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ESTIMATES FOR THE DIRICHLET-WAVE EQUATION AND APPLICATIONS TO NONLINEAR WAVE EQUATIONS

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1. Introduction.

In this article we shall go over recent work in proving dispersive and Strichartz estimates for the Dirichlet-wave equation. We shall discuss applications to existence questions outside of obstacles and discuss open problems.

The estimates that we shall discuss involve solutions of the Dirichlet-wave equation outside of a fixed obstacle $\mathcal{K}\subset\mathbb{R}^n$, i.e., if $\Box=\partial_t^2-\Delta$,

$$\begin{cases}\Box u(t,x)=F(t,x), \quad t>0, \quad x\in\mathbb{R}^n\backslash\mathcal{K}\\ u(t,x)=0, \quad t>0, \quad x\in\partial\mathcal{K}\\ u(0,x)=f(x), \quad \partial_t u(0,x)=g(x).\end{cases} \tag{1.1}$$

We shall assume throughout that $\mathcal{K}$ has C^∞ boundary. We also shall assume that $\mathcal{K}$ is compact, and, by rescaling, there is no loss of generality in assuming in what follows that

$$\mathcal{K}\subset\{x\in\mathbb{R}^n:\,|x|<1\}.$$

We shall mainly concern ourselves with the physically important case where the spatial dimension n equals 3. It is considerably easier to prove estimates for the wave equation in odd-spatial dimensions in part because of the fact that the sharp Huygens principle holds in this case for solutions of the boundaryless wave equation in Minkowski space $\mathbb{R}_+\times\mathbb{R}^n$. By this we mean that if v solves the Minkowski wave equation $\Box v(t,x)=0$ and if its initial data $(v(0,\,\cdot\,),\partial_t v(0,\,\cdot\,))$ vanish when $|x|>R$, then $v(t,x)=0$ if $|\,t-|x|\,|>R$.

Sharp Huygens principle of course does not hold for the obstacle case (1.1). On the other hand, for a wide class of obstacles, there is exponential decay of local energies for compactly supported data when the spatial dimension n is odd. Specifically, in this case, if $\mathcal{K}\subset\mathbb{R}^n$ is nontrapping and if v solves the homogeneous Dirichlet-wave equation

$$\begin{cases}\Box v(t,x)=0, \quad t>0, \quad x\in\mathbb{R}^n\backslash\mathcal{K}\\ v(t,x)=0, \quad t>0, \quad x\in\partial\mathcal{K},\end{cases} \tag{1.2}$$

then there is a constant $c>0$ so that if $R>1$ is fixed and if

$$v(0,x)=\partial_t v(0,x)=0, \quad \{x\in\mathbb{R}^n\backslash\mathcal{K}:\,|x|>R\}, \tag{1.3}$$

then

$$\Bigl(\int_{|x|<R}|v'(t,x)|^2\,dx\Bigr)^{1/2}\le Ce^{-ct}\|v'(0,\,\cdot\,)\|_2. \tag{1.4}$$

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Here, and in what follows,

$$v' = (\partial_t v, \nabla_x v)$$

denotes the space-time gradient of v and in the obstacle case the region $\{|x| < R\}$ is understood to mean $\{x \in \mathbb{R}^n \backslash \mathcal{K} : |x| < R\}$.

The exponential local decay of energies for nontrapping obstacles in odd dimensions is due to Morawetz, Ralston and Strauss [29], following earlier work for star-shaped obstacles of Lax, Morawetz and Phillips [22]. Estimate (1.4) will be a substitute for sharp Huygens principle that will allow us, in certain cases, to prove global estimates, such as Strichartz estimates, if local in time estimates hold for the obstacle case and if the corresponding global estimates hold for Minkowski space.

By using the local exponential decay of energy we can prove the following sharp weighted space-time estimate for solutions of (1.1)

$$\begin{aligned}(1.5)\quad \Big(\log(2+T)\Big)^{-1/2} \| (1+|x|)^{-1/2} u' \|_{L^2(\{(t,x)\in[0,T]\times\mathbb{R}^n\backslash\mathcal{K}\})} \\ \le C\|u'(0,\,\cdot\,)\|_2 + C\int_0^T \|F(t,\,\cdot\,)\|_2\,dt,\end{aligned}$$

if $\mathcal{K}$ is non-trapping and n is odd. In the region where $|x|$ is small compared to t, say $|x| < t/2$, this estimate is in some ways stronger than the usual energy estimate. For this reason, it plays an important role in applications to nonlinear problems involving obstacles. One uses (1.5) to handle various local terms near the boundary that arise in the proofs of the main pointwise and L^2 estimates.

Even though (1.4) cannot hold if there are trapped rays a weaker form of this inequality is valid when n is odd in certain situations where there are elliptic trapped rays. Indeed, a remarkable result of Ikawa [13], [14] says that if v solves (1.2) and if (1.3) holds then

$$(1.6)\qquad \|v'(t,\,\cdot\,)\|_{L^2(|x|<R)} \le Ce^{-ct} \sum_{|\alpha|\le 1} \|\partial_x^\alpha v'(0,\,\cdot\,)\|_{L^2(|x|<R)},$$

for some constant $c > 0$ if $\mathcal{K}$ is a finite union of convex obstacles. In the case of three or more obstacles Ikawa's result requires a technical assumption that the obstacles are sufficiently separated, but it is thought that (1.6) should hold in the case where there are no hyperbolic trapped rays. Also, just by interpolating with the standard energy estimate, one concludes that the variant of (1.6) holds if one replaces the L^2 norm of $v'(0,\,\cdot\,)$ by an H^ε norm with $\varepsilon > 0$ and the constant $c > 0$ in the exponential depending on ε. This fact would allow one to prove global Strichartz estimates with arbitrary small loss of derivatives if the local in time estimates were known (cf. [4]). For other local decay bounds see Burq [2].

In the rest of the paper we shall indicate how one can use the exponential local decay of energy to prove global estimates for solutions of (1.1) that have applications to nonlinear Dirichlet-wave equations. In the next section we shall go over the simplest situation of proving global Strichartz estimates in $\mathbb{R}^3\backslash\mathcal{K}$ when $\mathcal{K}$ is convex with smooth boundary. This argument will serve as a template for the more involved ones that are used to prove almost global and global existence for certain quasilinear wave equations. The most basic of these, which will be discussed in §3, will be to show that one can prove fixed-time L^2

estimates and weighted space-time L^2 estimates for $\Omega_{ij}u'$ if

$$\Omega_{ij} = x_i\partial_j - x_j\partial_i, \quad 1 \le i < j \le 3, \tag{1.7}$$

are angular-momentum operators for $\mathbb{R}^3$. As we shall see, by using these estimates one can prove almost global existence for semilinear wave equations in $\mathbb{R}^3\backslash\mathcal{K}$ if $\mathcal{K}$ is nontrapping. In the next section we shall see how one can prove a pointwise dispersive estimate for solutions of (1.1) if $\mathcal{K}$ is nontrapping or if it satisfies Ikawa's conditions. We shall also present related L^2 estimates that can be used to prove almost global existence results for quasilinear Dirichlet-wave equations and global existence for ones satisfying an appropriate null condition.

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2. Strichartz estimates outside convex obstacles.

In this section we shall show how local Strichartz estimates for obstacles, global ones for Minkowski space and the energy decay estimates (1.4) can be used to prove global Strichartz estimates for obstacles. This was first done in the case of odd dimensions by Smith and the author [36], and later for even dimension by Burq [3] and Metcalfe [25].

For simplicity, we shall only consider the special case where the spatial dimension n is equal to three. We shall also only treat the most basic Strichartz estimate in this case. The global Minkowski version, which will be used in the proof of the version for obstacles, says that

$$\|v\|_{L^4(\mathbb{R}_+\times\mathbb{R}^3)} \le C\Big(\|v(0,\,\cdot\,)\|_{\dot H^{1/2}(\mathbb{R}^3)} + \|\partial_t v(0,\,\cdot\,)\|_{\dot H^{-1/2}(\mathbb{R}^3)} + \|\Box v\|_{L^{4/3}(\mathbb{R}_+\times\mathbb{R}^3)}\Big). \tag{2.1}$$

Here $\dot H^\gamma(\mathbb{R}^3)$ denote the homogeneous Sobolev spaces on $\mathbb{R}^3$.

In addition to this, if $\mathcal{K}\subset\mathbb{R}^3$ is our compact obstacle, we shall need to assume that we have the local in time Strichartz estimates

$$\begin{aligned}\|u\|_{L^4([0,1]\times\mathbb{R}^3\backslash\mathcal{K})} &+ \sup_{0\le t\le 1}\Big(\|u(t,\,\cdot\,)\|_{H_D^{1/2}(\mathbb{R}^3\backslash\mathcal{K})} + \|u(t,\,\cdot\,)\|_{H_D^{-1/2}(\mathbb{R}^3\backslash\mathcal{K})}\Big) \\ &\le C\Big(\|u(0,\,\cdot\,)\|_{H_D^{1/2}(\mathbb{R}^3\backslash\mathcal{K})} + \|\partial_t u(0,\,\cdot\,)\|_{H_D^{-1/2}(\mathbb{R}^3\backslash\mathcal{K})} + \|F\|_{L^{4/3}([0,1]\times\mathbb{R}^3\backslash\mathcal{K})}\Big),\end{aligned} \tag{2.2}$$

assuming that the initial data is supported in the set $\{x\in\mathbb{R}^3\backslash\mathcal{K} : |x| < 4\}$. Here, $H_D^\gamma(\mathbb{R}^3\backslash\mathcal{K})$ are the usual Dirichlet-Sobolev spaces.

For the homogeneous case where the forcing term $F\equiv 0$ it was shown in [35] that (2.2) holds when $\mathcal{K}\subset\mathbb{R}^3$ is convex. An interesting problem would be to show that this estimate holds for a larger class of obstacles. In [35] more general Strichartz estimates for convex obstacles in all dimensions were also proved. In [36] estimates for the inhomogeneous wave equation were also obtained by using a lemma of Christ and Kiselev [5].

In addition to (2.1) and (2.2), we shall need a Sobolev space variant of (1.4). We suppose that $R > 1$ is given and that $\beta(x)$ is smooth and supported in $|x|\le R$. Then there is a $c > 0$ so that

$$\|\beta u(t,\,\cdot\,)\|_{H_D^{1/2}} + \|\beta\partial_t u(t,\,\cdot\,)\|_{H_D^{-1/2}} \le Ce^{-ct}\Big(\|u(0,\,\cdot\,)\|_{H_D^{1/2}} + \|\partial_t u(0,\,\cdot\,)\|_{H_D^{-1/2}}\Big), \tag{2.3}$$

if u solves (1.1) with vanishing forcing term F and has initial data satisfying $u(0,x) = \partial_t u(0,x) = 0$, $|x| > R$. This estimate just follows from (1.4) and a simple interpolation argument.

We claim that by using these three inequalities, we can prove the following result from [36].

Theorem 2.1. *Let u solve* (1.1) *when $\mathcal{K} \subset \mathbb{R}^3$ is a convex obstacle with smooth boundary. Then*

$$\|u\|_{L^4(\mathbb{R}_+\times\mathbb{R}^3\backslash\mathcal{K})} \le C\Big(\|f\|_{H_D^{1/2}} + \|g\|_{H_D^{-1/2}} + \|F\|_{L^{4/3}(\mathbb{R}_+\times\mathbb{R}^3\backslash\mathcal{K})}\Big). \tag{2.4}$$

Recall that we are assuming, as we may, that $\mathcal{K} \subset \{x \in \mathbb{R}^3 : |x| < 1\}$. The first step in the proof of this result will be to establish the following

Lemma 2.2. *Let u solve the Cauchy problem* (1.1) *with forcing term F replaced by $F+G$. Suppose that the initial data is supported in $\{\,|x| \le 2\}$ and that F, G are supported in $\{0 \le t \le 1\} \times \{\,|x| \le 2\}$. Then if $\rho < c$, where c is the constant in* (2.3),

$$\begin{aligned}(2.5)\quad &\Big\|e^{\rho(t-|x|)}u\|_{L^4(\mathbb{R}_+\times\mathbb{R}^3\backslash\mathcal{K})} \\ &\le C\Big(\|f\|_{H_D^{1/2}} + \|g\|_{H_D^{-1/2}} + \|F\|_{L^{4/3}(\mathbb{R}_+\times\mathbb{R}^3\backslash\mathcal{K})} + \int \|G(t,\,\cdot\,)\|_{H_D^{-1/2}(\mathbb{R}^3\backslash\mathcal{K})}\,dt\Big).\end{aligned}$$

Proof of Lemma 2.2: By (2.2) and Duhamel's principle, the inequality holds for the $L^4(dtdx)$ norm of u over $[0,1] \times \mathbb{R}^3\backslash\mathcal{K}$. Also, by (2.2),

$$\begin{aligned}(2.6)\quad &\|u(1,\,\cdot\,)\|_{H_D^{1/2}(\mathbb{R}^3\backslash\mathcal{K})} + \|\partial_t u(1,\,\cdot\,)\|_{H_D^{-1/2}(\mathbb{R}^3\backslash\mathcal{K})} \\ &\le C\Big(\|f\|_{H_D^{1/2}} + \|g\|_{H_D^{-1/2}} + \|F\|_{L^{4/3}(\mathbb{R}_+\times\mathbb{R}^3\backslash\mathcal{K})} + \int \|G(t,\,\cdot\,)\|_{H_D^{-1/2}(\mathbb{R}^3\backslash\mathcal{K})}\,dt\Big).\end{aligned}$$

By considering $t \ge 1$, we may take $F = G = 0$, with (f,g) now supported in $\{\,|x| \le 3\}$.

We next decompose $u = \beta u + (1-\beta)u$, where $\beta(x) = 1$ for $|x| \le 1$ and $\beta(x) = 0$ for $|x| \ge 2$. Let us first consider βu. We write

$$(\partial_t^2 - \Delta)(\beta u) = -2\nabla_x\beta\cdot\nabla_x u - (\Delta\beta)u = \tilde{G}(t,x),$$

and note that $\tilde{G}(t,x) = 0$ if $|x| \ge 2$. By (2.3) we have

$$\begin{aligned}(2.7)\quad &\|\tilde{G}(t,\,\cdot\,)\|_{H_D^{-1/2}} + \|\beta u(t,\,\cdot\,)\|_{H_D^{1/2}} + \|\partial_t(\beta u)(t,\,\cdot\,)\|_{H_D^{-1/2}} \\ &\qquad\qquad \le Ce^{-ct}\Big(\|f\|_{H_D^{1/2}} + \|g\|_{H_D^{-1/2}}\Big).\end{aligned}$$

By (2.2) and Duhamel's principle, it follows that

$$\|\beta u\|_{L^4([j,j+1]\times\mathbb{R}^3\backslash\mathcal{K})} \le Ce^{-cj}\Big(\|f\|_{H_D^{1/2}} + \|g\|_{H_D^{-1/2}}\Big),$$

which implies that βu satisfies the bounds in (2.5).

Now let us show that the same is true for $(1-\beta)u$. On the support of $(1-\beta)u$, we have

$$(\partial_t^2 - \Delta)u = -\tilde{G},$$

and by Duhamel's principle we have

$$u(t,x) = u_0(t,x) + \int_0^t u_s(t,x)\,ds,$$

where u_0 is the solution of the Minkowski wave equation on $\mathbb{R}_+ \times \mathbb{R}^3$ with initial data $\big((1-\beta)f, (1-\beta)g\big)$, and where $u_s(t,x)$ is the solution of the Minkowski space wave equation on the set $t > s$ with Cauchy data $(0, \tilde{G}(s,\,\cdot\,))$ on the hyperplane $t = s$. (Recall that $\tilde{G}$ and $(1-\beta)$ vanish near $\partial\mathcal{K}$.) Since the initial data of u_0 is supported in $\{x \in \mathbb{R}^3 : |x| \le 2\}$, by the sharp Huygens principle, u_0 must satisfy the bounds in (2.5). Additionally, on the support of $u_s(t,x)$ have $t \ge s$ and $t - |x| \in [s-3, s+3]$, so that by (2.1) and (2.7) we have

$$\|e^{\rho(t-|x|)}u_s\|_{L^4(dtdx)} \le Ce^{(\rho-c)s}\Big(\|f\|_{H^{1/2}_D} + \|g\|_{H^{-1/2}_D}\Big),$$

which leads to the desired estimate for the remaining part of u. □

We also require a simple consequence of Plancherel's theorem:

Lemma 2.3. *Let $\beta(x)$ be smooth and supported in $\{x \in \mathbb{R}^3 : |x| \le 2\}$. Then*

$$\int_{-\infty}^{+\infty} \|\beta(\,\cdot\,)\Big(e^{it|D|}f\Big)(t,\,\cdot\,)\|^2_{H^{1/2}(\mathbb{R}^3)}\,dt \le C\|f\|^2_{\dot{H}^{1/2}(\mathbb{R}^3)},$$

if $|D| = \sqrt{-\Delta}$.

Proof: By Plancherel's theorem over t, x, the left side can be written as

$$\int_0^\infty \int \Big|\int \hat{\beta}(\xi-\eta)\hat{f}(\eta)\delta(\tau-|\eta|)\,d\eta\Big|^2 (1+|\xi|^2)^{1/2}\,d\xi d\tau.$$

If we apply the Schwarz inequality in η we conclude that this is dominated by

$$\int_0^\infty \int \Big(\int |\hat{\beta}(\xi-\eta)|\delta(\tau-|\eta|)\,d\eta\Big)\Big(\int |\hat{\beta}(\xi-\eta)|\,|\hat{f}(\eta)|^2\delta(\tau-|\eta|)\,d\eta\Big) \times (1+|\xi|^2)^{1/2}\,d\xi d\tau.$$

This in turn is dominated by

$$\int |\hat{f}(\eta)|^2 \min\Big(|\eta|^2, (1+|\eta|^2)^{1/2}\Big)\,d\eta \le C\|f\|^2_{\dot{H}^{1/2}(\mathbb{R}^3)},$$

since

$$\sup_\xi (1+|\xi|^2)^{1/2}\Big(\int |\hat{\beta}(\xi-\eta)|\,\delta(\tau-|\eta|)\,d\eta\Big) \le C\min(\tau^2, (1+\tau^2)^{1/2}),$$

which completes the proof. □

Corollary 2.4. *Let β be as above, and let u solve the $\mathbb{R}_+ \times \mathbb{R}^3$ Minkowski wave equation $\Box u = F$ with initial data (f,g). Then*

$$\sum_{|\alpha|\le 1}\int_0^\infty \|\beta\partial^\alpha_{t,x}u(t,\,\cdot\,)\|^2_{\dot{H}^{-1/2}(\mathbb{R}^3)}\,dt \le C\Big(\|f\|_{\dot{H}^{1/2}} + \|g\|_{\dot{H}^{-1/2}} + \|F\|_{L^{4/3}(\mathbb{R}_+\times\mathbb{R}^3)}\Big)^2.$$

Proof: If $F=0$ then this is a direct consequence of the preceding lemma. If $f=g=0$ then the Minkowski Strichartz estimate (2.1), duality, and Huygens principle imply that for $t>0$

$$\sum_{|\alpha|\le 1} \|\beta\partial_{t,x}^\alpha u(t,\cdot)\|^2_{H^{-1/2}(\mathbf{R}^3)} \le C\|F\|^2_{L^{4/3}(\Gamma_t)},$$

where

$$\Gamma_t=\{(s,x):\, s\ge 0,\, s+|x|\in[t-2,t+2]\}.$$

Since $4/3\le 2$,

$$\int_0^\infty \|F\|^2_{L^{4/3}(\Gamma_t)}\,dt \le 4\|F\|^2_{L^{4/3}(\mathbf{R}_+\times\mathbf{R}^3)},$$

which finishes the proof. □

Proof of Theorem 2.1: By Lemma 2.2, we may without loss of generality assume that f and g vanish for $|x|\le 2$. If β is as above write

$$u=u_0-v=(1-\beta)u_0+\beta u_0-v,$$

where u_0 solves the Cauchy problem for the Minkowski wave equation, with data f,g,F, where we set $F=0$ in $\mathbb{R}_+\times\mathcal{K}$. By (2.1), u_0 satisfies the desired bounds, and so we just need to estimate βu_0-v. We write

$$(\partial_t^2-\Delta)(\beta u_0-v)=\beta F+G,$$

where $G=-2\nabla_x\beta\cdot\nabla_x u_0-(\Delta\beta)u_0$ vanishes for $|x|\ge 2$, and satisfies

$$\int_0^\infty \|G(t,\cdot)\|^2_{H_D^{-1/2}}\,dt \le C\Big(\|f\|_{\dot H^{1/2}}+\|g\|_{\dot H^{-1/2}}+\|F\|_{L^{4/3}}\Big)^2, \tag{2.8}$$

by Corollary 2.4. Note that the initial data of βu_0-v vanishes. Let F_j, G_j denote the restricitions of F,G to the set where $t\in[j,j+1]$, and write for $t>0$

$$\beta u_0-v=\sum_{j=0}^\infty u_j(t,x),$$

where u_j is the forward solution of $(\partial_t^2-\Delta)u_j=\beta F_j+G_j$.

By Lemma 2.2, the following holds

$$\Big\| e^{\rho(t-j-|x|)}u_j\Big\|_{L^4} \le C\Big(\|\beta F_j\|_{L^{4/3}}+\int_j^{j+1}\|G(t,\cdot)\|_{H^{-1/2}}\,dt\Big).$$

Furthermore, $u_j(t,x)$ is supported on the set where $t-j-|x|\ge -2$. Consequently, we have

$$\begin{aligned}
\|\beta u_0-v\|^2_{L^4(dtdx)} &\le C\sum_{j=0}^\infty \|e^{\rho(t-j-|x|)}u_j\|^2_{L^4(dtdx)}\\
&\le C\sum_{j=0}^\infty \|F_j\|^2_{L^{4/3}}+C\sum_{j=0}^\infty\Big(\int_j^{j+1}\|G(t,\cdot)\|_{H^{-1/2}}\,dt\Big)^2\\
&\le C\|F\|^2_{L^{4/3}}+C\int_0^\infty \|G(t,\cdot)\|^2_{H^{-1/2}}\,dt.
\end{aligned}$$

If we use (2.8), we conclude that βu_0-v also satisfies the desired bounds, which completes the proof. □

Remark: It would be very interesting to see whether the Strichartz estimates of Georgiev, Lindblad and the author [7] or Tataru [39] are valid for $\mathbb{R}_+ \times \mathbb{R}^3 \backslash \mathcal{K}$ when, as above $\mathcal{K}$ is convex.

3. Weighted space-time L^2 estimates.

In [16], the following weighted space-time estimate for Minkowski space was proved

$$(3.1) \quad \left(\log(2+T)\right)^{-1/2} \|(1+|x|)^{-1/2} v'\|_{L^2(\{(t,x):0\le t\le T,\, x\in\mathbb{R}^3\})} \\ \le C\|v'(0,\,\cdot\,)\|_{L^2(\mathbb{R}^3)} + C\int_0^T \|\Box v(t,\,\cdot\,)\|_{L^2(\mathbb{R}^3)}\, dt.$$

By using this estimate and the exponential local decay of energy, one can adapt the arguments of the previous section to prove the following analogous estimates for solutions of the Dirichlet-wave equation (1.1) if $\mathcal{K} \subset \mathbb{R}^3$ is non-trapping

$$(3.2) \quad \Big(\log(2+T)\Big)^{-1/2} \|\,(1+|x|)^{-1/2} u'\|_{L^2(\{(t,x)\in[0,T]\times\mathbb{R}^3\backslash\mathcal{K}\})} \\ \le C\|u'(0,\,\cdot\,)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} + C\int_0^T \|F(t,\,\cdot\,)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\, dt.$$

Indeed, the proof of Lemma 2.3 shows that this estimate is valid (without the log weight) when one replaces the L^2 norm in the left side of (3.2) by one over $\{(t,x) : 0 \le t \le T, x \in \mathbb{R}^3\backslash\mathcal{K}, |x| < 2\}$, with a constant that is independent of T. Using this and the Minkowski space estimates (3.1), one sees that the analog of (3.2) also holds when the norm is taken over the region where $|x| > 2$.

To handle applications to nonlinear wave equations, one requires a slight generalization of this estimate, which involves the operators

$$(3.3) \qquad Z = \{\partial_t, \partial_i, \Omega_{jk}, 1 \le i \le 3, 1 \le j < k \le 3\}.$$

Theorem 3.1. *If u is as in* (1.1) *has vanishing Cauchy data, then for any $N = 0, 1, 2, \ldots$*

$$(3.4) \\ \sum_{|\alpha|\le N} \Big(\|Z^\alpha u'(t,\,\cdot\,)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} + (\ln(2+t))^{-1/2} \|(1+|x|)^{-1/2} Z^\alpha u'\|_{L^2(\{(s,x)\in[0,t]\times\mathbb{R}^3\backslash\mathcal{K}\})} \Big) \\ \le C\sum_{|\alpha|\le N} \int_0^t \|Z^\alpha F(s,\,\cdot\,)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\, ds + C\sup_{0\le s\le t} \sum_{|\alpha|\le N-1} \|Z^\alpha F(s,\,\cdot\,)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} \\ + C\sum_{|\alpha|\le N-1} \|Z^\alpha F\|_{L^2(\{(s,x)\in[0,t]\times\mathbb{R}^3\backslash\mathcal{K}\})}.$$

Let us first see that the estimate holds when the norm in the left is taken over $|x| < 2$. Clearly the first term in the left is under control since

$$\sum_{|\alpha|\le N} \|Z^\alpha u'(t,\,\cdot\,)\|_{L^2(\{x\in\mathbb{R}^3\backslash\mathcal{K}):\, |x|<2\}} \le C_N \sum_{|\alpha|\le N} \|\partial_{t,x}^\alpha u'(t,\,\cdot\,)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})},$$

and standard arguments imply that the right hand side here is dominated by

$$\sum_{|\alpha|\le N} \|\partial_{t,x}^\alpha u'(t,\,\cdot\,)\|_{L^2(\mathbf{R}^3\backslash\mathcal{K})} \le C\sum_{|\alpha|\le N}\int_0^t \|\partial_{t,x}^\alpha F(s,\,\cdot\,)\|_{L^2(\mathbf{R}^3\backslash\mathcal{K})}\,ds + C\sum_{|\alpha|\le N-1}\|\partial_{t,x}^\alpha F(t,\,\cdot\,)\|_{L^2(\mathbf{R}^3\backslash\mathcal{K})}. \tag{3.5}$$

Indeed, if $N=0$, (3.5) is just the standard energy identity. To prove that (3.5) holds for N, assuming that it is valid when N is replaced by $N-1$, one notes that since $\partial_t w$ vanishes on the boundary one has

$$\sum_{|\alpha|\le N-1} \|\partial_{t,x}^\alpha \partial_t u'(t,\,\cdot\,)\|_{L^2(\mathbf{R}^3\backslash\mathcal{K})} \le C\sum_{|\alpha|\le N-1}\int_0^s \|\partial_{t,x}^\alpha \partial_s F(s,\,\cdot\,)\|_{L^2(\mathbf{R}^3\backslash\mathcal{K})} + C\sum_{|\alpha|\le N-2}\|\partial_{t,x}^\alpha \partial_t F(t,\,\cdot\,)\|_{L^2(\mathbf{R}^3\backslash\mathcal{K})}.$$

Since $\partial_t^2 w = \Delta w + F$, we get from this that

$$\sum_{|\alpha|\le N-1} \|\partial_{t,x}^\alpha \Delta u(t,\,\cdot\,)\|_{L^2(\mathbf{R}^3\backslash\mathcal{K})} \le C\sum_{|\alpha|\le N}\int_0^s \|\partial_{t,x}^\alpha F(s,\,\cdot\,)\|_{L^2(\mathbf{R}^3\backslash\mathcal{K})}\,ds + \sum_{|\alpha|\le N-1}\|\partial_{t,x}^\alpha F(t,\,\cdot\,)\|_{L^2(\mathbf{R}^3\backslash\mathcal{K})}.$$

By elliptic regularity, $\sum_{|\alpha|\le N}\|\partial_{t,x}^\alpha u'(t,\,\cdot\,)\|_{L^2(\mathbf{R}^3\backslash\mathcal{K})}$ is dominated by the left side of the last equation, which finishes the proof of (3.4), since

$$\sum_{|\alpha|\le N} \|\partial_{t,x}^\alpha u'(t,\,\cdot\,)\|_{L^2(\mathbf{R}^3\backslash\mathcal{K})} \le \sum_{|\alpha|\le N}\|\partial_x^\alpha u'(t,\,\cdot\,)\|_{L^2(\mathbf{R}^3\backslash\mathcal{K})} + \sum_{|\alpha|\le N-1}\|\partial_{t,x}^\alpha \partial_t u'(t,\,\cdot\,)\|_{L^2(\mathbf{R}^3\backslash\mathcal{K})}.$$

To handle the second term on the left side of (3.3), again when the left hand norm is taken over $|x|<2$, we shall need the following

Lemma 3.2. *If u is as in* (1.1) *then for any* $N=0,1,2,\dots$

$$\sum_{|\alpha|\le N} \|\partial_{t,x}^\alpha u'\|_{L^2(\{(s,x)\in[0,t]\times\mathbf{R}^3\backslash\mathcal{K}:\,|x|<2\})} \le C\sum_{|\alpha|\le N}\int_0^t \|\partial_{t,x}^\alpha F(s,\,\cdot\,)\|_{L^2(\mathbf{R}^3\backslash\mathcal{K})}\,ds + C\sum_{|\alpha|\le N-1}\|\partial_{t,x}^\alpha F\|_{L^2(\{(s,x)\in[0,t]\times\mathbf{R}^3\backslash\mathcal{K}\})}. \tag{3.6}$$

Clearly (3.6) implies that

$$\sum_{|\alpha|\le N} \|Z^\alpha u'\|_{L^2(\{(s,x)\in[0,t]\times\mathbf{R}^3\backslash\mathcal{K}:\,|x|<2\})} \le C\sum_{|\alpha|\le N}\int_0^t \|Z^\alpha F(s,\,\cdot\,)\|_{L^2(\mathbf{R}^3\backslash\mathcal{K})}\,ds + C\sum_{|\alpha|\le N-1}\|Z^\alpha F\|_{L^2(\{(s,x)\in[0,t]\times\mathbf{R}^3\backslash\mathcal{K}\})},$$

finishing the proof that the analog of (3.4) holds where the norms in the left are taken over $|x|<2$.

Proof of Lemma 3.2: By the proof of (3.5), (3.6) follows from the special case where $N = 0$:

$$\|u'\|_{L^2(\{(s,x)\in[0,t]\times\mathbb{R}^3\backslash\mathcal{K}:\,|x|<2\})} \le C\int_0^t \|F(s,\,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\,ds, \tag{3.7}$$

which, as we noted before, follows from the proof of Lemma 2.3. □

End of proof of Theorem 3.1: We need to see that

(3.8)

$$\begin{aligned}\sum_{|\alpha|\le N}\Big(\|Z^\alpha u'(t,\,\cdot)\|_{L^2(|x|>2)} + (\ln(2+t))^{-1/2}\|(1+|x|)^{-1/2}Z^\alpha u'\|_{L^2(\{[0,t]\times\{x:\,|x|>2\}\})}\Big)\\ \le C\sum_{|\alpha|\le N}\int_0^t \|Z^\alpha F(s,\,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\,ds + C\sup_{0\le s\le t}\sum_{|\alpha|\le N-1}\|Z^\alpha F(s,\,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\\ + C\sum_{|\alpha|\le N-1}\|Z^\alpha F\|_{L^2(\{(s,x)\in[0,t]\times\mathbb{R}^3\backslash\mathcal{K}\})}.\end{aligned}$$

For this we fix $\beta \in C^\infty(\mathbb{R}^3)$ satisfying $\beta(x) = 1$, $|x| \ge 2$ and $\beta(x) = 0$, $|x| \le 3/2$. Then since, by the assumption that the obstacle is contained in the set $|x| < 1$, it follows that $v = \beta u$ solves the boundaryless wave equation

$$\Box v = \beta F - 2\nabla_x\beta\cdot\nabla_x u - (\Delta\beta)u$$

with zero initial data, and satisfies $u(t, x) = v(t, x)$, $|x| \ge 2$. If we split $v = v_1 + v_2$, where $\Box v_1 = \beta F$, and $\Box v_2 = -2\nabla_x\beta\cdot\nabla_x u - (\Delta\beta)u$, it then suffices to prove that

(3.9)

$$\begin{aligned}\sum_{|\alpha|\le N}\Big(\|Z^\alpha v_2'(t,\,\cdot)\|_{L^2(|x|>2)} + (\ln(2+t))^{-1/2}\|(1+|x|)^{-1/2}Z^\alpha v_2'\|_{L^2(\{[0,t]\times\{x:\,|x|>2\}\})}\Big)\\ \le C\sum_{|\alpha|\le N}\int_0^t \|Z^\alpha F(s,\,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\,ds + C\sup_{0\le s\le t}\sum_{|\alpha|\le N-1}\|Z^\alpha F(s,\,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\\ + C\sum_{|\alpha|\le N-1}\|Z^\alpha F\|_{L^2(\{(s,x)\in[0,t]\times\mathbb{R}^3\backslash\mathcal{K}\})}.\end{aligned}$$

This is because by (3.1) we have

$$\begin{aligned}\sum_{|\alpha|\le N}\Big(\|Z^\alpha v_1'(t,\,\cdot)\|_{L^2(|x|>2)} + (\ln(2+t))^{-1/2}\|(1+|x|)^{-1/2}Z^\alpha v_1'\|_{L^2(\{[0,t]\times\{x:\,|x|>2\}\})}\Big)\\ \le C\sum_{|\alpha|\le N}\int_0^t \|Z^\alpha F(s,\,\cdot)\|_2\,ds,\end{aligned}$$

due to the fact that

$$\sum_{|\alpha|\le N}\int_0^t \|Z^\alpha(\beta F)(s,\,\cdot)\|_2\,ds \le C\sum_{|\alpha|\le N}\int_0^t \|Z^\alpha F(s,\,\cdot)\|_2\,ds.$$

To prove (3.9) we note that $G = -2\nabla_x\beta\cdot\nabla_x u - (\Delta\beta)u = \Box v_2$, vanishes unless $1 < |x| < 2$. To use this, fix $\chi \in C_0^\infty(\mathbb{R})$ satisfying $\chi(s) = 0$, $|s| > 2$, and $\sum_j \chi(s-j) =$

1. We then split $G = \sum_j G_j$, where $G_j(s,x) = \chi(s-j)G(s,x)$, and let $v_{2,j}$ be the solution of the corresponding inhomogeneous wave equation $\Box v_{2,j} = G_j$ with zero initial data in Minkowski space. By sharp Huygen's principle we have that $|Z^\alpha v_2(t,x)|^2 \le C\sum_j |Z^\alpha v_{2,j}(t,x)|^2$ for some uniform constant C. Therefore, by (3.1) we have that the square of the left side of (3.9) is dominated by

$$\begin{aligned}
&\sum_{|\alpha|\le N}\sum_j\Big(\int_0^t \|Z^\alpha G_j(s,\cdot)\|_2 ds\Big)^2 \\
&\quad\le C\sum_{|\alpha|\le N}\|Z^\alpha G\|^2_{L^2(\{(s,x):\,0\le s\le t,\,1<|x|<2\})} \\
&\quad\le C\sum_{|\alpha|\le N}\|Z^\alpha u'\|^2_{L^2(\{(s,x):\,0\le s\le t,\,1<|x|<2\})} + C\sum_{|\alpha|\le N}\|Z^\alpha u\|^2_{L^2(\{(s,x):\,0\le s\le t,\,1<|x|<2\})} \\
&\quad\le C\sum_{|\alpha|\le N}\|Z^\alpha u'\|^2_{L^2(\{(s,x)\in[0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|<2\})} \\
&\quad\le C\sum_{|\alpha|\le N}\|\partial^\alpha_{t,x} u'\|^2_{L^2(\{(s,x)\in[0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|<2\})}.
\end{aligned}$$

Consequently, (3.9) follows from (3.6), which finishes the proof. $\square$

To handle almost global existence, in addition to (3.4), we need the following consequence of the Sobolev estimates for $S^2 \times [0,\infty)$

$$\|h\|_{L^2(\{x\in\mathbb{R}^3\backslash\mathcal{K}:\,|x|\in[R-1,R]\})} \le \frac{C}{R}\sum_{|\alpha|\le 2}\|Z^\alpha h\|_{L^2(\{x\in\mathbb{R}^3\backslash\mathcal{K}:\,|x|\in[R-2,R+1]\})},\quad R\ge 1. \tag{3.10}$$

Let us conclude this section by showing how (3.4) and (3.10) can be used to prove almost global existence of semilinear wave equations outside of non-trapping obstacles. We shall consider semilinear systems of the form

$$\begin{cases}\Box u = Q(u'), \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3\backslash\mathcal{K}\\ u(t,\cdot)|_{\partial\mathcal{K}} = 0\\ u(0,\cdot) = f,\ \partial_t u(0,\cdot) = g.\end{cases} \tag{3.11}$$

Here

$$\Box = \partial_t^2 - \Delta$$

is the D'Alembertian, with $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ being the standard Laplacian. Also, Q is a constant coefficient quadratic form in $u' = (\partial_t u, \nabla_x u)$.

In the non-obstacle case we shall obtain almost global existence for equations of the form

$$\begin{cases}\Box u = Q(u'), (t,x)\in\mathbb{R}_+\times\mathbb{R}^3\\ u(0,\cdot) = f,\ \partial_t u(0,\cdot) = g.\end{cases} \tag{3.12}$$

In order to solve (3.11) we must also assume that the data satisfies the relevant compatibility conditions. Since these are well known (see e.g., [15]), we shall describe them briefly. To do so we first let $J_k u = \{\partial_x^\alpha u : 0\le|\alpha|\le k\}$ denote the collection of all spatial derivatives of u of order up to k. Then if m is fixed and if u is a formal H^m solution of (3.11) we can write $\partial_t^k u(0,\cdot) = \psi_k(J_k f, J_{k-1} g)$, $0 \le k \le m$, for certain compatibility functions ψ_k which depend on the nonlinear term Q as well as $J_k f$ and $J_{k-1} g$. Having done this, the compatibility condition for (3.11) with $(f,g) \in H^m \times H^{m-1}$ is just the requirement that the ψ_k, vanish on $\partial\mathcal{K}$ when $0 \le k \le m-1$. Additionally, we shall say that $(f,g) \in C^\infty$ satisfy the compatibility conditions to infinite order if this condition holds for all m.

If $\{\Omega\}$ denotes the collection of vector fields $x_i\partial_j - x_j\partial_i$, $1 \le i < j \le 3$, then we can now state our existence theorem.

Theorem 3.3. *Let $\mathcal{K}$ be a smooth compact nontrapping obstacle and assume that $Q(u')$ is above. Assume further that $(f,g) \in C^\infty(\mathbb{R}^3\backslash\mathcal{K})$ satisfies the compatibility conditions to infinite order. Then there are constants $c, \varepsilon_0 > 0$ so that if $\varepsilon \le \varepsilon_0$ and*

$$\sum_{|\alpha|+j\le 10} \|\partial_x^j \Omega^\alpha f\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} + \sum_{|\alpha|+j\le 9} \|\partial_x^j \Omega^\alpha g\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} \le \varepsilon, \tag{3.13}$$

then (3.11) *has a unique solution $u \in C^\infty([0,T_\varepsilon] \times \mathbb{R}^3\backslash\mathcal{K})$, with*

$$T_\varepsilon = \exp(c/\varepsilon). \tag{3.14}$$

We shall actually establish existence of limited regularity almost global solutions u for data $(f,g) \in H^9 \times H^8$ satisfying the relevant compatibility conditions and smallness assumptions (3.13). The fact then that u must be smooth if f and g are smooth and satisfy the compatibility conditions of infinite order follows from standard local existence theorems (see §9, [15]).

As in [15], to prove this theorem it is convenient to show that one can solve an equivalent nonlinear equation which has zero initial data to avoid having to deal with issues regarding compatibility conditions for the data. We can then set up an iteration argument for this new equation that is similar to the one used in the proof of Theorem 3.3.

To make the reduction we first note that by local existence theory (see, e.g., [15]) if the data satisfies (3.13) with ε small we can find a local solution u to $\Box u = Q(u')$ in $0 < t < 1$ that satisfies

$$\sup_{0\le t\le 1} \sum_{|\alpha|\le 10} \Big(\|Z^\alpha u'(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} + \|(1+|x|)^{-1/2} Z^\alpha u'\|_{L^2(\{(s,x)\in[0,t]\times\mathbb{R}^3\backslash\mathcal{K}\})}\Big) \le C\varepsilon, \tag{3.15}$$

for some uniform constant C.

Using this local solution we can set up our iteration. We first fix a bump function $\eta \in C^\infty(\mathbb{R})$ satisfying $\eta(t) = 1$ if $t \le 1/2$ and $\eta(t) = 0$ if $t > 1$. If we set

$$u_0 = \eta u$$

then

$$\Box u_0 = \eta Q(u') + [\Box, \eta]u.$$

So u will solve $\Box u = Q(u')$ for $0 < t < T_\varepsilon$ if and only if $w = u - u_0$ solves

$$\begin{cases} \Box w = (1-\eta)Q((u_0+w)') - [\Box,\eta](u_0+w) \\ w|_{\partial\mathcal{K}} = 0 \\ w(0,x) = \partial_t w(0,x) = 0 \end{cases} \tag{3.16}$$

for $0 < t < T_\varepsilon$.

We shall solve this equation by iteration. We set $w_0 = 0$ and then define w_k, $k = 1,2,3,\dots$ recursively by requiring that

$$\begin{cases} \Box w_k = (1-\eta)Q((u_0+w_{k-1})') - [\Box,\eta](u_0+w_k) \\ w_k|_{\partial\mathcal{K}} = 0 \\ w_k(0,x) = \partial_t w_k(0,x) = 0. \end{cases} \tag{3.17}$$

To proceed, we let

$$M_k(T) = \sup_{0\le t\le T} \sum_{|\alpha|\le 10} \Big(\|Z^\alpha w_k'(t,\,\cdot\,)\|_2 + (\ln(2+t))^{-1/2} \|(1+|x|)^{-1/2} Z^\alpha w_k'\|_{L^2(\{(s,x):\,0\le s\le t\})}\Big).$$

Then, if we use (3.4), (3.10) and (3.15), we conclude that there is a uniform constant C_1 so that

$$M_k(T_\varepsilon) \le C_1\varepsilon + C_1 \ln(2+T_\varepsilon)(\varepsilon + M_{k-1}(T_\varepsilon))^2 + C_1(\varepsilon + M_{k-1}(T_\varepsilon))^2,$$

for some uniform constant C_1, if ε is small. Since $M_0 \equiv 0$, an induction argument implies that, if the constant c occurring in the definition of T_ε is small then

$$M_k(T_\varepsilon) \le 2C_1, \quad k = 1,2,\dots, \tag{3.18}$$

for small $\varepsilon > 0$.

If we let

$$A_k(T) = \sup_{0\le t\le T} \sum_{|\alpha|\le 10} \Big(\|Z^\alpha (u_k' - u_{k-1}')(t,\,\cdot\,)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} + (\ln(2+t))^{-1/2} \|(1+|x|)^{-1/2} Z^\alpha (u_k' - u_{k-1}')\|_{L^2(\{(s,x):0\le s\le t,x\in\mathbb{R}^3\setminus\mathcal{K}\})}\Big),$$

then the preceding argument can be modified to show that

$$A_k(T_\varepsilon) \le \tfrac{1}{2} A_{k-1}(T_\varepsilon), \quad k = 1,2,\dots. \tag{3.19}$$

Estimates (3.18) and (3.19) imply Theorem 3.3. $\Box$

4. Pointwise estimates.

To prove existence theorems for quasilinear wave equations we need some pointwise estimates for solutions of inhomogeneous wave equations, as well as some weighted Sobolev inequalities. To describe the bounds for the wave equation, let us start out by considering pointwise estimates for solutions of the inhomogeneous wave equation in Minkowski space,

$$\begin{cases} (\partial_t^2 - \Delta) w_0(t,x) = G(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \\ w_0(0,x) = \partial_t w_0(0,x) = 0. \end{cases} \tag{4.1}$$

If

$$L = t\partial_t + \langle x, \nabla_x \rangle$$

is the scaling operator, then in [17] the following estimate was proved

$$(1+t)|w_0(t,x)| \le C \sum_{\substack{|\alpha|+\mu\le 3 \\ \mu\le 1}} \int_0^t \int_{\mathbb{R}^3} |L^\nu Z^\alpha G(s,y)| \frac{dyds}{|y|}. \tag{4.2}$$

Using this estimate and arguments from §2, we can obtain related estimates for solutions of the inhomogeneous wave equation,

$$\begin{cases} (\partial_t^2 - \Delta) w(t,x) = F(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \backslash \mathcal{K} \\ w(t,x) = 0, & x \in \partial\mathcal{K} \\ w(t,x) = 0, & t \le 0. \end{cases} \tag{4.3}$$

outside of obstacles satisfying Ikawa's local energy decay bounds (1.6). If we assume, as before, that $\mathcal{K} \subset \{x \in \mathbb{R}^3 : |x| < 1\}$ and that $\mathcal{K}$ satisfies (1.4) or (1.6), then the following pointwise estimate was proved in [17] and [27], respectively.

Theorem 4.1. *Let w be a solution to* (4.3), *and suppose that the local energy decay bounds* (1.4) *hold for $\mathcal{K}$. Then,*

$$\begin{aligned} (1+t+|x|)|L^\nu Z^\alpha w(t,x)| \le C \int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+7 \\ \mu\le\nu+1}} |L^\mu Z^\beta F(s,y)| \frac{dy\,ds}{|y|} \\ + C \int_0^t \sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+4 \\ \mu\le\nu+1}} \|L^\mu \partial^\beta F(s,\,\cdot\,)\|_{L^2(|y|<2)}\, ds. \end{aligned} \tag{4.4}$$

The estimate for non-trapping obstacles (in which case one can take one less derivative in the right side of (4.4)) was proved in [17]. It was observed in [27] that the same arguments will give (4.4) for obstacles satisfying Ikawa's bounds (1.6).

In [26], it was observed that one has the following estimates for w'.

Theorem 4.2. *Let w be a solution to* (4.3). *Suppose that $F(t,x)=0$ when $|x|>10t$. Then, if $|x|<t/10$ and $t>1$,*

(4.5)
$$(1+t+|x|)|L^\nu Z^\alpha w'(t,x)| \le C \sum_{\substack{\mu+|\beta|\le\nu+|\alpha|+3\\ \mu\le\nu+1}} \int_{t/100}^{t}\int_{\mathbb{R}^3\backslash\mathcal{K}} |L^\mu Z^\beta F'(s,y)|\,\frac{dy\,ds}{|y|}$$
$$+C\sup_{0\le s\le t}(1+s)\sum_{\substack{|\beta|+\mu\le|\alpha|+4+\nu\\ \mu\le\nu}} \|L^\mu Z^\beta F(s,\cdot\,)\|_\infty$$
$$+C\sup_{0\le s\le t}(1+s)\sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+6\\ \mu\le\nu}} \int_0^s\int_{\substack{||y|-(s-\tau)|\le 10\\ |y|\le(1000+\tau)/2}} |L^\mu Z^\beta F(\tau,y)|\,\frac{dy\,d\tau}{|y|}$$
$$+C\sup_{0\le s\le t}\sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+7\\ \mu\le\nu+1}} \int_{s/100}^{s}\int_{|y|\ge(1+\tau)/10} |L^\mu Z^\beta F(\tau,y)|\,\frac{dy\,d\tau}{|y|}.$$

To prove either of these two estimates we realize that inequality (4.2) yields

$$(4.6)\quad (1+t)|L^\nu Z^\alpha w(t,x)| \le C\int_0^t\int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+3\\ \mu\le\nu+1}} |L^\mu Z^\beta F(s,y)|\,\frac{dy\,ds}{|y|}$$
$$+C\sup_{|y|\le 2,\,0\le s\le t}(1+s)\sum_{\substack{|\beta|+\mu\le\nu+|\alpha|+2\\ \mu\le\nu}} \|L^\mu \partial^\beta w(s,\cdot\,)\|_{L^2(|x|<2)}.$$

The proof of (4.6) is exactly like that of Lemma 4.2 in [17]. The last term in (4.6) can be estimated using the local exponential decay of energy and the free space estimates. This is the term that is responsible for the last term in (4.4) and the last three terms in (4.5).

As we mentioned before, we also need some weighted Sobolev estimates. The first is an exterior domain analog of results of Klainerman-Sideris [20].

Lemma 4.3. *Suppose that $u(t,x)\in C_0^\infty(\mathbb{R}\times\mathbb{R}^3\backslash\mathcal{K})$ vanishes for $x\in\partial\mathcal{K}$. Then if $|\alpha|=M$ and ν are fixed*

$$(4.7)\quad \|\langle t-r\rangle L^\nu Z^\alpha \partial^2 u(t,\cdot\,)\|_2 \le C\sum_{\substack{|\beta|+\mu\le M+\nu+1\\ \mu\le\nu+1}} \|L^\mu Z^\beta u'(t,\cdot\,)\|_2$$
$$+C\sum_{\substack{|\beta|+\mu\le M+\nu\\ \mu\le\nu}} \|\langle t+r\rangle L^\mu Z^\beta(\partial_t^2-\Delta)u(t,\cdot\,)\|_2 + C(1+t)\sum_{\mu\le\nu}\|L^\mu u'(t,\cdot\,)\|_{L^2(|x|<2)}.$$

The other such estimate that we need is an exterior domain analog of an estimate of Hidano and Yokoyama [10].

Lemma 4.4. *Suppose that $u(t,x) \in C_0^\infty(\mathbb{R}\times\mathbb{R}^3\backslash\mathcal{K})$ vanishes for $x \in \partial\mathcal{K}$. Then*

$$r^{1/2}\langle t-r\rangle|\partial L^\nu Z^\alpha u(t,x)| \le C \sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+2\\ \mu\le\nu+1}} \|L^\mu Z^\beta u'(t,\,\cdot\,)\|_2 \tag{4.8}$$

$$+C \sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+1\\ \mu\le\nu}} \|\langle t+r\rangle L^\mu Z^\beta(\partial_t^2-\Delta)u(t,\,\cdot\,)\|_2 + C(1+t)\sum_{\mu\le\nu}\|L^\mu u'(t,\,\cdot\,)\|_{L^\infty(|x|<2)}.$$

5. L^2 Estimates.

In addition to the pointwise estimates, to prove global and almost global existence results for quasilinear wave equations outside of obstacles, we require certain energy-type estimates. Since the operators $\{Z\}$ and L do not preserve the Dirichlet boundary conditions, these are considerably more technical than the estimates that are used for the Minkowski space setting, which just follow from standard energy estimates and the fact that the Z operators commute with the D'Alembertian, while $[\Box, L] = 2\Box$.

The existence theorems involve possibly non-diagonal systems. Because of this we are led to proving L^2 estimates for solutions $u \in C^\infty(\mathbb{R}_+\times\mathbb{R}^3\backslash\mathcal{K})$ of the Dirichlet-wave equation

$$\begin{cases} \Box_\gamma u = F \\ u|_{\partial\mathcal{K}} = 0 \\ u|_{t=0} = f, \quad \partial_t u|_{t=0} = g \end{cases} \tag{5.1}$$

where

$$(\Box_\gamma u)^I = (\partial_t^2 - c_I^2\Delta)u^I + \sum_{J=1}^{D}\sum_{j,k=0}^{3}\gamma^{IJ,jk}(t,x)\partial_j\partial_k u^J, \quad 1\le I\le D.$$

We shall assume that the $\gamma^{IJ,jk}$ satisfy the symmetry conditions

$$\gamma^{IJ,jk} = \gamma^{JI,jk} = \gamma^{IJ,kj} \tag{5.2}$$

as well as the size condition

$$\sum_{I,J=1}^{D}\sum_{j,k=0}^{3}\|\gamma^{IJ,jk}(t,x)\|_\infty \le \delta/(1+t), \tag{5.3}$$

for δ sufficiently small (depending on the wave speeds). The energy estimate will involve bounds for the gradient of the perturbation terms

$$\|\gamma'(t,\,\cdot\,)\|_\infty = \sum_{I,J=1}^{D}\sum_{j,k,l=0}^{3}\|\partial_l\gamma^{IJ,jk}(t,\,\cdot\,)\|_\infty,$$

and the energy form associated with $\Box_\gamma$, $e_0(u) = \sum_{I=1}^{D} e_0^I(u)$, where

$$e_0^I(u) = (\partial_0 u^I)^2 + \sum_{k=1}^{3} c_I^2(\partial_k u^I)^2 + 2\sum_{J=1}^{D}\sum_{k=0}^{3}\gamma^{IJ,0k}\partial_0 u^I\partial_k u^J - \sum_{J=1}^{D}\sum_{j,k=0}^{3}\gamma^{IJ,jk}\partial_j u^I\partial_k u^J. \tag{5.4}$$

The most basic estimate will lead to a bound for

$$E_M(t) = E_M(u)(t) = \int \sum_{j=0}^{M} e_0(\partial_t^j u)(t,x)\,dx.$$

Lemma 5.1. *Fix $M = 0,1,2,\dots$, and assume that the perturbation terms $\gamma^{IJ,jk}$ are as above. Suppose also that $u \in C^\infty$ solves* (5.1) *and for every t, $u(t,x) = 0$ for large x. Then there is an absolute constant C so that*

$$\partial_t E_M^{1/2}(t) \le C \sum_{j=0}^{M} \|\Box_\gamma \partial_t^j u(t,\,\cdot\,)\|_2 + C\|\gamma'(t,\,\cdot\,)\|_\infty E_M^{1/2}(t). \tag{5.5}$$

This estimate is standard, and for this estimate one can weaken (5.3) by replacing the right side with δ for $\delta > 0$ sufficiently small. It is important to note that there is no "loss" of derivatives here in (5.5). On the other hand, if we wish to prove bounds involving the $\{Z, L\}$ operators our techniques lead to estimates where there is an additional local term which unfortunately involves a loss of one derivative. To be more specific, if we let

$$Y_{N_0,\nu_0}(t) = \int \sum_{\substack{|\alpha|+\mu \le N_0+\nu_0 \\ \mu \le \nu_0}} e_0(L^\mu Z^\alpha u)(t,x)\,dx. \tag{5.6}$$

then, if (5.3) holds, we have

$$\begin{aligned}\partial_t Y_{N_0,\nu_0} \le C Y_{N_0,\nu_0}^{1/2} &\sum_{\substack{|\alpha|+\mu \le N_0+\nu_0 \\ \mu \le \nu_0}} \|\Box_\gamma L^\mu Z^\alpha u(t,\,\cdot\,)\|_2 + C\|\gamma'(t,\,\cdot\,)\|_\infty Y_{N_0,\nu_0} \\ &+ C \sum_{\substack{|\alpha|+\mu \le N_0+\nu_0+1 \\ \mu \le \nu_0}} \|L^\mu \partial^\alpha u'(s,\,\cdot\,)\|^2_{L^2(|x|<1)}.\end{aligned} \tag{5.7}$$

In the arguments that are used to prove the existence theorems we are able to handle the contributions of the last term in (5.7) by using the following result from [27].

Lemma 5.2. *Suppose that* (1.6) *holds, and suppose that $u \in C^\infty$ solves* (5.1) *and satisfies $u(t,x) = 0$ for $t < 0$. Then, for fixed N_0 and ν_0 and $t > 2$,*

$$\begin{aligned}\sum_{\substack{|\alpha|+\mu \le N_0+\nu_0 \\ \mu \le \nu_0}} &\int_0^t \|L^\mu \partial^\alpha u'(s,\,\cdot\,)\|_{L^2(|x|<2)}\,ds \\ &\le C \sum_{\substack{|\alpha|+\mu \le N_0+\nu_0+1 \\ \mu \le \nu_0}} \int_0^t \left(\int_0^s \|L^\mu \partial^\alpha \Box u(\tau,\,\cdot\,)\|_{L^2(||x|-(s-\tau)|<10)}\,d\tau \right) ds.\end{aligned} \tag{5.8}$$

These are the main L^2 estimates that are needed in the proof of the existence results. Using them and variations of the weighted space-space time norms described in §3 that involve L as well as the operators $\{Z\}$ we can prove existence theorems for certain quadratic, quasilinear systems of the form

$$\begin{cases} \Box u = Q(du, d^2u), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \backslash \mathcal{K} \\ u(t,\,\cdot\,)|_{\partial\mathcal{K}} = 0 \\ u(0,\,\cdot\,) = f, \quad \partial_t u(0,\,\cdot\,) = g. \end{cases} \tag{5.9}$$

Here

$$\Box = (\Box_{c_1}, \Box_{c_2}, \dots, \Box_{c_D})$$

is a vector-valued multiple speed D'Alembertian with

$$\Box_{c_I} = \partial_t^2 - c_I^2 \Delta.$$

We will assume that the wave speeds c_I are positive and distinct. This situation is referred to as the nonrelativistic case. Straightforward modifications of the argument give the more general case where the various components are allowed to have the same speed. Also, $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the standard Laplacian. Additionally, when convenient, we will allow $x_0 = t$ and $\partial_0 = \partial_t$.

We shall assume that $Q(du, d^2u)$ is of the form

$$Q^I(du, d^2u) = B^I(du) + \sum_{\substack{0 \le j,k,l \le 3 \\ 1 \le J,K \le D}} B^{IJ,jk}_{K,l} \partial_l u^K \partial_j \partial_k u^J, \quad 1 \le I \le D \tag{5.10}$$

where $B^I(du)$ is a quadratic form in the gradient of u and $B^{IJ,jk}_{K,l}$ are real constants satisfying the symmetry conditions

$$B^{IJ,jk}_{K,l} = B^{JI,jk}_{K,l} = B^{IJ,kj}_{K,l}. \tag{5.11}$$

To obtain global existence, we shall also require that the equations satisfy the following null condition which only involves the self-interactions of each wave family. That is, we require that

$$\sum_{0 \le j,k,l \le 3} B^{JJ,jk}_{J,l} \xi_j \xi_k \xi_l = 0 \quad \text{whenever} \quad \frac{\xi_0^2}{c_J^2} - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0, \quad J = 1, \dots, D. \tag{5.12}$$

To describe the null condition for the lower order terms, we expand

$$B^I(du) = \sum_{\substack{1 \le J,K \le D \\ 0 \le j,k \le 3}} A^{I,jk}_{JK} \partial_j u^J \partial_k u^K.$$

We then require that each component satisfy the similar null condition

$$\sum_{0 \le j,k \le 3} A^{J,jk}_{JJ} \xi_j \xi_k = 0 \quad \text{whenever} \quad \frac{\xi_0^2}{c_J^2} - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0, \quad J = 1, \dots, D. \tag{5.13}$$

Thus, the null condition (5.12)-(5.13) is one that only involves interactions of components with the same wave speed.

We can now state the main result in [26]:

Theorem 5.3. *Let $\mathcal{K}$ be a fixed compact obstacle with smooth boundary that satisfies (1.6). Assume that $Q(du, d^2u)$ and $\Box$ are as above and that $(f, g) \in C^\infty(\mathbb{R}^3 \backslash \mathcal{K})$ satisfy* *the compatibility conditions to infinite order. Then there is a constant $\varepsilon_0 > 0$, and an integer $N > 0$ so that for all $\varepsilon < \varepsilon_0$, if*

$$\sum_{|\alpha|\le N} \| < x >^{|\alpha|} \partial_x^\alpha f\|_2 + \sum_{|\alpha|\le N-1} \| < x >^{1+|\alpha|} \partial_x^\alpha g\|_2 \le \varepsilon \tag{5.14}$$

then (5.9) *has a unique solution $u \in C^\infty([0,\infty) \times \mathbb{R}^3\backslash\mathcal{K})$.*

This result extended earlier ones of [15] and [27]. In [27] a weaker theorem was proved where instead of assuming the null conditions (5.12) and (5.13), the authors assumed that for every I one has

$$\sum_{0\le j,k,l\le 3} B_{J,l}^{IJ,jk}\xi_j\xi_k\xi_l = 0 \quad \text{whenever} \quad \frac{\xi_0^2}{c_J^2} - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0, \quad J = 1,\dots,D,$$

and

$$\sum_{0\le j,k\le 3} A_{JK}^{I,jk}\xi_j\xi_k = 0 \quad \text{for all} \quad \xi \in \mathbb{R}\times\mathbb{R}^3, \quad 1 \le J,K \le D.$$

respectively.

The nonrelativistic system satisfying the above null condition that we study serves as a simplified model for the equations of elasticity. In Minkowski space, such equations were studied and shown to have global solutions by Sideris-Tu [34], Agemi-Yokoyama [1], and Kubota-Yokoyama [21].

One can also, as in [17], prove almost global existence for solutions of equations of the form (5.9) that do not involve null conditions.

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