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Strong random Clarkson inequality and its extension

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Strong random Clarkson 不等式の成立する Banach 空間を strong type p の空間として特徴づけ, その重み定数を考察する。また Lebesgue-Bochner 空間 $L_r(X)$, Banach 空間 X_i の l_r-sum である $l_r(X_i)$ への遺伝性を示す. さらにはその重み定数が strong type 不等式の最良定数と一致するような拡張型の strong random Clarkson 不等式を導入する.

1. p-uniform smoothness and q-uniform convexity

Definition 1. A Banach space X is called p-uniformly smooth $(1 < p \leq 2)$ if there exists $K > 0$ such that $\rho_X(\tau) \leq K\tau^p$ for all $\tau \geq 0$, where $\rho_X(\tau)$ is the modulus of smootheness of X, i.e.

$$\rho_X(\tau) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1,\ \|y\| = \tau\right\}.$$

Definition 2. A Banach space X is called q-uniformly convex $(2 \leq q < \infty)$ if there exists $C > 0$ such that $\delta_X(\epsilon) \geq C\epsilon^q$ for all $0 < \epsilon \leq 2$, where $\delta_X(\epsilon)$ is the modulus of convexity of X, i.e.

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1,\ \|x-y\| = \epsilon\right\}.$$

Proposition 1. A Banach space X is p-uniformly smooth if and only if for any $1 \leq s < \infty$,

$$\left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \leq (\|x\|^p + \|Ky\|^p)^{1/p} \qquad (1)$$

holds in X with some $K > 0$.

Proposition 2. A Banach space X is q-uniformly convex if and only if for any $1 < t \leq \infty$,

$$\left(\frac{\|x+y\|^t + \|x-y\|^t}{2}\right)^{1/t} \geq (\|x\|^q + \|Cy\|^q)^{1/q} \tag{2}$$

holds in X with some $C > 0$.

2. Strong type p and strong cotype q

Definition 3. A Banch space X is called *of strong type* p $(1 \leq p \leq 2)$, provided there exists a constant $K > 0$ and $1 \leq s < \infty$,

$$\left(\mathbf{E}\left\|\sum_{j=1}^{n} \epsilon_j x_j\right\|^s\right)^{1/s} \leq \left(\|x_1\|^p + \sum_{j=2}^{n} \|Kx_j\|^p\right)^{1/p} \tag{3}$$

holds for all finite systems $x_1, \cdots, x_n \in X$, where $\{\epsilon_j\}$ is a Rademacher sequence and $\mathbf{E}$ denotes the mathematical expectation.

Definition 4. A Banach space X is called *of strong cotype* q $(2 \leq q \leq \infty)$, provided there exists a constant $C > 0$ and $1 < t \leq \infty$,

$$\left(\mathbf{E}\left\|\sum_{j=1}^{n} \epsilon_j x_j\right\|^t\right)^{1/t} \geq \left(\|x_1\|^q + \sum_{j=2}^{n} \|Cx_j\|^q\right)^{1/q} \tag{4}$$

holds for all finite systems $x_1, \cdots, x_n \in X$.

Theorem 1. Let X be a Banach space and let $1 \leq u \leq p \leq 2$. If

$$\left(\mathbf{E}\left\|\sum_{j=1}^{n} \epsilon_j x_j\right\|^{p'}\right)^{1/p'} \leq \left(\|x_1\|^p + \sum_{j=2}^{n} \|Kx_j\|^p\right)^{1/p}$$

holds in X with a constant K. Then

$$\left(\mathbf{E}\left\|\sum_{j=1}^{n} \epsilon_j x_j\right\|^{u'}\right)^{1/u'} \leq \left(\|x_1\|^u + \sum_{j=2}^{n} \|Kx_j\|^u\right)^{1/u}$$

holds in X with the same K, where $1/p + 1/p' = 1/u + 1/u' = 1$.

(3),(4) で $n = 2$ の場合はそれぞれ (1),(2) となる. また (3) の右辺で x_1 を Kx_1, (4) の右辺で x_1 を Cx_1 と置き換えれば, それぞれ type p, cotype q を定義する type 不等式, cotype 不等式となる. さらに $K \geq 1$, $0 < C \leq 1$ で, (3),(4) の右辺は K, C について非減

少である．そこで (3) をみたす K の最小値を strong type $p(s)$ 定数といい $ST_{p(s)}(X)$ で表し，(4) をみたす C の最大値を strong cotype $q(t)$ 定数といい $SC_{q(t)}(X)$ で表すことにする．Theorem 1 からは X : strong type $p \Rightarrow X$: strong type u , $ST_{u(u')}(X) \leq ST_{p(p')}(X)$ がわかる．

3. Strong random Clarkson inequality

Theorem 2. Let X be a Banach space and let $1 < p \leq 2$. Then the following are equivalent.

(i) X is p-uniformly smooth.

(ii) X is of strong type p.

(iii) (**Strong random Clarkson inequality**) For any $1 \leq r, s \leq \infty$ and for all $n \in \mathbb{N}$

$$\mathbf{E}\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n}\epsilon_{ij}x_j\right\|^s\right)^{1/s} \leq n^{c(r,s;p)}\left(\|x_1\|^r + \sum_{j=2}^{n}\|Kx_j\|^r\right)^{1/r} \tag{5}$$

holds in X with some constant K independent of n, r and s, where $c(r,s;p) = \max\{1/r' + 1/s - 1/p', 1/r', 1/s\}$, $1/p + 1/p' = 1/r + 1/r' = 1$ and (ϵ_{ij}) is an $n \times n$ matrix whose coefficients are independent identically distributed random variables taking the values ± 1 with equal probability $1/2$. Further, let $K_p(X)$ denote the smallest value of K in (5). Then

$$ST_{p(1)}(X) \leq K_p(X) \leq ST_{p(p')}(X). \tag{6}$$

(5) の右辺で x_1 を Kx_1 に置き換えた不等式は Tonge [3] が L_p 空間でその成立を示した random Clarkson 不等式である．したがって (5) は random Clarkson 不等式より強いタイプの不等式であると考えられる．random Clarkson 不等式は type p を特徴づける不等式であることが Kato, Persson, Takahashi [2] によって示されたが，この定理は彼らの結果の strong version である．

4. Heredity to $L_r(X)$, $l_r(X_i)$

Theorem 3. Let $L_r(X) = L_r(\Omega, \mathfrak{B}, \mu; X)$ be a Lebesgue-Bochner space on an arbitrary measure space $(\Omega, \mathfrak{B}, \mu)$. Let $1 < p \leq 2$ and $p \leq r \leq p'$ $(1/p + 1/p' = 1)$. Then X is of strong type p if and only if $L_r(X)$ is of strong type p; in this case

$$ST_{p(p')}(X) = ST_{p(p')}(L_r(X)).$$

Theorem 4. Let $l_r(X_i)$, $1 \le r < \infty$, be the l_r-sum of Banach spaces $(X_i, \|\cdot\|_i)$ with the norm $\|x\|_{(r)} = (\sum_i \|x_i\|_i^r)^{1/r}$ for $x = (x_i) \in l_r(X_i)$. Let $1 < p \le 2$, $p \le r \le p'$. Then $l_r(X_i)$ is of strong type p if and only if all X_i are of strong type p and $\sup_i ST_{p(p')}(X_i) < \infty$. In this case

$$ST_{p(p')}(l_r(X_i)) = \sup_i ST_{p(p')}(X_i).$$

5. Extended strong random Clarkson inequality

Thoerem 5. Let X be a Banach space. Let $1 < p \le 2$, $1/p + 1/p' = 1$ and $1 \le K < \infty$. Then the following are equivalent.

(i)X is p-uniformly smooth:

$$\left(\frac{\|x+y\|^{p'} + \|x-y\|^{p'}}{2}\right)^{1/p'} \le (\|x\|^p + \|Ky\|^p)^{1/p}. \tag{7}$$

(ii) X is of strong type p: For any $n \in \mathbb{N}$

$$\left(\mathbf{E}\left\|\sum_{j=1}^n \epsilon_j x_j\right\|^{p'}\right)^{1/p'} \le \left(\|x_1\|^p + \sum_{j=2}^n \|Kx_j\|^p\right)^{1/p}. \tag{8}$$

(iii) For any $1 \le r, s, t \le \infty$ and for any $n \in \mathbb{N}$

$$\left\{\mathbf{E}\left(\sum_{i=1}^n \left\|\sum_{j=1}^n \epsilon_{ij} x_j\right\|^s\right)^{t/s}\right\}^{1/t} \le n^{c(r,s,t;p)} \left(\|x_1\|^r + \sum_{j=2}^n \|Kx_j\|^r\right)^{1/r}, \tag{9}$$

where $c(r,s,t;p) = \max\{1/r' + 1/s - 1/p',\ 1/r',\ 1/s,\ 1/r' + 1/s - 1/t\}$, $1/r + 1/r' = 1$.

(9) は $t = 1$ の場合 $c(r,s,1;p) = c(r,s;p)$ より strong random Clarkson 不等式 (5) となり, (5) を拡張した不等式である. さらに p-uniformly smoothness 不等式, strong type p 不等式もそれぞれ重み K をもち, Thoerem 2 においては strong random Clarkson 不等式も含めた 3 つの不等式でそれぞれ独立した重み K をとる. しかし拡張型の strong ramdom Clarkson 不等式を使ったこの定理ではすべて同じ重みの値をとることができる. こうして Theorem 5 の不等式の重み定数 K の最良値 $(\min K)$ は $ST_{p(p')}(X)$ となる.

参考文献

[1] J. A. Clarkson, Uniformly convex spaces, Trans. Amer. Math. Soc. **40** (1936), 396-414.

[2] M. Kato, L. E. Persson and Y. Takahashi, Clarkson type inequalities and their relations to the concepts of type and cotype, Collect. Math. **51** (2000), 327-346.

[3] A. Tonge, Random Clarkson inequalities and L_p-version of Grothendieck's inequality, Math. Nachr. **131** (1987), 335-343.

[4] Y. Takahashi, K. Hashimoto and M. Kato, On sharp uniform convexity, smoothness, and strong type, cotype inequalities, J. Nonlinear. Convex Anal. **3**, No.2, (2002), 267-281.

[5] Y. Takahashi and M. Kato, Clarkson and random Clarkson inequalities for $L_r(X)$, Math. Nachr. **188** (1997), 341-348.

[6] Y. Yamada, M. Kato and Y. Takahashi, On strong ramdom Clarkson inequality and strong Rademacher type, Bull Kyushu Inst. Tech. Pure Appl. No.51, (2004), 1-15.

[7] J. Wenzel, Strong martingale type and uniform smoothness, preprint.