数理解析研究所講究録 1389 巻 2004 年 78-89

On some generalization of the weighted Strichartz estimates for the wave equation and self-similar solutions to nonlinear wave equations

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1 Weighted Strichartz estimates

This note is based on our recent joint work of the same title [7].

Let w be a solution to the following Cauchy problem of the inhomogeneous wave equation with zero data,

$$\partial_t^2 w - \Delta w = F, \quad (t,x) \in (0,\infty) \times \mathbf{R}^n \equiv \mathbf{R}_+^{1+n}, \tag{1.1}$$

$$w|_{t=0} = 0, \ \partial_t w|_{t=0} = 0, \quad x \in \mathbf{R}^n. \tag{1.2}$$

We consider the associated time-space weighted L^q-$L^{q'}$ estimates for the solution w of the form

$$\| |t^2 - |x|^2|^a w \|_{L^q(\mathbf{R}_+^{1+n})} \le C \| |t^2 - |x|^2|^b F \|_{L^{q'}(\mathbf{R}_+^{1+n})}, \quad 2 \le q \le \frac{2(n+1)}{n-1}, \tag{1.3}$$

which is called the weighted Strichartz estimates. Here, q' is the conjugate exponent to q. We notice that estimates (1.3) are recognized as the hyperbolic version of the following Carleman type estimates

$$\| |x|^{-a} f \|_{L^q(\mathbf{R}^n)} \le \| |x|^{-b} \Delta f \|_{L^{q'}(\mathbf{R}^n)}, \quad 2 \le q \le \frac{2n}{n-2}.$$

See [5], for example.

Estimates (1.3) were proved by Georgiev-Lindblad-Sogge [3] under the following conditions

$$a < \frac{n-1}{2} - \frac{n}{q}, \quad b > \frac{1}{q}, \quad \mathrm{supp} F \subset \{(t,x); |x| < t-1\}. \tag{1.4}$$

Using these estimates, they solved part of Strauss' conjecture concerning the existence of time-global solutions to the Cauchy problem of nonlinear wave equation with compactly supported, smooth, small initial data. Later, D'Ancona-Georgiev-Kubo [1] removed the assumption on the support of F in (1.4). Tataru [22] proved (1.3) when

$$a-b+\frac{n+1}{q}=\frac{n-1}{2}, \quad b<\frac{1}{q}, \quad \mathrm{supp}F\subset\{(t,x);\ |x|<t\}, \tag{1.5}$$

where the first one is related to the scale invariance.

The purpose of this note is to show the estimates (1.3) without the support assumption on F in the *scale invariant* case, which have an application to the existence of the self-similar solutions to nonlinear wave equations as we shall see below. Concerning this, in [9, 10], it was shown that the estimates (1.3) hold if F is radial in space variables without the support assumption on F. Precisely, it was proved that the estimates (1.3) hold if

$$a-b+\frac{n+1}{q}=\frac{n-1}{2}, \quad \frac{n}{q}-\frac{n-1}{2}<b<\frac{1}{q}, \quad F(t,x)=\widetilde{F}(t,|x|). \tag{1.6}$$

except the borderline case $q=2(n+1)/(n-1)$. As compared with the condition (1.5) the assumption on the support of F is removed at the cost of the additional lower bound on b, namely, $b>n/q-(n-1)/2$.

In this note, we remove the assumption of radial symmetry on F in (1.6):

Theorem 1.1. *Let $n\geq 2$. Let q, a, b satisfy $2<q<2(n+1)/(n-1)$,*

$$a-b+\frac{n+1}{q}=\frac{n-1}{2}, \quad \frac{n}{q}-\frac{n-1}{2}<b<\frac{1}{q}. \tag{1.7}$$

Then, for the solution w to (1.1), (1.2), *the estimate*

$$\big\||t^2-|x|^2|^a w\big\|_{L^q_{t,r}L^2_\omega}\leq C\big\||t^2-|x|^2|^b F\big\|_{L^{q'}_{t,r}L^2_\omega} \tag{1.8}$$

holds.

Here, for $G=G(t,x)$, the norm $\|\cdot\|_{L^p_{t,r}L^2_\omega}$ is defined by

$$\|G\|_{L^p_{t,r}L^2_\omega}=\Big\{\int_0^\infty\int_0^\infty\|G(t,r\cdot)\|^p_{L^2(S^{n-1})}r^{n-1}dr\,dt\Big\}^{1/p}, \tag{1.9}$$

using polar coordinates $x=r\omega$, $r>0$, $\omega\in S^{n-1}$. Theorem 1.1 says that, the introduction of L^2 space on the sphere enables us to remove the assumption of radial symmetry on F.

In odd space dimensions, we are able to improve the above result. Namely, we obtain a gain of regularity with respect to angular variables.

Theorem 1.2. *Let $n\geq 3$ be odd. Let q, a, b satisfy $4(n-1)/(2n-3)<q<2(n+1)/(n-1)$,*

$$a-b+\frac{n+1}{q}=\frac{n-1}{2}, \quad \frac{n+1}{2q}-\frac{n-1}{4}<b<\frac{1}{q}. \tag{1.10}$$

Then, for the solution w to (1.1), (1.2),

$$\big\||t^2-|x|^2|^a w\big\|_{L^q_{t,r}H^{1/2}_\omega}\leq C\big\||t^2-|x|^2|^b F\big\|_{L^{q'}_{t,r}L^2_\omega} \tag{1.11}$$

holds.

Remark 1.3. The lower bound on b in Theorem 1.2 is strictly greater than the one in Theorem 1.1 for $q > 2$.

Here, H^s_ω denotes the Sobolev space on S^{n-1} of fractional order s and the norm $\|\cdot\|_{L^q_{t,r}H^s_\omega}$ is defined similarly to (1.9).

The idea of the proof of Theorems 1.1, 1.2 is based on the expansion by spherical harmonics. We derive the expansion of the solution w with respect to spherical harmonics and reduce the estimates essentially to radial case. This idea is due to [13], which treats end point Strichartz estimates for the wave equation in 3 space dimensions using the norm with respect to angular variables. We also notice that similar type of Strichartz estimates are treated in [12].

This note is organized as follows. In Section 2 we prove Theorem 1.1 and give the outline of the proof of 1.2. In Section 3 we give the application of these theorems to the existence of self-similar solutions to nonlinear wave equations.

2 Proof of theorems

The proof of Theorem 1.1, 1.2 is based on the expansion of the solution w with respect to the spherical harmonics. We first describe its expansion precisely.

For $k \geq 0$, We denote by $\mathcal{H}_k$ the space of spherical harmonics of degree k on S^{n-1}, by α_k its dimension, and by $\{Y^k_1, \cdots, Y^k_{\alpha_k}\}$ the orthonormal basis of $\mathcal{H}_k$. It is well known that $L^2(S^{n-1}) = \oplus_{k=0}^\infty \mathcal{H}_k$ and that $F(t,x) = F(t, r\theta)$ has the expansion

$$F(t, r\omega) = \sum_{k=0}^{\infty} \sum_{l=1}^{\alpha_k} F^k_l(t,r)\, Y^k_l(\omega). \tag{2.1}$$

Then, by orthogonality, we observe that $\|F(t,r\cdot)\|_{L^2(S^{n-1})} = (\sum_{k,l} |F^k_l(t,r)|^2)^{1/2}$ and more generally,

$$\|F(t,r\cdot)\|_{H^s(S^{n-1})} = \Big\{ \sum_{k,l} (1 + k(k+n-2))^s |F^k_l(t,r)|^2 \Big\}^{1/2}. \tag{2.2}$$

Note that $(-\Delta_{S^{n-1}})Y^k = k(k+n-2)Y^k$ for $Y^k \in \mathcal{H}_k$, where $\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on S^{n-1}.

In the following, we set

$$W_n(t) = (-\Delta)^{-1/2} \sin[(-\Delta)^{1/2} t],$$

where we specially denote the space dimension n for later use. Then, the solution w to (1.1), (1.2) is given by

$$w(t, r\omega) = \int_0^t [W_n(t-s)F(s,\cdot)](r\omega)\, ds,$$

which is written in terms of (2.1) by

$$w(t,r\omega)=\int_0^t \big[W_n(t-s)\{\sum_{k=0}^{\infty}\sum_{l=1}^{\alpha_k}F_l^k(s,\lambda)\,Y_l^k(\theta)\}\big](r\omega)\,ds$$
$$=\sum_{k=0}^{\infty}\sum_{l=1}^{\alpha_k}\int_0^t \big[W_n(t-s)\{F_l^k(s,\lambda)\,Y_l^k(\theta)\}\big](r\omega)\,ds. \tag{2.3}$$

Then, we use the following lemma.

Lemma 2.1. *Let $Y^k\in\mathcal{H}_k$. Then, for $f\in C_0^\infty((0,\infty))$,*

$$W_n(t)\big[f(\lambda)Y^k(\theta)\big](r\omega)=r^k\,W_{n+2k}(t)\big[\lambda^{-k}f(\lambda)\big](r)\,Y^k(\omega). \tag{2.4}$$

Remark 2.2. We apply Lemma 2.1 to compute (2.3). To prove Theorems 1.1, 1.2 it suffices to show for $F\in C_0^\infty(\mathbf{R}_+^{1+n}\setminus\{|x|=0\})$. In fact, such space is dense in the weighted Lebesgue spaces in question, and then, for each $t\geq 0$,

$$F_l^k(t,r)=\int_{S^{n-1}}F(t,r\theta)Y_l^k(\theta)\,d\sigma(\theta)\in C_0^\infty((0,\infty)).$$

Note that since $F\in C_0^\infty(\mathbf{R}_+^{1+n}\setminus\{|x|=0\})$, $F_l^k(t,r)$ vanishes when r is sufficiently small.

Proof of Lemma 2.1. Since $f\in C_0^\infty((0,\infty))$, the left hand side of (2.4) is a classical solution of the Cauchy problem of the wave equation

$$\partial_t v-\Delta v=0, \tag{2.5}$$
$$v(0,x)=0,\quad \partial_t v(0,x)=f(|x|)Y^k(x/|x|). \tag{2.6}$$

Thus, if we show that the right hand side of (2.4)

$$z(t,r\omega)=r^k\,\tilde{z}(t,r)Y^k(\omega)$$

is also a classical solution of (2.5), (2.6), where $\tilde{z}(t,r)=W_{n+2k}(t)[\lambda^{-k}f(\lambda)](r)$, then by the uniqueness of classical solutions we obtain (2.4). Obviously, z is regular and satisfies (2.6). Therefore, it suffices to show that z satisfies (2.5), which is observed as follows.

$$\begin{aligned}
&(\partial_t^2-\Delta)z\\
=&\Big(\partial_t^2-\partial_r^2-\frac{n-1}{r}\partial_r-\frac{1}{r^2}\Delta_{S^{n-1}}\Big)r^k\,\tilde{z}\,Y^k\\
=&r^k\Big(\partial_t^2\,\tilde{z}-\partial_r^2\,\tilde{z}-\frac{n+2k-1}{r}\partial_r\tilde{z}-\frac{k(k+n-2)}{r^2}\tilde{z}\Big)Y^k+r^k\,\tilde{z}\frac{k(k+n-2)}{r^2}Y^k\\
=&r^k\Big(\partial_t^2\,\tilde{z}-\partial_r^2\,\tilde{z}-\frac{n+2k-1}{r}\partial_r\,\tilde{z}\Big)Y^k=0.
\end{aligned}$$

This completes the proof of Lemma 2.1. □

Applying Lemma 2.1, we obtain the expansion of w,

$$w(t, r\omega) = \sum_{k=0}^{\infty} \sum_{l=1}^{\alpha_k} S_k(F_l^k)(t,r)\, Y_l^k(\omega), \tag{2.7}$$

where

$$S_k(G)(t,r) = r^k \int_0^t W_{n+2k}(t-s)\big[\lambda^{-k} G(s,\lambda)\big](r)\, ds. \tag{2.8}$$

Using this expansion, we prove Theorems 1.1, 1.2.

Proof of Theorem 1.1. By the expansion (2.7), the crucial point of the proof of Theorem 1.1 is to derive the estimate on the coefficients $S_k(F_l^k)$. But the estimate on $S_k(F_l^k)$ needed for the proof of Theorem 1.1 are derived by a similar argument in [9, 10], where the weighted Strichartz estimates under the assumption of radial symmetry are considered. In particular, the following estimates hold.

Lemma 2.3. *Let $n \geq 2$. Let q, a, and b be as in Theorem* 1.1. *Then, there exists a constant $C > 0$ independent of k such that*

$$\big\| |t^2 - r^2|^a r^{(n-1)/q} S_k(G) \big\|_{L^q_{t,r}} \leq C \big\| |t^2 - r^2|^b r^{(n-1)/q'} G \big\|_{L^{q'}_{t,r}}. \tag{2.9}$$

Proof. We first consider the case the space dimension n is odd. From (2.8) and the representation of the radial solution (see for instance [21, Lemma 2.2]), we have

$$S_k(G)(t,r) = r^{-(n-1)/2} \int_0^t \int_{|t-s-r|}^{t-s+r} P_{k+(n-3)/2}(\mu) \lambda^{(n-1)/2} G(s,\lambda)\, d\lambda ds, \tag{2.10}$$

where P_m is the Legendre polynomial of degree m and

$$\mu = \frac{r^2 + \lambda^2 - (t-s)^2}{2r\lambda}. \tag{2.11}$$

Then, from the estimate of the Legendre polynomials

$$|P_m(z)| \leq 1, \quad |z| \leq 1, \ m \geq 0 \tag{2.12}$$

and the fact that $|\mu| \leq 1$ if $\lambda \geq |t-s-r|$, we estimate

$$|S_k(G)(t,r)| \leq r^{-(n-1)/2} \int_0^t \int_{|t-s-r|}^{t-s+r} \lambda^{(n-1)/2} |G(s,\lambda)|\, d\lambda ds. \tag{2.13}$$

Thus, to derive the estimate (2.9) it is sufficient to apply the same argument as in [9, Lemma 3.3]. Note that the right hand side of (2.13) is independent of k.

We next consider the case n is even. In this case we need two types of representations and estimates of $S_k(G)(t,r)$ to apply the argument in [10]. From (2.8) and the representation of the radial solution (see for instance [21, Lemma 2.3]), we have

$$\begin{aligned} S_k(G)(t,r) = \frac{2}{\pi} r^{-n/2+1} \int_0^t \int_0^{t-s} \frac{\rho}{\sqrt{(t-s)^2-\rho^2}} \\ \times \Big(\int_{|r-\rho|}^{r+\rho} \frac{T_{k+(n-2)/2}(\widetilde{\mu})}{\sqrt{\lambda^2-(r-\rho)^2}\sqrt{(r+\rho)^2-\lambda^2}} \lambda^{n/2} F(s,\lambda) d\lambda \Big) d\rho\, ds, \end{aligned} \tag{2.14}$$

where T_m is the Tschebysheff polynomial of degree m and $\widetilde{\mu} = (\lambda^2 + r^2 - \rho^2)/2r\lambda$. Since $|T_m(z)| \le 1$ for $|z| \le 1$, $m \ge 0$, and $|\widetilde{\mu}| \le 1$ for $\lambda \ge |r-\rho|$, we obtain the pointwise estimate of $S_k(G)(t,r)$ independent of k. Similarly, from (2.8) and the representation of radial solution (see for instance [11, Theorem 3.4]), we have

$$\begin{aligned} S_k(G)(t,r) &= r^{-k-n+2} \int_0^t \int_{|t-s-r|}^{t-s+r} \lambda^{k+n-1} K_{k+(n-2)/2}(\lambda, r, t-s) F(s,\lambda)\, d\lambda\, ds \\ &+ r^{-k-n+2} \int_0^{\max(t-r,0)} \int_0^{t-s-r} \lambda^{k+n-1} \widetilde{K}_{k+(n-2)/2}(\lambda, r, t-s) F(s,\lambda)\, d\lambda\, ds. \end{aligned}$$

Here the kernels have the estimates (see [11, Lemma 4.2], [10, Lemma 3.1])

$$\begin{aligned} r^{-k}\lambda^k |K_{k+(n-2)/2}(\lambda,r,\tau)| &\le C r^{(n-3)/2} \lambda^{-(n-1)/2} \min(r^{1/2}, \lambda^{1/2}) (\lambda - \tau + r)^{-1/2}, \\ &\qquad |\tau - r| < \lambda < \tau + r, \\ r^{-k}\lambda^k |\widetilde{K}_{k+(n-2)/2}(\lambda,r,\tau)| &\le C r^{(n-3)/2+\sigma} (\tau - r)^{-(n-2)/2-\sigma} (\tau - r - \lambda)^{-1/2}, \\ &\qquad 0 < \lambda < \tau - r, \quad 0 \le \sigma \le 1/2, \end{aligned}$$

where the constants are independent of k. These representations and estimates of $S_k(G)(t,r)$ enable us to apply the argument in [10] to derive the estimate (2.9). □

Then, the estimate (1.8) is obtained as follows. By the expansion (2.7) and Lemma 2.3, we have

$$\begin{aligned} \big\| |t^2 - |x|^2|^a w \big\|_{L^q_{t,r} L^2_\omega} &= \big\| |t^2 - r^2|^a r^{(n-1)/q} \big(\sum_{k,l} |S_k(F^k_l)|^2 \big)^{1/2} \big\|_{L^q_{t,r}} \\ &\le \big(\sum_{k,l} \big\| |t^2 - r^2|^a r^{(n-1)/q} S_k(F^k_l) \big\|^2_{L^q_{t,r}} \big)^{1/2} \\ &\le C \big(\sum_{k,l} \big\| |t^2 - r^2|^b r^{(n-1)/q'} F^k_l \big\|^2_{L^{q'}_{t,r}} \big)^{1/2} \\ &\le C \big\| |t^2 - r^2|^b r^{(n-1)/q'} \big(\sum_{k,l} |F^k_l|^2 \big)^{1/2} \big\|_{L^{q'}_{t,r}} \\ &\le C \big\| |t^2 - |x|^2|^b F \big\|_{L^{q'}_{t,r} L^2_\omega}, \end{aligned}$$

where we have used Minkowski's integral inequality repeatedly, since $q > 2$ and $q' < 2$. □

Proof of Theorem 1.2. For the proof of Theorem 1.2, we need improved estimates on $S_k(F_l^k)$ instead of those in Lemma 2.4 and such estimates are derived at least in odd space dimensions.

Lemma 2.4. *Let $n \geq 3$ be odd. Let q, a, and b be as in Theorem 1.2. Then, there exists a constant $C > 0$ independent of $k \geq 1$ such that*

$$\left\| |t^2 - r^2|^a \, r^{(n-1)/q} S_k(G) \right\|_{L^q_{t,r}} \leq C \, k^{-1/2} \left\| |t^2 - r^2|^b \, r^{(n-1)/q'} G \right\|_{L^{q'}_{t,r}}. \tag{2.15}$$

Outline of the proof of Lemma 2.4. We recall that $S_k(G)$ is given by (2.10) in odd space dimensions. To derive the estimate (2.15) we use another estimate of the Legendre polynomials instead of (2.12). Namely,

$$|P_m(z)| \leq Cm^{-1/2}(1 - |z|^2)^{-1/4}, \quad |z| < 1, \; m \geq 1. \tag{2.16}$$

(See [2, §1.6].) Then, from (2.10), we have

$$\begin{aligned} |S_k(G)(t,r)| \leq\, & C k^{-1/2} r^{-(n-1)/2} \\ & \times \int_0^t \int_{|t-s-r|}^{t-s+r} (1 - \mu^2)^{-1/4} \, \lambda^{(n-1)/2} |G(s,\lambda)| \, d\lambda ds. \end{aligned} \tag{2.17}$$

Note that μ is given by (2.11), and thus

$$\begin{aligned} &(1 - \mu^2)^{-1/4} \\ &= \frac{\sqrt{2}\, r^{1/2} \lambda^{1/2}}{(r + \lambda + t - s)^{1/4} (r + \lambda - t + s)^{1/4} (t - s + r - \lambda)^{1/4} (t - s - r + \lambda)^{1/4}}. \end{aligned}$$

We observe that the estimate of $S_k(G)(t,r)$ in this case is similar to that of even space dimensions (see (2.14)). In fact, applying the similar method in [10], which treats the weighted Strichartz estimates in even space dimensions, we are able to reduce the estimate (2.15) to the following weighted Hardy-Littlewood-Sobolev inequality.

Lemma 2.5 **([19]).** *Let $0 < \lambda < n$, $1 < r, s < \infty$. Let $\alpha < n/s'$ and $\beta < n/r'$ with $\alpha + \beta \geq 0$ satisfy $1/s + 1/r + (\lambda + \alpha + \beta)/n = 2$. Then,*

$$\left| \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{f(x) g(y)}{|x|^\alpha \, |x - y|^\lambda \, |y|^\beta} dx \, dy \right| \leq C \|f\|_{L^s(\mathbf{R}^n)} \|g\|_{L^r(\mathbf{R}^n)}.$$

□

Using Lemma 2.4, estimates (1.11) are obtained as follows. By the expansion (2.7) and (2.2), we have

$$\begin{aligned}\big\||t^2-|x|^2|^a w\big\|_{L^q_{t,r}H^{1/2}_\omega} &= \Big\||t^2-r^2|^a r^{(n-1)/q}\big\{\sum_{k,l}(1+k(k+n-2))^{1/2}|S_k(F^k_l)|^2\big\}^{1/2}\Big\|_{L^q_{t,r}}\\ &\le C\big(\sum_{k,l}(1+k)\big\||t^2-r^2|^a r^{(n-1)/q}S_k(F^k_l)\big\|^2_{L^q_{t,r}}\big)^{1/2}\\ &\le C\big(\sum_{k,l}\big\||t^2-r^2|^b r^{(n-1)/q'}F^k_l\big\|^2_{L^{q'}_{t,r}}\big)^{1/2}\\ &\le C\,\Big\||t^2-r^2|^b r^{(n-1)/q'}\big(\sum_{k,l}|F^k_l|^2\big)^{1/2}\Big\|_{L^{q'}_{t,r}}\\ &\le C\,\big\||t^2-|x|^2|^b F\big\|_{L^{q'}_{t,r}L^2_\omega},\end{aligned}$$

where we have used Minkowski's integral inequality repeatedly, since $q > 2$ and $q' < 2$. □

3 Existence of self-similar solutions

As an application of Theorems 1.1, 1.2, we are able to show the existence of self-similar solutions to the nonlinear wave equation

$$\partial_t^2 u - \Delta u = |u|^p, \quad (t,x) \in (0,\infty)\times \mathbf{R}^n. \tag{3.1}$$

The solution u to (3.1) is called a *self-similar solution* if u satisfies

$$u(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x) \tag{3.2}$$

for all $\lambda > 0$. Letting $\lambda = 1/t$, $u(1,\cdot) = W(\cdot)$, we observe that self-similar solutions are solutions of the following form

$$u(t,x) = t^{-\frac{2}{p-1}} W(x/t).$$

From such scaling properties, it is known that self-similar solutions are useful to investigate the asymptotic behavior of the time-global solutions as $t \to \infty$ (See [14], for example).

It is known that there is a close connection between the existence of the self-similar solutions to (3.1) and the power p of the nonlinear term. In fact, in three space dimensions, Pecher [17] proved that if $p > 1+\sqrt{2}$, there exist self-similar solutions, and if $p \le 1+\sqrt{2}$, self-similar solutions do not exist. We intended to extend such sharp existence results of self-similar solutions to higher dimensions. We denote by $p_0(n)$ the positive root of

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

Then, $p_0(3) = 1+\sqrt{2}$ and we expect $p_0(n)$ to be the critical power concerning the existence of self-similar solutions to the equation (3.1). We notice that $p_0(n)$ is the critical exponent concerning the existence of time-global solutions to the Cauchy problem of the equation (3.1) with compactly supported, small, smooth initial data. (See John [6], Georgiev-Lindblad-Sogge [3] and references therein.) So, it is natural to expect $p_0(n)$ to be the one because self-similar solutions are also time-global solutions.

Concerning this problem, in 2 and 3 space dimensions, Hidano [4] proved the existence of self-similar solutions when $p > p_0(n)$. In [9, 10] the first and the third author proved the existence of radially symmetric self-similar solutions when $p > p_0(n)$ for $n \geq 2$.

Remark 3.1. Precisely, the above results show the existence of self-similar solutions when $p_0(n) < p < (n+3)/(n-1)$. The existence of self-similar solutions for large p was treated in [16, 18].

As an application of Theorems 1.1, 1.2, we have the following result.

Theorem 3.2. *Let $2 \leq n \leq 5$. For any p with $p_0(n) < p < (n+3)/(n-1)$, let ϕ, $\psi \in C^\infty(\mathbf{R}^n \setminus \{0\})$ be homogeneous of degree $-2/(p-1)$, $-2/(p-1)-1$, respectively. Then, if $\varepsilon > 0$ is sufficiently small, there exists a unique time-global solution u to* (3.1) *with*

$$u(0,x) = \varepsilon\phi(x), \quad \partial_t u(0,x) = \varepsilon\psi(x) \tag{3.3}$$

satisfying

$$\left\| |t^2 - |x|^2|^\gamma u; \mathcal{L}_{t,r}^{p+1} H_\omega^{(n-1)/2+\delta} \right\| \leq C\varepsilon,$$

where $\gamma = 1/(p-1) - (n+1)/2(p+1)$ and $\delta > 0$ sufficiently small.

Here, $\mathcal{L}_{t,r}^q$ denotes the weak Lebesgue spaces on $\mathbf{R}_+ \times \mathbf{R}_+$ and $\| \cdot \|_{\mathcal{L}_{t,r}^q H_\omega^s}$ is defined by

$$\|G\|_{\mathcal{L}_{t,r}^q H_\omega^s} \equiv \sup_{\lambda>0} \lambda \left| \{(t,r);\ \|G(t,r\cdot)\|_{H^s(S^{n-1})} > \lambda\} \right|^{1/p}.$$

Remark 3.3. By the homogeneity of the data (3.3) and the uniqueness of solutions, the solution obtained in Theorem 3.2 is to be self-similar. That is, self-similar solutions to (3.1) are shown to exist when $2 \leq n \leq 5$, $p > p_0(n)$.

Remark 3.4. Sobolev type embedding theorem on the unit sphere

$$H^s(S^{n-1}) \hookrightarrow L^\infty(S^{n-1}) \quad \text{for } s > \frac{n-1}{2} \tag{3.4}$$

is basic to our estimates on the nonlinear term, which in turn causes the restriction $n \leq 5$.

The proof of Theorem 3.2 is essentially the same as that of [9, Theorem 1.1], which shows the existence of radially symmetric self-similar solutions by the standard fixed point arguments using weighted Strichartz estimates of radial case. In fact, we can translate those proofs into $H^{(n-1)/2+\delta}(S^{n-1})$-valued functions. For example, we have the following theorem interpolating the estimates in Theorems 1.1 and 1.2, respectively.

Theorem 3.5. *Let $n \geq 2$ and let $s \geq 0$. For $2 < q < 2(n+1)/(n-1)$ and $(n-1)/q < \alpha < (n-1)/q'$, we set*

$$a = \frac{\alpha}{2} - \frac{n+1}{2q}, \quad b = \frac{\alpha}{2} + \frac{n+1}{2q} - \frac{n-1}{2}. \tag{3.5}$$

Then, there exists a constant $C > 0$ such that for any function F which is homogeneous of degree $-\alpha - 2$, i.e.

$$F(\lambda t, \lambda x) = \lambda^{-\alpha-2} F(t,x), \quad (t,x) \in \mathbf{R}^{1+n}_+, \ \lambda > 0,$$

we have

$$\| |t^2 - |x|^2|^a w \|_{\mathcal{L}^q_{t,r} H^s_\omega} \leq C \| |t^2 - |x|^2|^b F \|_{\mathcal{L}^{q'}_{t,r} H^s_\omega}. \tag{3.6}$$

Theorem 3.6. *Let $n \geq 3$ be odd and let $s \geq 1/2$. For $4(n-1)/(2n-3) < q < 2(n+1)/(n-1)$ and $(n-1)/2 < \alpha < (n-1)/q'$, we set a and b as in (3.5). Then, there exists a constant $C > 0$ such that for any function F which is homogeneous of degree $-\alpha - 2$, we have*

$$\| |t^2 - |x|^2|^a w \|_{\mathcal{L}^q_{t,r} H^s_\omega} \leq C \| |t^2 - |x|^2|^b F \|_{\mathcal{L}^{q'}_{t,r} H^{s-1/2}_\omega}. \tag{3.7}$$

So, we omit the proof of Theorem 3.2 here without illustrating the different point, that is, the estimate of the nonlinear term of the equation (3.1) in terms of $H^{(n-1)/2+\delta}(S^{n-1})$-norm. For that estimate we use the following proposition.

Proposition 3.7. *Let $n \geq 2$ and let $p > 1$. For $s \geq s_0$, we assume $p > s_0$ and $s > (n-1)/2$. Then,*

$$\| |g|^p \|_{H^{s_0}(S^{n-1})} \leq C \|g\|^p_{H^s(S^{n-1})}, \tag{3.8}$$

$$\| |g|^p - |h|^p \|_{L^2(S^{n-1})} \leq C(\|g\|^{p-1}_{H^s(S^{n-1})} + \|h\|^{p-1}_{H^s(S^{n-1})}) \|g - h\|_{L^2(S^{n-1})}. \tag{3.9}$$

Proof. The estimate (3.8) follows from the Moser type estimate

$$\| |g|^p \|_{H^{s_0}(S^{n-1})} \leq C \|g\|^{p-1}_{L^\infty(S^{n-1})} \|g\|_{H^{s_0}(S^{n-1})}$$

with $p > \max(s_0, 1)$ and the Sobolev embedding (3.4). The estimate (3.9) follows from the Hölder inequality and also the Sobolev embedding (3.4). □

Combining the above proposition with $s_0 = s$ with (3.6), we are able to estimate the nonlinear term when $2 \leq n \leq 4$. In fact,

$$p_0(n) > \frac{n-1}{2} \quad \text{for } 2 \leq n \leq 4$$

and thus

$$\||u|^p\|_{H^s(S^{n-1})} \le C\|u\|^p_{H^s(S^{n-1})}$$

holds if $p > p_0(n)$, $p > s > (n-1)/2$. When $n = 5$, we use (3.7) and Proposition 3.7 with $s_0 = s - 1/2$. In fact,

$$p_0(5) > \left(\frac{n-1}{2} - \frac{1}{2}\right) = \frac{3}{2} \quad \text{for } n = 5,$$

and thus

$$\||u|^p\|_{H^{s-1/2}(S^4)} \le C\|u\|^p_{H^s(S^4)}$$

holds if $p > p_0(5)$, $p > s - 1/2 > 3/2$. Thus, a gain of regularity in (3.7) is used effectively.

Remark 3.8. $p_0(n)$ is monotone decreasing as $n \to \infty$, which goes to 1. Note that $p_0(4) = 2$ and $p_0(5) = (3 + \sqrt{17})/4$ ($\fallingdotseq 1.75$).

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