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On minimal fields

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Abstract

An infinite structure M is minimal if every definable subset (using parameters in M) is finite or cofinite. An algebraically closed field is minimal, since it allows quantifier elimination. Podewski [Po] conjectured that minimal fields are algebraically closed. Wagner [Wa] has shown that a minimal field of nonzero characteristics is algebraically closed. We discuss minimal fields in all characteristics.

1 Minimal fields with infinite bases

It is well-known that a minimal structure M with an infinite basis is strongly minimal. For we can easily show that M is saturated, and it follows that M is strongly minimal. Since strongly minimal fields are algebraically closed, we get that a minimal field with an infinite basis is algebraically closed. Nevertheless we give a direct proof that a minimal field with an infinite basis is algebraically closed in order to use the arguments of that proof in the next section.

The following facts are well-known. See Hodges [Ho], and Wagner [Wa].

Fact 1 *Let M be a minimal structure. Then*

1. (M, acl_M) *is a pregeometry.*
2. *Every subset of M has a dimension over any subset.*
3. *For every independent (over $\emptyset$) subsets A, B of M with $|A| = |B|$, any bijective map of A to B can be extended to an automorphism of M.*
4. *If X is an algebraically closed (in the sense of model theory) infinite subset of M, then $X \prec K$.*

Every infinite field with quantifier elimination is algebraically closed. Direct proofs were given by several authors. We prove the following lemma in the same manner as its proof given by Wheeler [Wh].

Lemma 2 *A minimal field K with an infinite basis is algebraically closed.*

Proof. Consider the definable set $A = \{a \in K : K \models \exists y(a = y^n)\}$ for $n \in \mathbb{N}^*$. Then A must be infinite, hence cofinite. Suppose $K \setminus A \neq \emptyset$, and let $b \in K \setminus A$. Then $\{bc^n : c \in K^*\} \subseteq K \setminus A$, a contradiction. Therefore $A = K$. It follows that K is perfect and $\mathrm{acl}_K(\{\emptyset\})$ is infinite, hence $\mathrm{acl}_K(\{\emptyset\}) \prec K$.

Claim A *Let $n \in \mathbb{N}^*$ and $a_1, \dots, a_n$ be independent elements of K (over $\emptyset$). Then the polynomial $X^n + a_1X^{n-1} + \cdots + a_n$ has a root in K.*

For suppose that $b_1, \dots, b_n$ be independent elements of K, and let $p(X)$ be the polynomial $\prod_{1 \leq i \leq n}(X - b_i)$. Then $p(X) = X^n + s_1X^{n-1} + \cdots + s_n$, where s_i is the i-th elementary symmetric polynomial. $s_1, \dots, s_n$ are independent, since $\mathrm{acl}_K(\{b_1, \dots, b_n\}) = \mathrm{acl}_K(\{s_1, \dots, s_n\})$. In minimal structures there is an automorophism which takes s_i to a_i, hence the polynomial $X^n + a_1X^{n-1} + \cdots + a_n$ has a root in K.

Now suppose that K is not algebraicaly closed. Then $\mathrm{acl}_K(\{\emptyset\})$ is not algebraically closed, since $\mathrm{acl}_K(\{\emptyset\}) \prec K$. Let $K_0 = \mathrm{acl}_K(\{\emptyset\})$. Then there is $\alpha \in \overline{K_0} \setminus K_0$, where $\overline{K_0}$ is the algebraic closure of K_0. α is seperable over K_0, since K_0 is perfect. Let α has degree n over K_0 $(n > 1)$ and $\alpha = \alpha_0, \dots, \alpha_{n-1}$ be the distinct conjugates of α. Choose independent elements $t_0, \dots, t_{n-1}$ of K (over $\emptyset$), and form the polynomial $F(X) = \prod_{i<n}(X - \sum_{j<n} t_j\alpha_i^j)$. We can write this polynomial as $F(X) = X^n + g_1X^{n-1} + \cdots + g_n$ where each g_i is in $K_0[\bar{t}] \subset K$.

Claim B *The g_i are independent (over $\emptyset$).*

Let the roots of $F(X)$ be $r_0, \dots, r_{n-1}$, that is, $r_j = t_0 + t_1\alpha_j + \cdots + t_{n-1}\alpha_j^{n-1}$. Then

$$\begin{pmatrix} 1 & \alpha_0 & \cdots & \alpha_0^{n-1} \\ 1 & \alpha_1 & \cdots & \alpha_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \alpha_{n-1} & \cdots & \alpha_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \\ \vdots \\ t_{n-1} \end{pmatrix} = \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \end{pmatrix}$$

Since the matrix M on the left is invertible, we get the t_j as linear combinations of the roots r_j with coefficients in $P(\bar{\alpha})$ where P is the prime field of K. Then the t_j are algebraic over $K_0(\bar{g})$ in the sense of field theory, since each r_i is algebraic over $P(\bar{g})$ and each α_i is algebraic over K_0 in the sense of field theory. It follows that $\mathrm{acl}_K(\{\bar{t}\}) = \mathrm{acl}_K(\{\bar{g}\})$, and we conclude that the g_i are independent (over $\emptyset$).

By Claim A, $F(X) = 0$ has a root in K. Therefore some $r_j = t_0 + t_1\alpha_j + \cdots + t_{n-1}\alpha_j^{n-1}$ is in K, hence α_j has degree at most $n - 1$ over K.

On the other hand, α must have degree n over K. For K is separable over K_0 since K_0 is perfect, and K_0 is algebraically closed in K in the sense of field theory. Then K is a regular extension of K_0, hence α has degree n over K. Thus we get a contradiction.

□

Uncountable minimal fields have infinite bases over $\emptyset$, hence we get:

Corollary 3 *An uncoutable minimal field is algebraically closed.*

Remark. Note that the proof of lemma2 also gives a direct proof that strongly minimal fields are algebraically closed fields.

2 Minimal elementary extensions

In this section, assuming some common property of minimal fields we show that any minimal field K has a proper minimal elementary extension. Then, under this assumption, we get a minimal field $L \succ K$ which has a dimension $> n$ (over $\emptyset$) for any $n \in \mathbb{N}$.

We assume the following property.

(†) For any minimalfield, there is no $L(K)$-formula $\psi(x, y)$ such that $\exists^{\geq n} x\psi(x, K)$ and $\exists^{\geq n} x \neg\psi(x, K)$ are cofinite for all $n \in \mathbb{N}$.

Note that algebraically closed fields have the property (†) since they admit quantifier elimination.

Lemma 4 *Assuming (†), any minimal field K has a proper minimal elementary extension.*

Proof. Let $T = \mathrm{Th}(K)_{a \in K}$, c a new constant symbol other than $L(K)$, and $T_0 = T \cup \{x \neq a\}_{a \in K}$. Then T_0 is a consistent $L(K \cup \{c\})$-theory. Let $L \models T_0$, and $L_0 = \mathrm{acl}_L(K \cup \{c\})$. We show that L_0 is a desired structure.

First we show that $L_0 \prec L$, which implies that $K \prec L_0$. Consider a sentence $\exists x\varphi(x, \bar{b})$ with $\bar{b} \in L_0$ where $\varphi(x, \bar{y})$ is an L-formula, and suppose that $L \models \exists x\varphi(x, \bar{b})$. We show that there exists $a \in L_0$ such that $L \models \varphi(a, \bar{b})$. Since each b_i is algebraic over $K \cup \{c\}$, there is an algebraic $L(K \cup \{c\})$-formula $\psi(\bar{x})$ such that $L \models \psi(\bar{b}) \wedge \exists^{=n_0} \bar{x}\psi(\bar{x})$ for some $n_0 \in \mathbb{N}$. We choose ψ to make n_0 as small as possible. Clearly $L \models \exists x\varphi(x, \bar{b}) \leftrightarrow \exists x \exists \bar{y}(\varphi(x, \bar{y}) \wedge \psi(\bar{y}))$, and hence $L \models \exists x \exists \bar{y}(\varphi(x, \bar{y}) \wedge \psi(\bar{y}))$ Since $\exists \bar{y}(\varphi(x, \bar{y}) \wedge \psi(\bar{y}))$ is an $L(K \cup \{c\})$-formula, we write it as $\varphi_0(x, c)$ where $\varphi_0(x, y)$ is an $L(K)$-formula.

We show that there exists $a \in L_0$ such that $L \models \varphi_0(a, c)$. If $\varphi_0(x, c)$ dose not involve c, then we are done since $K \prec L$. Suppose that $\varphi_0(x, c)$ actually involves c.

If $\varphi_0(x, c)$ is algebraic in L, then we are done. Suppose not. Then $L \models \exists^{\geq n} x \varphi_0(x, c)$ for all n. It follows that $\exists^{\geq n} x \varphi_0(x, K)$ is cofinite for all n, since c does not satisfy algebraic formulas of K. By (†), $\exists^{\geq n} x \neg\varphi_0(x, K)$ is finite for some n, which implies that $L \models \exists^{<n} x \neg\varphi_0(x, c)$ for some n. Therefore $\neg\varphi_0(L, c)$ is finite. It follows that there exists $a \in L_0$ such that $L \models \varphi_0(a, c)$.

Now take $\bar{d} \in L_0$ such that $L \models \varphi(a, \bar{d}) \wedge \psi(\bar{d})$. Since $a \in L_0$, there is an algebraic $L(K \cup \{c\})$-formula $\eta(x)$ such that $L \models \eta(a) \wedge \exists^{=n_1} x \eta(x)$ for some $n_1 \in \mathbb{N}$. Again we choose η to make n_1 as small as possible. Consider $L(K \cup \{c\})$-formula $\theta(\bar{y}) = \exists x(\varphi(x, \bar{y}) \wedge \psi(\bar{y}) \wedge \eta(x))$. Then $L \models \theta(\bar{d})$, which implies that $L \models \theta(\bar{b})$. It follows that there exists $a' \in L_0$ such that $L \models \varphi(a', \bar{b})$.

The above argument also shows that L_0 is minimal. This completes the proof.

□

Theorem 5 *Assuming (†), any minimal field K is algebraically closed.*

Proof. Suppose that K is not algebraicaly closed. Again there is $\alpha \in \overline{K_0} \setminus K_0$, where $\overline{K_0}$ is the algebraic closure of $K_0 = \mathrm{acl}_K(\{\emptyset\})$. Let α has degree n over K_0 $(n > 1)$. By lemma4, there is a minimal elementary extension L of K which has dimension $> n$ (over $\emptyset$). Noting that $\mathrm{acl}_K(\{\emptyset\}) = \mathrm{acl}_L(\{\emptyset\})$, we get a desired contradiction as before.

□

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