数理解析研究所講究録 1389 巻 2004 年 90-92

RECENT PROGRESS ON THE NONLINEAR SCHRÖDINGER EVOLUTION PROBLEM

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This note outlines the talk given at the Kyoto meeting on harmonic analysis and nonlinear PDEs in July 2003. We describe part of the joint work [2] with Michael Christ and Terry Tao. The description below is somewhat informal and more precise statements can be found in [2].

We consider the initial value problem

$$\text{(NLSp)} \qquad \begin{cases} -iu_t + \Delta_x u = \omega |u|^{p-1} u \\ u(0,x) = u_0(x) \in H^s(\mathbb{R}^d) \end{cases}$$

and ask:

Question: What is the minimal H^s regularity for which local well-posedness holds?

Scaling invariance:
u_λ is a solution of NLSp if and only if u is a solution of NLSp where

$$u_\lambda(t,x) = \lambda^{-2/(p-1)} u(t/\lambda^2, x/\lambda), \ \lambda > 0.$$

$$\|u_\lambda(t)\|_{\dot{H}^s_x} = \lambda^{s_c - s} \|u(t)\|_{\dot{H}^s_x}, \ s_c = \frac{d}{2} - \frac{2}{p-1}.$$

Galilean invariance:
$g_v u$ is a solution of NLSp if and only if u is a solution of NLSp where

$$(g_v u)(t,x) = e^{-i\frac{v \cdot x}{2}} e^{i\frac{t^2}{4}} u(t, x - vt), \ v \in \mathbb{R}^d.$$

$$\|g_v u\|_{\dot{H}^s} \sim (1 + |v|)^{s - s_g} \|u\|_{\dot{H}^s}, \ s_g = 0.$$

Local Well-Posedness Theorem (Cazenave-Weissler):
NLSp is locally well-posed[1] in H^s if $s \geq \max(s_g, s_c)$.

Our work addresses the optimality of this restriction on the Sobolev index s.

Let $B^s_R(\psi)$ denote the ball $\{\phi \in H^s : \|\phi - \psi\| < R\}$. Let $\mathcal{S}$ denote the Schwartz functions.

[1]When $s = s_c$, the lifetime of local existence depends upon the profile of the initial data; in the other cases the lifetime is determined by the H^s norm of the data.

Theorem 1. *Let $p > 1$ be odd, $d \geq 1$, $\omega = \pm 1$. If $s < \max(s_g, s_c)$ then for all $t > 0$ there exits $u_1(0), u_2(0) \in B^s_\epsilon \cap S$ such that for any $0 < \delta \ll \epsilon < 1$*

$$u_1(0) - u_2(0) \in B^s_\delta(0),$$
$$u_1(t) - u_2(t) \in H^s \backslash B^s_{\epsilon/4}(0).$$

where $u_i(0) \longmapsto u_i(t)$ are solutions of NLSp for $i = 1, 2$.

Thus, the solution map $S_{NLSp}(t)$ which maps initial data u_0 to the solution $u(t)$ is not locally uniformly continuous on H^s when $s < \max(s_g, s_c)$.

- For the focusing case $\omega = -1$, this was established using soliton or blow-up solutions in works by Birnir-Kenig-Ponce-Svanstedt-Vega, Kenig-Ponce-Vega.
- For the defocusing $\omega = +1$, $p = 3$, $d = 1$ case, this was established in [1] using modified scattering solutions found by Ozawa.
- The general case is obtained using approximation via small and zero dispersion versions of NLSp.
- We have also obtained similar results [2] for certain semilinear wave equations.

Theorem 2. *Let $p > 1$ be odd, $d \geq 1$, $\omega = \pm 1$. If $0 < s < s_c$ or $s \leq -\frac{d}{2}$ then for any $\epsilon > 0$ there exists $u(0) \in S$ and there exists a $t \in (0, \epsilon)$ such that $u(0) \longmapsto u(t)$ solves NLSp and*

$$\|u(0)\|_{H^s} < \epsilon,$$
$$\|u(t)\|_{H^s} > \frac{1}{\epsilon}.$$

Thus, the solution map S_{NLSp} is not continuous at 0 for s in these regimes. The situation when $0 < s < s_c$ exploits a low to high frequency mass transfer. For $s \leq -\frac{d}{2}$, a high to low transfer is used.

We briefly describe the construction of a parametrized family of solutions used to prove Theorems 1 and 2.

ϕ solves zNLSp if and only if $u(t, x) = \phi(t, x\nu)$ solves NLSp where

$$\text{(zNLSp)} \qquad \begin{cases} -i\phi_t + \nu^2 \Delta_y \phi = \omega |\phi|^{p-1} \phi \\ \qquad \phi(0, x) = \phi_0(x) \in H^s(\mathbb{R}^d) \end{cases}$$

For small dispersion ($\nu \ll 1$ or $\nu \to 0$), we expect that zNLSp solutions are approximable by ODEp solutions

$$\text{(ODEp)} \qquad \begin{cases} -i\phi_t + = \omega |\phi|^{p-1} \phi \\ \phi(0, x) = \phi_0(x) \in H^s(\mathbb{R}^d). \end{cases}$$

The ODEp solution is given by

$$\phi^{(0)}(t, y) = \phi_0(y) e^{i\omega t |\phi_0(y)|^{p-1}}$$

which reveals an explicit phase behavior.

Using the energy method, we prove that there are indeed solutions of zNLSp which may be approximated by solutions of ODEp on a time interval whose length goes to infinity as $\nu \to 0$. We return these zNLSp solutions to the original setting and apply Galilean and scaling symmetries to obtain a parametrized family of solutions of NLSp. We then adapt the Birnir-Kenig-Ponce-Svanstedt-Vega, Kenig-Ponce-Vega arguments to force phase decoherence of the ODEp solutions and use the approximation result to transfer this decoherence to the NLSp solutions. This is how we prove Theorem 1.

A direct calculation shows the explicit ODEp solutions have H^j_y norms which grow with time, for $j = 0, 1, \dots, p-1$. This reveals a low to high frequency mass transfer. The ODEp solution emerging from initial data whose Fourier transform is zero near the zero frequency may also be shown to excite frequencies near the zero frequency thus revealing a high to low frequency mass transfer. The scaling invariance is used to speed up these mass transfers to prove Theorem 2.

REFERENCES

[1] M. Christ, J. Colliander, and T. Tao, *Asymptotics, frequency modulation, and low regularity illposedness for canonical defocusing equations*, Amer. J. Math. 125 (2003), 1235–1293.

[2] M. Christ, J. Colliander, and T. Tao, *Ill-Posedness for nonlinear Schrödinger and wave equations*.