V_{\lambda,p}(\Omega)$ a.e. respectively.

Then we see

Proposition 3.1 *Let u_λ be the minimal solution. Then, $L_p(\cdot)$ is differentiable at u_λ in the direction to $\forall\varphi \in \tilde{V}_{\lambda,p}(\Omega)$.*

Definition 3.5 *Let us set for $\forall$ compact set $F \subset \Omega$*

$$Cap(F, |\nabla u_\lambda|^{p-2}) = \inf\left[\int_\Omega |\nabla u_\lambda|^{p-2}|\nabla\varphi|^2\,dx : \varphi \in C_0^\infty(\Omega), \varphi \geq 1 \text{ on } F\right]$$

Then we see

Proposition 3.2 *If $Cap(F_{\lambda,p}, |\nabla u_\lambda|^{p-2}) = 0$, then $\overline{\tilde{V}_{\lambda,p}(\Omega)} = V_{\lambda,p}(\Omega)$.*

Corollary 3.1 *If $Cap(F_{\lambda,p}, |\nabla u_\lambda|^{p-2}) = 0$, then $L_p(\cdot)$ is differentiable at u_λ in $V_{\lambda,p}(\Omega)$ a.e.*

Remark 3.1 *The denseness of $\tilde{V}_{\lambda,p}$ in $V_{\lambda,p}$ is not completely essential in this talk. In most cases it is sufficient that a first eigenfunction can be approximated by elements in $\tilde{V}_{\lambda,p}$.*

Remark 3.2 *In the case that $p \geq 2$, we have $W_0^{1,p}(\Omega) \subset V_{\lambda,p}(\Omega)$. But we can not take $W_0^{1,p}(\Omega)$ as S in the definition . Because $L_p(u_\lambda + t\varphi)$ with $\varphi \in W_0^{1,p}(\Omega)$ does not belong to $[V_{\lambda,p}(\Omega)]'$ but to $[W_0^{1,p}(\Omega)]'$ in general.*

But $L_p'(u_\lambda)$ is continuous from $W_0^{1,p}(\Omega)$ to its dual $[W_0^{1,p}(\Omega)]'$, hence we can give an alternative definition of differentiability of $L_p(\cdot)$ in $[W_0^{1,p}(\Omega)]'$.

Definition 3.6 (Differentiability in $W_0^{1,p}(\Omega)$)

Let $p \in [2, +\infty)$ and let u_λ be the minimal solution for $\lambda \in (0, \lambda^*)$. $L_p(\cdot)$ is said to be differentiable at u_λ in $W_0^{1,p}(\Omega)$, if for any $\varphi \in W_0^{1,p}(\Omega)$ it holds that as $t \to 0$

$$\frac{1}{t}\left(L_p(u_\lambda + t\varphi) - L_p(u_\lambda) - L_p'(u_\lambda)\varphi\right) = o(1), \quad \text{in } \left[W_0^{1,p}(\Omega)\right]'.$$

Proposition 3.3 *Let u_λ be the minimal solution for $\lambda \in (0, \lambda^*)$. If $p \in [2, +\infty)$, then $L_p(\cdot)$ is differentiable at u_λ in direction to $W_0^{1,p}(\Omega)$.*

4 The linearlized operator $L'_p(u_\lambda)$

Let $u_\lambda \in C^{1,\sigma}(\Omega)$ be the minimal solution.

$$\begin{cases} -\operatorname{div}(|\nabla u_\lambda|^{p-2}\nabla u_\lambda) = \lambda f(u_\lambda) & \text{in } \Omega \\ u_\lambda = 0 \quad \text{on } \partial\Omega, \end{cases}$$

Lemma 4.1 *For* $\forall\lambda \in (0,\lambda^*)$, *we have for* $\forall\varphi \in C_0^1(\Omega)$ *:*

$$\int_\Omega |\nabla u_\lambda|^{p-1}|\nabla\varphi|\,dx \geq C\int_\Omega |\varphi|\,dx \tag{4.1}$$

$$\int_\Omega |\nabla u_\lambda|^{2(p-1)}|\nabla\varphi|^2\,dx \geq C\int_\Omega \varphi^2\,dx \tag{4.2}$$

$$\int_\Omega |\nabla u_\lambda|^{p-2}|\nabla\varphi|^2 \geq C\int_\Omega \varphi^2\,dx \tag{4.3}$$

Here C *is a positive number independent of each* φ.

Let us recall $F_{\lambda,p} = \{x \in \Omega : |\nabla u_\lambda| = 0\}$.

Corollary 4.1

1. $F_{\lambda,p}$ *is discrete in* Ω.
2. $L'_p(u_\lambda)$: $V_{\lambda,p} \to [V_{\lambda,p}]'$ *is invertible.*
3. $L'_p(u_\lambda)$ *is extended to a self-adjoint operator on* $L^2(\Omega)$.

Definition 4.1 *By* I *we denote the imbedding operator from* $V_{\lambda,p}(\Omega)$ *into* $L^2(\Omega)$ *defined by*

$$I : \varphi \in V_{\lambda,p}(\Omega) \longrightarrow \varphi \in L^2(\Omega)$$

Then we can show

Proposition 4.1 *The imbedding operator*

$$I : \varphi \in V_{\lambda,p}(\Omega) \longrightarrow \varphi \in L^2(\Omega)$$

is compact.

Corollary 4.2 *The operator*

$$M_{\lambda,p} \equiv I_{V\to L^2} \circ (L'_p(u_\lambda))^{-1}\big|_{L^2}$$

is compact from $L^2(\Omega)$ *into* $L^2(\Omega)$.

5 Differentiability of u_λ w.r.t. λ ($p \geq 2$)

Theorem 5.1 *Assume* $2 \leq p < \infty$ *and the operator* $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ *on* $L^2(\Omega)$ *has a positive first eigenvalue for* $\forall \lambda \in (0, \lambda^*)$.

Then u_λ *is left differentiable with respect to* $\forall \lambda \in (0, \lambda^*)$, *and* $v_\lambda \equiv \left(\frac{du_\lambda}{d\lambda}\right)_- \in V_{\lambda,p}(\Omega)$ *satisfies*

$$\begin{cases} L_p'(u_\lambda)v_\lambda - \lambda f'(u_\lambda)v_\lambda = f(u_\lambda), & \text{in } \Omega \\ v_\lambda = 0, & \text{on } \partial\Omega. \end{cases}$$

Remark 5.1 *1.*

$$\frac{1}{p-1} u_\lambda \leq \lambda v_\lambda, \qquad \text{if } v_\lambda \text{ exists.}$$

6 Behaviors of u_λ and $\frac{du_\lambda}{d\lambda}$ near $\lambda = 0$

Let $\varphi_0 \geq 0$ be the unique solution of

$$L_p(\varphi_0) = 1 \quad \text{in } \Omega; \quad \varphi_0 = 0 \quad \text{on } \partial\Omega.$$

Lemma 6.1 *For* $\forall \varepsilon_0 \in (0, \lambda^*)$, $\exists C > 0$ *such that for* $\forall \lambda \in [0, \varepsilon_0]$:

(1) $\int_\Omega |\nabla u_\lambda|^q \, dx \leq C \lambda^{\frac{q}{p-1}}$ *for* $\forall q \geq 0$.

(2) $|\nabla u_\lambda| \leq C \lambda^{\frac{1}{p-1}}$ *a.e.*

(3) $\lambda^{\frac{1}{p-1}} \varphi_0 \leq u_\lambda \leq C \lambda^{\frac{1}{p-1}}$

Lemma 6.2 *For* $\forall \varepsilon_0 \in (0, \lambda^*)$, $\exists C > 0$ *such that we have :*

If $p \geq 2$, *then for* $\forall \lambda \in [0, \varepsilon_0]$

(1) $\int_\Omega v_\lambda \, dx \geq C \lambda^{-\frac{p-2}{p-1}}$

(2) $\int_\Omega |\nabla v_\lambda| \, dx \geq C \lambda^{-\frac{p-2}{p-1}}$

If $1 < p < 2$, *then for* $\forall \lambda \in [0, \varepsilon_0]$

(3) $\int_\Omega v_\lambda \, dx \leq C \lambda^{\frac{2-p}{p-1}}$.

(4) $\int_\Omega |\nabla v_\lambda|^2 \, dx \leq C \lambda^{2\frac{2-p}{p-1}}$.

7 Positivity of $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ for a small λ

Theorem 7.1 *$L_p'(u_\lambda) - \lambda f'(u_\lambda)$ has a positive first eigenvalue if λ is sufficiently small.*

In other words, $\exists \mu > 0$ such that

$$\langle (L_p'(u_\lambda) - \lambda f'(u_\lambda))\varphi, \varphi \rangle_{V_{\lambda,p}' \times V_{\lambda,p}} \geq \mu \int_\Omega \varphi^2 \, dx,$$

for any $\varphi \in V_{\lambda,p}(\Omega)$.

Proof: A scaling arguement;

$$u_\lambda = \lambda^{\frac{1}{p-1}} w_\lambda$$

Then as $\lambda \to 0$

$$w_\lambda \to w_0 :$$

$$\begin{cases} L_p(w_0) = 1 & \text{in } \Omega \\ w_0 = 0 & \text{on} \quad \partial\Omega \end{cases}$$

The linearlized operator at w_0 has a positive first eigen value! From this fact we can show the assertion.

8 Nonnegativity of $L_p'(u_\lambda) - \lambda f'(u_\lambda)$

Definition 8.1 *Let $\hat{\varphi}_\lambda \in V_{\lambda,p}(\Omega)$ be the first eigenfunction of $L_p'(u_\lambda) - \lambda f'(u_\lambda)$*

Definition 8.2 (Accessibility Condition) *The first eigenfunction $\hat{\varphi}^\lambda$ is said to satisfy (AC) if for $\forall \varepsilon > 0$ there exists a nonnegative $\varphi \in \tilde{V}_{\lambda,p}(\Omega)$ such that*

$$L_p'(u_\lambda)(\varphi - \hat{\varphi}^\lambda) + |\varphi - \hat{\varphi}^\lambda| \leq \varepsilon \max(\hat{\varphi}^\lambda, dist(x, \partial\Omega)) \quad \textit{in } \Omega.$$

Theorem 8.1 *Assume (AC). Then the 1st eigenvalue of $L_p'(u_\lambda) - \lambda f'(u_\lambda)$ is nonnegative.*

Remark 8.1 (1) *In case that Ω is radially symmetric, the minimal solution is also radial. Hence this condition is easily verified.*
(2) *Since L_p is not Frechet differentiable in general, we need Lemma which combines L_p with its linearized operator $L'(u_\lambda)$.*

A Sketchof proof of Theorem :

Assume that $L_p'(u_\lambda) - \lambda f'(u_\lambda)$has a negative first eigenvalue μ

$$L_p'(u_\lambda)\varphi - \lambda f'(u_\lambda)\varphi = \mu\varphi, \quad (\mu < 0, \varphi \in \tilde{V}_{\lambda,p}(\Omega)).$$

$$\downarrow$$

Lemma 8.1 (*Key Lemma*) *Assume $\varphi \in \tilde{V}_{\lambda,p}(\Omega)$. Then $^{\exists 1}\psi_t \in C^0([0,T], V_{\lambda,p}(\Omega))$ s.t.*

$$\begin{cases} L_p(u_\lambda - t\psi_t(x)) = L_p(u_\lambda) - tL_p'(u_\lambda)\varphi & in\ \Omega, \\ \psi_t = 0 & on\ \partial\Omega, \end{cases}$$

Moreover for a small $\rho > 0$ and $\Omega_\rho = \{a \in \Omega : dist(x\partial\Omega) < \rho\}$

$$\lim_{t\to 0} ||\psi_t - \varphi||_{C^1(\overline{\Omega_\rho})=0}.$$

$$\downarrow$$

For small $^\forall t > 0, {}^\exists x_t \in \Omega$ and $^\exists r_t > 0$ s.t.

$$L_p(u_\lambda) - tL_p'(u_\lambda)\varphi \le \lambda f(u_\lambda - t\psi_t) \quad \text{in } B_{r_t}(x_t).$$

$$\downarrow$$

$$0 \le \lambda f'(u_\lambda)(\varphi - \psi_t) + \mu\varphi + o(1)|\psi_t| \quad \text{in } B_{r_t}(x_t).$$

Or, $$0 \le \lambda f'(u_\lambda)\Big(1 - \frac{\psi_t}{\varphi}\Big) + \mu + o(1)\frac{|\psi_t|}{\varphi} \quad \text{in } B_{r_t}(x_t).$$

$$\downarrow$$

Since Ω is bounded, we can assume $\lim_{t\to+0} x_t = {}^\exists x^0 \in \overline{\Omega}$.

$$\downarrow$$

$$0 \le \mu$$

Contradiction!!

9 Proof of Key lemma

Lemma 9.1 (*Key Lemma*) *Assume* $\varphi \in \tilde{V}_{\lambda,p}(\Omega)$. *Then* $\exists_1 \psi_t \in C^0([0,T], V_{\lambda,p}(\Omega))$ *s.t.*

$$\begin{cases} L_p(u_\lambda - t\psi_t(x)) = L_p(u_\lambda) - tL_p'(u_\lambda)\varphi & \text{in } \Omega, \\ \eta_t = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover for a small number $\rho > 0$

$$\lim_{t\to 0} ||\psi_t - \varphi||_{C^1(\overline{\Omega_\rho})=0}.$$

Extremely rough sketch of Proof:

The former part follows from the invertibility of $L'(u_\lambda)$ and monotonicity of L_p.

The latter part follows from the energy inequalities

$$||W_t||_{W^{n,2}(\Omega_{\rho'})} \leq C(n,\rho,\rho')||W_t||_{V_{\lambda,p}(\Omega)} + t] \to 0 \text{ as } t \to +0.$$

involving $W_t = \psi_t - \varphi$ After all, from Sobolev imbedding theorem the assertion follows.

10 The extremal solution

Theorem 10.1 *Let* u_{λ^*} *be the singular extremal solution.*
Moreover, assume that $f(t)$ *satisfies*

$$\frac{f'(t)}{f(t)^{\frac{p-2}{p-1}}} \quad \text{is nondecreasing on } [0,\infty).$$

Then if $\lambda > \lambda^*$, *there is no solution even in the weak sense.*

Lemma 10.1 *Let* $u \in W_0^{1,p}(\Omega)$ *be a solution. Let* $\Psi \in C^2(\mathbb{R})$ *be concave, with* Ψ' *bounded and* $\Psi(0) = 0$. *Then* $v = \Psi(u)$ *satisfies*

$$L_p(v) \geq \lambda|\Psi'(u)|^{p-2}\Psi'(u)f(u).$$

For a given $\varepsilon \in (0,1)$ we set

$$\tilde{f} = (1-\varepsilon)f.$$

$$h(u) = \int_0^u \frac{ds}{f(s)^{\frac{1}{p-1}}} \quad \text{and} \quad \tilde{h}(u) = \int_0^u \frac{ds}{\tilde{f}(s)^{\frac{1}{p-1}}}.$$

Lemma 10.2 *Assuming (10.1), we set*

$$\Psi(u) = \tilde{h}^{-1}(h(u)).$$

then

(1) $\Psi(0) = 0$ *and* $0 \leq \Psi(u) \leq u$ *for all* $u \geq 0$.
(2) If $h(+\infty) < +\infty$ *and* $\tilde{f} \neq f$, *then* $\Psi(+\infty) < +\infty$.
(3) Ψ *is increasing, concave, and* $\Psi' \leq 1$ *for all* $u \geq 0$.

Proof of Theorem: Assume that $\exists u$;solution for some $\lambda > \lambda^*$. Set $v = \Psi(u) = \tilde{h}^{-1}(h(u))$. Then v satisfies

$$\begin{cases} L_p(v) \geq \lambda(1-\varepsilon)f(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence v is a supersolution.

Proposition 10.1 *Assume that* $p \geq 2$. *For any* $\varphi \in V_{\lambda^*,p}(\Omega)$

$$\langle (L_p'(u_{\lambda^*}) - \lambda^* f'(u_\lambda^*))\varphi, \varphi \rangle_{V'_{\lambda^*,p} \times V_{\lambda^*,p}} \geq 0.$$

A weaker result holds for $1 < p < 2$.

Proposition 10.2 *Assume* $1 < p \leq 2$. *Let* $u \in W_0^{1,p}(\Omega)$ *be a singular solution such that for any* $\varphi \in V_{\lambda,p}(\Omega)$

$$\langle (L_p'(u_\lambda) - \lambda f'(u_\lambda))\varphi, \varphi \rangle_{V'_{\lambda,p} \times V_{\lambda,p}} \geq 0.$$

Moreover we assume that

$$|\nabla u| \geq |\nabla u_\lambda| \qquad \text{in } \Omega \quad (p \neq 2).$$

Then we have $\lambda = \lambda^*$ *and* $u = u_{\lambda^*}$

A weaker result holds for $p > 2$.

11 Weighted Hardy's inequality in a ball

Theorem 11.1 *Suppose that a positive integer N and a real number α satisfy $N+\alpha>2$. Then it holds that for any $u \in W_0^1(\Omega)$*

$$\int_\Omega |\nabla u|^2|x|^\alpha\,dx \geq H(N,\nabla,\alpha)\int_\Omega |u|^2|x|^{\alpha-2}\,dx + \lambda_1\Big(\frac{\omega_N}{|\Omega|}\Big)^{\frac{2}{N}}\int_\Omega |u|^2|x|^\alpha\,dx.$$

Here

$$H(N,\nabla,\alpha)=\Big(\frac{n-2+\alpha}{2}\Big)^2,$$

ω_N is a volume of N-dimensional unit ball, and λ_1 is the first eigenvalue of of the Dirichlet problem given by:

$$\lambda_1=\inf\left[\int_{B_1^2}|\nabla_2 v|^2\,dx : v\in W_0^{1,2}(B_1^2), \int_{B_1^2} v^2\,dx=1\right],$$

where by B_1^2 and ∇_2 we denote the two dimensional unit ball and the gradient.

Remark 11.1 *When $\alpha=0$, this result was initially established in [3] by H. Brezis and J.L. Vázquez. They also investigated in [3] fundamental properties of blow-up solutions of some nonlinear elliptic problems.*

For the sake of the self-containedness, we give a proof of Theorem in the case $\alpha=0$. By the spherically symmetric decreasing rearragement, it suffices to show the inequality in the case that $\Omega=B$; a unit ball in $\mathbb{R}^N$ and $u\in C_0^1(B)$ is radiall symmetric. Set $u=r^{-\beta}v$ for $u\in C_0^1(B)$ and $\beta=\frac{N-2}{2}$.

$$\begin{aligned}
&\int_B |\nabla u|^2\,dx - H(N,\nabla,0)\int_B\frac{u^2}{|x|^2}\,dx \qquad (11.1)\\
&= N\omega_N\Big(\int_0^1|u'|^2r^{N-1}\,dr - H(N,\nabla,0)\int_0^1 u^2r^{N-3}\,dr\Big)\\
&= N\omega_N\Big(\int_0^1|v'|^2r\,dr\Big)\geq \lambda_1 N\omega_N\int_0^1 v^2r\,dr\\
&= \lambda_1\int_B u^2\,dx
\end{aligned}$$

This proves the assertion.

12 Example

$$\begin{cases} f_q(u) = (1+u)^q, & (q > p-1) \\ f_e(u) = e^u. \end{cases}$$

$$\begin{cases} \lambda_N(p,q) = \left(\frac{p}{q-p+1}\right)^{p-1}\left(N - \frac{pq}{q-p+1}\right), \\ \lambda_N(p) = p^{p-1}(N-p). \end{cases}$$

$$\begin{cases} U_{p,q}(r) = r^{-Q} - 1, \quad Q = \frac{p}{q-p+1} \\ U_p(r) = -p\log r. \end{cases}$$

Lemma 12.1 *$U_p \in W_0^{1,p}(B)$ if $N > p$ and $U_{p,q} \in W_0^{1,p}(B)$ if $N > p + pQ$. Moreover :*

$$\begin{cases} L_p(U_{p,q}) = \lambda_N(p,q)(U_{p,q}+1)^q & in B \\ U_{p,q} = 0 & on\ \partial B, \end{cases}$$

$$\begin{cases} L_p(U_p) = \lambda_N(p)e^{U_p} & in B \\ U_p = 0 & on\ \partial B. \end{cases}$$

As $q \to +\infty$, for any $r \in (0,1)$

$$\left(f_q(U_{p,q}(r)), q\lambda_N(p,q), qU_{p,q}(r)\right) \to \left(f_e(U_p(r)), \lambda_N(p), U_p(r)\right)$$

Proposition 12.1 *(Exponetial case) Assume that $1 < p \leq 2$. Then U_p is the singular extremal, iff $N \geq p\frac{p+3}{p-1}$.*

Proposition 12.2 *(Exponetial case) Assume $p > 2$. Then U_p is the singular extremal, if $N > 5p$.*

Proposition 12.3 *(Polynomial case) Assume $1 < p \leq 2$. Then $U_{p,q}$ is the singular extremal, iff*

$$N \geq \frac{p(1+qQ) + 2\sqrt{pqQ}}{p-1}.$$

Proposition 12.4 *(Polynomial case) Assume $p > 2$. Then $U_{p,q}$ is the singular extremal with $f = f_p$, if*

$$N \geq Q(3q - 1 + 2\sqrt{q(q-1)}).$$

Remark 12.1 *(1) When $p > 2$, it is unknown if U_p; $5p > N \geq p\frac{p+3}{p-1}$ ($U_{p,q}$; $Q(3q-1+2\sqrt{q(q-1)}) > N \geq \frac{p(1+qQ)+2\sqrt{pqQ}}{p-1}$) becomes the extremal.*
(2) $1 < p \leq 2$. If $N > p\frac{p+3}{p-1}$, then

$$L_p'(U_p) - \lambda_N(p)e^{U_p}$$

has a positive first eigenvalue $\mu(\lambda_N(p))$.

If $N = p\frac{p+3}{p-1}$, then this does not have a 1st eigenfunction in $W_0^{1,p}(B)$. However, the weighted Hardy inequality gives a positive value for $\mu(\lambda_N(p))$ defined as

$$\mu(\lambda_{N(p)}) = \lim_{\lambda \to \lambda_{N(p)}} \mu(\lambda) = \lambda_1 p^{p-2}(p-1).$$

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