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On the Singularities of Solutions of Nonlinear Partial Differential Equations in the Complex Domain

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Let us consider the following nonlinear partial differential equation

$$\frac{\partial u}{\partial t}=F\Big(t,x,u,\frac{\partial u}{\partial x}\Big) \tag{E}$$

in the complex domain $\mathbb{C}_t\times\mathbb{C}_x^n$. The structure of holomorphic solutions of (E) can be understood completely by the Cauchy-Kowalevsky theorem. But the structure of singualr solutions (that is, solutions with some singularities) of (E) has not yet been studied well. In this paper the author will consider the following problem:

Problem. Does (E) have a solution which possesses singularities only on the hypersurface $\{t=0\}$?

Result. Under suitable conditions we can find such a negative real number σ that the following (1) and (2) are satisfied: (1) (E) has no solutions with singularities only on $\{t=0\}$ of the growth order $o(|t|^{\sigma})$ (as $t\longrightarrow 0$), but (2) (E) has a solution with singularities only on $\{t=0\}$ of the growth order $O(|t|^{\sigma})$ (as $t\longrightarrow 0$).

The proof of (1) will be done by examining the possibility of analytic continuation of solutions, and (2) by actually constructing solutions that possess singularities only on $\{t=0\}$ with growth order $O(|t|^{\sigma})$ (as $t\longrightarrow 0$).

In the case of equations of the general order, many parts of this paper are valid also for

$$\Big(\frac{\partial}{\partial t}\Big)^m u=F\Big(t,x,\Big\{\Big(\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^{\alpha}u\Big\}_{j+|\alpha|\le m,j<m}\Big);$$

but the study of this case has not been completed yet.

§1. Equation and problem

Let $(t,x)=(t,x_1,\ldots,x_n)\in\mathbb{C}\times\mathbb{C}^n$, $y\in\mathbb{C}$, $z=(z_1,\ldots,z_n)\in\mathbb{C}^n$, denote $\partial/\partial x=(\partial/\partial x_1,\ldots,\partial/\partial x_n)$, and let $F(t,x,y,z)$ be a holomorphic function defined in a neighborhood of the origin of $\mathbb{C}_t\times\mathbb{C}_x^n\times\mathbb{C}_y\times\mathbb{C}_z^n$.

In this paper we will consider the following nonlinear first order partial differential equation

$$\frac{\partial u}{\partial t}=F\Big(t,x,u,\frac{\partial u}{\partial x}\Big) \tag{1.1}$$

where $u=u(t,x)$ is the unknown function.

It is well-known by Cauchy-Kowalevsky theorem that for any holomorphic function $\varphi(x)$ in a neighborhood of $x=0$ the equation (1.1) has a unique holomorphic solution $u(t,x)$ in a neighborhood of the origin $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x^n$ satisfying $u(0,x)=\varphi(x)$. Thus, the holomorphic solutions of (1.1) in a neighborhood of the origin $(0,0)$ are completely characterized by the initial data $\varphi(x)$.

But if we include into consideration the singualr solutions (that is, the solutions with some singularities) the structure of solutions of (1.1) will become much more interesting.

In this paper we will study the following problem:

Problem 1.1. Does (1.1) admit solutions which possess singularities only on the hypersurface $\{t=0\}$?

One method of arguing the non-existence of such solutions is by means of analytic continuation. Let Ω be a neighborhood of the origin $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x^n$, and set $\Omega_+ = \{(t,x) \in \Omega;\ \mathrm{Re}\, t > 0\}$.

If the equation (1.1) is linear, then Zerner's Theorem ([15], 1971) states that any solution which is holomorphic in Ω_+ can be analytically extended to some neighborhood of the origin $(0,0)$. In other words, there does not exist a solution with singularities only on $\{t=0\}$.

If the equation (1.1) is nonlinear, we have the following nonlinear analogue of Zerner's theorem due to Tsuno (1975).

Theorem 1.2 ([14]). *If a holomorphic solution $u(t,x)$ of (1.1) in Ω_+ satisfies $u(t,x) = O(1)$ (as $t \longrightarrow 0$) uniformly in x in some neighborhood of $x=0$, then $u(t,x)$ can be analytically continued up to a neighborhood of the origin.*

The assumption that $u(t,x)$ be bounded in some neighborhood of the origin seemed too strong to other researchers at that time. Some might have believed that Zerner's result can be extended to the nonlinear case without any additional assumption. However, this is not possible if the equation is nonlinear, as can be seen in the following example:

Example 1.3. Let $(t,x) \in \mathbb{C}^2$. The equation

$$\frac{\partial u}{\partial t} = u\left(\frac{\partial u}{\partial x}\right)^m \quad \text{with } m \in \mathbb{N}^* \,(=\{1,2,\ldots\}) \tag{1.2}$$

has a family of solutions $u(t,x) = (-1/m)^{1/m}(x+c)/t^{1/m}$ with an arbitrary constant $c \in \mathbb{C}$. Clearly, this is holomorphic in Ω_+ but has singularities on $\{t=0\}$.

Thus, in the case of equation (1.2) we see the following: (1) singularities on $\{t=0\}$ of order $u(t,x)=O(1)$ (as $t \longrightarrow 0$) do not appear in the solutions of (1.2), but (2) there really appear singularities on $\{t=0\}$ of order $u(t,x) = O(|t|^{-1/m})$ (as $t \longrightarrow 0$) in the solutions of (1.2).

Hence, for nonlinear equations, it seems better to reformulate our problem into the following form:

Problem 1.4. Let σ be a real number. Does (1.1) admit solutions which possess singularities on $\{t=0\}$ with growth order $O(|t|^{\sigma})$ (as $t \longrightarrow 0$)?

If σ is a non-negative real number, by Tsuno's result we conclude that such singularities do not appear in the solutions of (1.1). Therefore we may assume from now that σ is a negative real number. Then, in general the solution may tend to ∞ (as $t \longrightarrow 0$) and so we need to suppose:

(A) $F(t,x,y,z)$ is a holomorphic function on $\Omega \times \mathbb{C}_y \times \mathbb{C}_z^n$.

§2. Non-existence of singularities

Recently Kobayashi [7] gave a precise formulation on the non-existence part of the problem 1.4. In this section we will follow his argument and give its improved form.

Suppose the condition (A). We may expand the function $F(t,x,y,z)$ into the Taylor series with respect to (y,z):

$$F(t,x,y,z) = \sum_{(j,\alpha)\in\mathbb{N}\times\mathbb{N}^n} a_{j,\alpha}(t,x)\, y^j z^\alpha$$

where $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, $a_{j,\alpha}(t,x)$ are holomorphic functions on Ω, and $z^\alpha = {z_1}^{\alpha_1}\cdots {z_n}^{\alpha_n}$.

Let $\Delta = \{(j,\alpha) \in \mathbb{N}\times\mathbb{N}^n;\, a_{j,\alpha}(t,x) \not\equiv 0\}$ and $\Delta_2 = \{(j,\alpha)\in\Delta\,;\, j+|\alpha|\geq 2\}$ (where $|\alpha| = \alpha_1+\cdots+\alpha_n$). We remark that the equation (1.1) is linear if and only if $\Delta_2 = \emptyset$; it is nonlinear otherwise. Since we already have Zerner's result for the linear case, we will assume henceforth that (1.1) is nonlinear, that is, Δ_2 is non-empty. In the following, we will write the coefficients as

$$a_{j,\alpha}(t,x) = t^{k_{j,\alpha}}\, b_{j,\alpha}(t,x) \quad \text{for } (j,\alpha)\in\Delta$$

where $k_{j,\alpha}$ is a non-negative integer and $b_{j,\alpha}(0,x)\not\equiv 0$. Using the above, the equation (1.1) may now be written as

$$\frac{\partial u}{\partial t} = \sum_{(j,\alpha)\in\Delta} t^{k_{j,\alpha}} b_{j,\alpha}(t,x)\, u^j \left(\frac{\partial u}{\partial x}\right)^\alpha \tag{2.1}$$

where $(\partial u/\partial x)^\alpha = (\partial u/\partial x_1)^{\alpha_1}\cdots(\partial u/\partial x_n)^{\alpha_n}$.

Set

$$\sigma_{\mathrm{K}} = \sup_{(j,\alpha)\in\Delta_2} \frac{-k_{j,\alpha}-1}{j+|\alpha|-1}. \tag{2.2}$$

Note that σ_{K} is a non-positive real number and that it is calculated only by looking at the form of the equation. For a neighborhood ω of $x = 0\in\mathbb{C}^n_x$ and a function $f(t,x)$ we define the norm $\|f(t)\|_\omega = \sup_{x\in\omega}|f(t,x)|$. The following result is originally due to Kobayashi [7] and improved by Lope-Tahara [8]:

Theorem 2.1 ([7], [8]). *Suppose the conditions* (A) *and* $\Delta_2\neq\emptyset$. *If a holomorphic solution* $u(t,x)$ *of* (1.1) *in* Ω_+ *satisfies* $\|u(t)\|_\omega = o(|t|^{\sigma_{\mathrm{K}}})$ *(as* $t\longrightarrow 0$*), then* $u(t,x)$ *can be extended analytically up to a neighborhood of the origin.*

Hence we can get the following result on the non-existence of the singularities on $\{t=0\}$.

Corollary 2.2. *Suppose the conditions* (A) *and* $\Delta_2\neq$ *emptyset. Let* σ_{K} *be the real number given in* (2.2). *Then, there appear no singularities on* $\{t=0\}$ *with growth order* $o(|t|^{\sigma_{\mathrm{K}}})$ *(as* $t\longrightarrow 0$*) in the solutions of* (1.1).

In the equation (1.2) the number σ_{K} may be verified to be equal to $-1/m$. Hence, by the above result we see that the singularities of order $o(|t|^{-1/m})$ do not appear in the solutions of (1.2). Note further that the singularities of the solution $u(t,x) = (-1/m)^{1/m}(x+c)/t^{1/m}$ has growth order $O(|t|^{-1/m})$ (as $t\longrightarrow 0$). Thus in the case (1.2) the number $\sigma_{\mathrm{K}} = -1/m$ is just the critical value of the order of singularities.

Is this true in the general case? This is our next question.

Problem 2.3. Suppose $\Delta_2\neq\emptyset$. Let σ_{K} be the one in (2.2). Then, does (1.1) admit solutions which possess singularities on $\{t=0\}$ with growth order $O(|t|^{\sigma_{\mathrm{K}}})$ (as $t\longrightarrow 0$)?

Set

$$\mathcal{M} = \left\{ (j,\alpha) \in \Delta_2 \,;\, \frac{-k_{j,\alpha} - 1}{j + |\alpha| - 1} = \sigma_{\mathrm{K}} \right\}. \tag{2.3}$$

If $\mathcal{M} = \emptyset$, we have the following result on the problem 2.3.

Theorem 2.4 ([8]). *Suppose the conditions* (A) *and* $\Delta_2 \neq \emptyset$. *If* $\mathcal{M} = \emptyset$ *and if a holomorphic solution* $u(t,x)$ *of* (1.1) *in* Ω_+ *satisfies* $\|u(t)\|_\omega = O(|t|^{\sigma_{\mathrm{K}}})$ *(as* $t \longrightarrow 0$*), then* $u(t,x)$ *can be extended analytically up to a neighborhood of the origin.*

This implies that in the case $\mathcal{M} = \emptyset$ there appear no singularities on $\{t = 0\}$ with growth order $O(|t|^{\sigma_{\mathrm{K}}})$ (as $t \longrightarrow 0$) in the solutions of (1.1).

The following equation gives an example with $\mathcal{M} = \emptyset$: let $(t,x) \in \mathbb{C}^2$ and consider the first-order nonlinear equation $\partial u/\partial t = e^u(\partial u/\partial x)$. In this case, it is easily checked that $\sigma_{\mathrm{K}} = 0$ and $\mathcal{M} = \emptyset$. Therefore by theorem 2.4 we see that this equation has no singular solutions with growth order $O(1)$ (as $t \longrightarrow 0$), which is just the same result as in Tsuno's theorem.

§3. Existence of singularities

In this section we will show that the answer to the problem 2.3 is affirmative if $\Delta_2 \neq \emptyset$ and $\mathcal{M} \neq \emptyset$ hold; the proof was given in Tahara [9] and [10]. Some parts were due to Ishii [6] and Kobayashi [7].

Suppose $\Delta_2 \neq \emptyset$, $\mathcal{M} \neq \emptyset$ and set

$$P(x,y,z) = \sum_{(j,\alpha)\in\mathcal{M}} b_{j,\alpha}(0,x)\, y^j z^\alpha. \tag{3.1}$$

It is easy to see that $P(x,y,z) \not\equiv 0$ and that $P(x,y,z)$ is a holomorphic function on $\{x \in \mathbb{C}\,;(0,x) \in \Omega\} \times \mathbb{C}_y \times \mathbb{C}_z^n$; moreover in (3.1) we have $j + |\alpha| \geq 2$. Since $\mathcal{M} \neq \emptyset$, we have $\sigma = (-k_{j,\alpha} - 1)/(j + |\alpha| - 1)$ for any $(j,\alpha) \in \mathcal{M}$: this implies that σ is a negative rational number.

We write

$$\frac{\partial P}{\partial x} = \left(\frac{\partial P}{\partial x_1}, \ldots, \frac{\partial P}{\partial x_n}\right) \quad \text{and}$$
$$\frac{\partial P}{\partial z} = \left(\frac{\partial P}{\partial z_1}, \ldots, \frac{\partial P}{\partial z_n}\right).$$

In this section, we will present four sufficient conditions for the existence of singularities of the growth order $|t|^{\sigma_{\mathrm{K}}}$ only on $\{t = 0\}$. The four conditions correspond to the following four cases:

$$\text{Case (0)} : \frac{\partial P}{\partial x}(x,y,0) \equiv (0,\ldots,0) \;\text{ and }\; \frac{\partial P}{\partial z}(x,y,0) \equiv (0,\ldots,0)\,;$$
$$\text{Case (1)} : \frac{\partial P}{\partial z}(x,y,z) \equiv (0,\ldots,0)\,;$$
$$\text{Case (2)} : \frac{\partial P}{\partial z}(0,y,z) \not\equiv (0,\ldots,0)\,;$$
$$\text{Case (3)} : \frac{\partial P}{\partial z}(0,y,z) \equiv (0,\ldots,0) \;\text{ and }\; \frac{\partial P}{\partial z}(x,y,z) \not\equiv (0,\ldots,0)\,.$$

In the classification, the three cases (1),(2) and (3) are enough to cover all the cases. But, to compute examples, the case (0) is also very convenient: this is the reason why we add the extra case (0).

Theorem 3.1 (Case (0)). *Suppose $\Delta_2 \neq \emptyset$, $\mathcal{M} \neq \emptyset$ and the conditions in* Case (0). *Set $\Sigma^* = \{y \in \mathbb{C} \setminus \{0\}\,;\, P(0,y,0) = \sigma_K y\}$. Then, if $\Sigma^* \neq \emptyset$ the equatoin* (1.1) *has a solution which possesses singularities only on $\{t=0\}$ with the growth order $|t|^{\sigma_K}$.*

Example (0). Let $(t,x) \in \mathbb{C}^2$ and let us consider

$$\frac{\partial u}{\partial t} = u^2 + b(x)\Big(\frac{\partial u}{\partial x}\Big)^2 + c(t,x),$$

where $b(x)$ and $c(t,x)$ are holomorphic functions. Then $\sigma_K = -1$, $P = y^2 + b(x)z^2$ and so the conditions in Case (0) are satisfied. Since $P(0,y,0) = y^2$ we have $\Sigma^* = \{y \in \mathbb{C} \setminus \{0\}\,;\, P(0,y,0) = -y\} = \{-1\} \neq \emptyset$. Thus we can apply Theorem 3.1 to this case and obtain the following: this equation has a solution with singularities only on $\{t=0\}$ of order $|t|^{-1}$.

Theorem 3.2 (Case (1)). *Suppose $\Delta_2 \neq \emptyset$, $\mathcal{M} \neq \emptyset$ and the condition in* Case (1). *Set $\Sigma^* = \{y \in \mathbb{C} \setminus \{0\}\,;\, P(0,y,0) = \sigma_K y\}$. Then, if*

$$\Sigma^* \neq \emptyset \quad \text{and} \quad \frac{\partial P}{\partial y}(0,y,0) \not\equiv \sigma_K \quad \text{on } \Sigma^* \tag{3.2}$$

the equatoin (1.1) *has a solution which possesses singularities only on $\{t=0\}$ with the growth order $|t|^{\sigma_K}$.*

Example (1). Let $(t,x) \in \mathbb{C}^2$ and let us consider

$$\frac{\partial u}{\partial t} = a(x)\,u^2 + t\,\Big(\frac{\partial u}{\partial x}\Big)^2 + c(t,x),$$

where $a(x)$ and $c(t,x)$ are holomorphic functions. Then $\sigma_K = -1$, $P = a(x)y^2$ and so the condition in Case (1) is satisfied. Since $P(0,y,0) = a(0)y^2$ we have $\Sigma^* = \{y \in \mathbb{C} \setminus \{0\}\,;\, a(0)y^2 = -y\}$; if $a(0) \neq 0$ we have $\Sigma^* = \{-1/a(0)\} \neq \emptyset$ and $(\partial P/\partial y)(0,-1/a(0),0) = -2 \neq \sigma_K$. Thus, if $a(0) \neq 0$ we can apply Theorem 3.2 to this case and obtain the following: this equation has a solution with singularities only on $\{t=0\}$ of order $|t|^{-1}$.

Theorem 3.3 (Case (2)). *Suppose $\Delta_2 \neq \emptyset$, $\mathcal{M} \neq \emptyset$ and the condition in* Case (2). *Set $\Sigma = \{(y,z) \in \mathbb{C} \times \mathbb{C}^n\,;\, P(0,y,z) = \sigma_K y\}$. Then, if*

$$\frac{\partial P}{\partial z}(0,y,z) \not\equiv (0,\dots,0) \quad \text{on } \Sigma \tag{3.3}$$

the equatoin (1.1) *has a solution which possesses singularities only on $\{t=0\}$ with the growth order $|t|^{\sigma_K}$.*

Example (2). i) Let $(t,x) \in \mathbb{C}^2$ and let us consider

$$\frac{\partial u}{\partial t} = u\Big(\frac{\partial u}{\partial x}\Big)^m, \quad m \in \mathbb{N}^*.$$

Then $\sigma_K = -1/m$, $P = yz^m$, $\partial P/\partial z = myz^{m-1}$, and $\Sigma = \{(y,z) \in \mathbb{C} \times \mathbb{C}\,;\, yz^m = (-1/m)y\}$. If we take $(1,(-1/m)^{1/m}) \in \Sigma$ we have $(\partial P/\partial z)(0,1,(-1/m)^{1/m}) = -(-m)^{1/m} \neq 0$. Thus, we can apply Theorem 3.3 to this case and obtain the following: this equation has a solution with singularities only on $\{t=0\}$ of order $|t|^{-1/m}$. Compare this with Example 1.3.

ii) Let us consider

$$\frac{\partial u}{\partial t} = a(x)\,u^2 + b(x)\left(\frac{\partial u}{\partial x}\right)^2 + c(t,x),$$

where $a(x)$, $b(x)$ and $c(t,x)$ are holomorphic functions. Then $\sigma_K = -1$, $P = a(x)y^2 + b(x)z^2$ and so if $b(0) \neq 0$ the condition in Case (2) is satisfied. We have $\Sigma = \{(y,z) \in \mathbb{C}\times\mathbb{C}\,;\, a(0)y^2 + b(0)z^2 = -y\}$. If we take α so that $\alpha + a(0)\alpha^2 \neq 0$ and define β by $\beta^2 = -(\alpha + a(0)\alpha^2)/b(0)$, then we have $\beta \neq 0$, $(\alpha,\beta) \in \Sigma$ and $(\partial P/\partial z)(0,\alpha,\beta) = 2b(0)\beta$. Thus, if $b(0) \neq 0$ we can apply Theorem 3.3 to this case and obtain the following: this equation has a solution with singularities only on $\{t = 0\}$ of order $|t|^{-1}$.

Lastly, let us consider the case (3). We will give a sufficient condition only in the case $n = 1$; in the general case $n \geq 2$ we have no good results. Suppose $n = 1$ and set

$$\Sigma = \left\{ y \in \mathbb{C}\,;\, \frac{\partial P}{\partial x}(0,0,y) = \sigma_K\, y \right\},$$
$$\frac{\partial^2 P}{\partial z \partial x}(0,0,\Sigma) = \left\{ \frac{\partial^2 P}{\partial z \partial x}(0,0,y)\,;\, y \in \Sigma \right\}.$$

Theorem 3.4 (Case (3)). *Suppose $n = 1$, $\Delta_2 \neq \emptyset$, $\mathcal{M} \neq \emptyset$ and the conditions in* Case (3). *If*

$$\frac{\partial^2 P}{\partial z \partial x}(0,0,\Sigma) \not\subset [0,\infty) \cup \left\{ \frac{\sigma_K}{2}, \frac{\sigma_K}{3}, \frac{\sigma_K}{4}, \ldots \right\} \tag{3.4}$$

in $\mathbb{C}$, the equatoin (1.1) *has a solution which possesses singularities only on $\{t = 0\}$ with the growth order $|t|^{\sigma_K}$.*

Example (3). Let $(t,x) \in \mathbb{C}^2$ and let us consider

$$\frac{\partial u}{\partial t} = a(x)\,u^2 + x\left(\frac{\partial u}{\partial x}\right)^2 + c(t,x),$$

where $a(x)$ and $c(t,x)$ are holomorphic functions. Then $\sigma_K = -1$, $P = a(x)y^2 + xz^2$ and so the conditions in Case (3) are satisfied. We have $\Sigma = \{\alpha \in \mathbb{C}\,;\, \alpha^2 = -\alpha\} = \{0,-1\}$ and $(\partial^2 P/\partial z\partial x)(0,0,\Sigma) = \{0,-2\}$. Since $-2 \notin [0,\infty) \cup \{-1/2, -1/3, \ldots\}$ we have the condition (3.4). Thus, we can apply Theorem 3.4 to this case and obtain the following: this equation has a solution with singularities only on $\{t = 0\}$ of order $|t|^{-1}$.

§4. Way of Constructing a singular solution

The proofs of Theorems 3.1 - 3.4 are given in Tahara [9] and [10]. In this section, we will give only a sketch of the construction of a singular solution with the growth order $|t|^{\sigma_K}$.

Suppose $\Delta_2 \neq \emptyset$, $\mathcal{M} \neq \emptyset$ and let $P(x,y,z)$ be the one in (3.1). Let us construct a solution of (1.1) of the form

$$u(t,x) = t^{\sigma_K}(\varphi(x) + w(t,x)) \tag{4.1}$$

where $\varphi(x)$ is a holomorphic function in a neighborhood of $x = 0$ with $\varphi(x) \not\equiv 0$, and $w(t,x)$ is a function belonging in the class $\widetilde{\mathcal{O}}_+$ which is defined by the following:

Definition 4.1. A function $w(t,x)$ is said to be in the class $\widetilde{\mathcal{O}}_+$ if it satisfies the conditions c$_1$) and c$_2$): c$_1$) $w(t,x)$ is a holomorphic function in a domain $\{(t,x) \in \mathcal{R}(\mathbb{C}_t \setminus \{0\}) \times \mathbb{C}^n_x\,;\, 0 <$

$|t| < \eta(\arg t), |x| < R\}$ for some positive-valued continuous function $\eta(s)$ on $\mathbb{R}_s$ and some $R > 0$; c$_2$) there is an $a > 0$ such that for any $\theta > 0$ we have $\sup_{|x|<R} |w(t,x)| = O(|t|^a)$ (as $t \longrightarrow 0$ under $|\arg t| < \theta$). Here $\mathcal{R}(\mathbb{C}_t \setminus \{0\})$ denotes the universal covering space of $\mathbb{C}_t \setminus \{0\}$.

Since $\sigma_K < 0$, $\varphi(x) \not\equiv 0$ and $w(t,x) \longrightarrow 0$ (as $t \longrightarrow 0$), we easily see that this function (4.1) has really singularities of order $|t|^{\sigma_K}$ on $\{t = 0\}$. Hence, if we can construct such a solution as in (4.1), we can conclude that singulasrities of order $|t|^{\sigma_K}$ on $\{t = 0\}$ appear in the solutions of (1.1).

Substituting this (4.1) into (1.1), we get

$$t^{\sigma_K - 1}\Big(\sigma_K \varphi + \Big(t\frac{\partial}{\partial t} + \sigma_K\Big)w\Big) = \sum_{(j,\alpha)\in\Delta} t^{k_{j,\alpha}+\sigma_K(j+|\alpha|)} b_{j,\alpha}(t,x)\,(\varphi + w)^j \Big(\frac{\partial \varphi}{\partial x} + \frac{\partial w}{\partial x}\Big)^{\alpha}$$

and by cancelling the factor $t^{\sigma_K - 1}$ we have

$$\sigma_K \varphi + \Big(t\frac{\partial}{\partial t} + \sigma_K\Big)w = \sum_{(j,\alpha)\in\Delta} t^{k_{j,\alpha}+1+\sigma_K(j+|\alpha|-1)} b_{j,\alpha}(t,x)\,(\varphi + w)^j \Big(\frac{\partial \varphi}{\partial x} + \frac{\partial w}{\partial x}\Big)^{\alpha}. \tag{4.2}$$

Here we remark that

$$\begin{aligned} &k_{j,\alpha} + 1 + \sigma_K(j + |\alpha| - 1) = 0 \quad \text{if } (j,\alpha) \in \mathcal{M}, \\ &k_{j,\alpha} + 1 + \sigma_K(j + |\alpha| - 1) > 0 \quad \text{if } (j,\alpha) \in \Delta \setminus \mathcal{M}. \end{aligned}$$

Therefore, by using the function $P(x, y, z)$ we can write the equation (4.2) in the following form:

$$\begin{aligned} \sigma_K \varphi + \Big(t\frac{\partial}{\partial t} + \sigma_K\Big)w = {} & P\Big(x,\, \varphi + w,\, \frac{\partial \varphi}{\partial x} + \frac{\partial w}{\partial x}\Big) \\ & + t \sum_{(j,\alpha)\in\mathcal{M}} c_{j,\alpha}(t,x)\,(\varphi + w)^j \Big(\frac{\partial \varphi}{\partial x} + \frac{\partial w}{\partial x}\Big)^{\alpha} \\ & + \sum_{(j,\alpha)\in\Delta\setminus\mathcal{M}} t^{k_{j,\alpha}+1+\sigma_K(j+|\alpha|-1)} b_{j,\alpha}(t,x)\,(\varphi + w)^j \Big(\frac{\partial \varphi}{\partial x} + \frac{\partial w}{\partial x}\Big)^{\alpha} \end{aligned} \tag{4.3}$$

where $c_{j,\alpha}(t,x) = (b_{j,\alpha}(t,x) - b_{j,\alpha}(0,x))/t$. Since we are now considering a $w(t,x) \in \widetilde{\mathcal{O}}_+$, we have $w(t,x) = o(1)$ (as $t \longrightarrow 0$) and so by letting $t \longrightarrow 0$ in the above equation we have

$$\sigma_K \varphi = P\Big(x, \varphi, \frac{\partial \varphi}{\partial x}\Big). \tag{4.4}$$

Then by subtracting the equation (4.4) from (4.3) we obtain

$$\begin{aligned} & \Big(t\frac{\partial}{\partial t} + \sigma_K\Big)\, w \\ & = \frac{\partial P}{\partial y}\Big(x, \varphi, \frac{\partial \varphi}{\partial x}\Big)\, w + \sum_{j=1}^{n} \frac{\partial P}{\partial z_j}\Big(x, \varphi, \frac{\partial \varphi}{\partial x}\Big)\frac{\partial w}{\partial x_j} + G_2\Big(x, \varphi, \frac{\partial \varphi}{\partial x}, w, \frac{\partial w}{\partial x}\Big) \\ & \quad + t \sum_{(j,\alpha)\in\mathcal{M}} c_{j,\alpha}(t,x)\,(\varphi + w)^j \Big(\frac{\partial \varphi}{\partial x} + \frac{\partial w}{\partial x}\Big)^{\alpha} \\ & \quad + \sum_{(j,\alpha)\in\Delta\setminus\mathcal{M}} t^{k_{j,\alpha}+1+\sigma_K(j+|\alpha|-1)} b_{j,\alpha}(t,x)\,(\varphi + w)^j \Big(\frac{\partial \varphi}{\partial x} + \frac{\partial w}{\partial x}\Big)^{\alpha}. \end{aligned} \tag{4.5}$$

Here, G_2 is the remainder term of the Taylor expansion of P with respect to $(w, \partial w/\partial x)$. To summarize our goal, we have the following proposition:

Proposition 4.2. *If the equation* (4.4) *has a holomorphic solution* $\varphi(x)$ *which is not identically zero and the equation* (4.5) *has a solution* $w(t,x) \in \widetilde{\mathcal{O}}_+$, *then we have succeeded in constructing a solution* $u(t,x)$ *of* (1.1) *with singularities of order* $|t|^{\sigma_\kappa}$ *on* $\{t = 0\}$.

Thus, to prove the existence of singularities of order $|t|^{\sigma_\kappa}$ on $\{t = 0\}$, it is sufficient to study about when Proposition 4.2 is valid. For the concrete construction we need to use results in [2], [12], [1], [11] and the Cauchy-kowalevsky theorem.

§5. On higher order case

Lastly let us give some comments on the following higher order case:

$$\left(\frac{\partial}{\partial t}\right)^m u = F\left(t, x, \left\{\left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{(j,\alpha)\in\Lambda}\right), \tag{5.1}$$

Here, for convenience, we have denoted by Λ the set of multi-indices $\{(j,\alpha) \in \mathbb{N}\times\mathbb{N}^n\,;\, j+|\alpha| \leq m,\ j < m\}$; let N be the cardinality of Λ. In describing the function F, the variable $Z_{j,\alpha}$ will correspond to $(\partial/\partial t)^j(\partial/\partial x)^\alpha u$ and the totality of the $Z_{j,\alpha}$'s will be denoted by Z, that is,

$$Z = \{Z_{j,\alpha}\}_{(j,\alpha)\in\Lambda} \in \mathbb{C}^N.$$

Let Ω be an open neighborhood of the origin $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x^n$. We suppose the following condition: *$F(t,x,Z)$ is a holomorphic function on $\Omega \times \mathbb{C}^N$.*

Since the function $F(t,x,Z)$ is holomorphic, we may expand it into the following convergent power series in Z:

$$\begin{aligned} F(t,x,Z) &= \sum_{\mu\in\Delta} a_\mu(t,x) Z^\mu \\ &= \sum_{\mu\in\Delta} t^{k_\mu} b_\mu(t,x) Z^\mu. \end{aligned}$$

In the summation above, the set Δ has elements of the form $\mu = (\mu_{j,\alpha})_{(j,\alpha)\in\Lambda}$ and is a subset of $\mathbb{N}^N$; we have omitted from Δ those multi-indices μ for which $a_\mu(t,x) \equiv 0$. The expression Z^μ is

$$Z^\mu = \prod_{(j,\alpha)\in\Lambda} (Z_{j,\alpha})^{\mu_{j,\alpha}}.$$

Moreover, we have taken out the maximum power of t from each coefficient $a_\mu(t,x)$, so that we have $b_\mu(0,x) \not\equiv 0$ for all $\mu \in \Delta$. Using this expansion, we can now write our partial differential equation as

$$\left(\frac{\partial}{\partial t}\right)^m u = \sum_{\mu\in\Delta} t^{k_\mu} b_\mu(t,x) \prod_{(j,\alpha)\in\Lambda} \left[\left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u\right]^{\mu_{j,\alpha}}.$$

Denote by $\gamma_t(\mu)$ the total number of derivatives with respect to t on the right-hand side of the equation above, i.e., let

$$\gamma_t(\mu) = \sum_{(j,\alpha)\in\Lambda} j\,\mu_{j,\alpha} \quad \text{for } \mu = (\mu_{j,\alpha})_{(j,\alpha)\in\Lambda} \in \mathbb{N}^N.$$

Since the highest order of differentiation with respect to t appearing on the right-hand side is $m-1$, we have $\gamma_t(\mu) \leq (m-1)|\mu|$. We set $\Delta_2 = \{\mu \in \Delta;\, |\mu| \geq 2\}$. If $\Delta_2 = \emptyset$, (5.1) is linear and we have Zerner's result. In the case $\Delta_2 \neq \emptyset$ we introduce the index σ_K due to Kobayashi:

$$\sigma_K = \sup_{\mu \in \Delta_2} \frac{-k_\mu - m + \gamma_t(\mu)}{|\mu| - 1}. \tag{5.2}$$

Using this index, we see:

(1) On the non-existence of singularities we have the same results as in section 2 also in higher order case.
(2) But, on the existence of singularities we have not yet completed to construct singular solutions in all the cases which appear in the discussion.

See Chen-Tahara [1], Gérard-Tahara [3],[4],[5], Kobayashi [7], Lope-Tahara [8], Tahara [11], and Tahara-Yamazawa [13].

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