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Propagation of microlocal solutions near a hyperbolic fixed point

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1 Introduction

This is a partial report of the work in progress with Jean-François Bony, Thierry Ramond and Maher Zerzeri about the quantum monodromy operator associated to a homoclinic trajectory. A major part of the results here was already reported by one of the collaborators in [3].

The notion of monodromy operator was introduced by J. Sjöstrand and M. Zworski in [4] for a periodic trajectory. It consists in continuing microlocal solutions of the semiclassical Schrödinger equaiton

$$-h^2\Delta u+V(x)u=Eu \tag{1}$$

along a Hamilton flow H_p on $p^{-1}(E)$ of the corresponding classical mechanics:

$$H_p=\sum_{j=1}^{d}\left(\frac{\partial p}{\partial\xi_j}\frac{\partial}{\partial x_j}-\frac{\partial p}{\partial x_j}\frac{\partial}{\partial\xi_j}\right),\quad p(x,\xi)=\xi^2+V(x). \tag{2}$$

Recall briefly the notion of microlocal solution according to [4]. If $dp\neq 0$ at a point $(x^0,\xi^0)\in p^{-1}(E)$, there exists a local canonical transformation κ defined in a neighborhood of (x^0,ξ^0) with $\kappa(x^0,\xi^0)=(0,0)$, and a semiclassical microlocal Fourier integral operator U associated to κ, such that $p=\kappa^*\xi_1$ and $UPU^{-1}=hD_{x_1}$. We can then define the space of microlocal solution at (x^0,ξ^0) by

$$\ker_{(x^0,\xi^0)}(P)=U^{-1}(\ker(hD_{x_1})),\quad \ker(hD_{x_1})=\{u\in\mathcal{D}'(\mathbb{R}^d):hD_{x_1}u=0\}$$

Since $\ker(hD_{x_1})$ is identified with $\mathcal{D}'(\mathbb{R}^{d-1})$, so is $\ker_{(x^0,\xi^0)}(P)$. If $(x^1,\xi^1) = \exp TH_p(x^0,\xi^0)$ is another point on this flow, we can naturally define the propagator of microlocal solutions from $\ker_{(x^0,\xi^0)}(P)$ to $\ker_{(x^1,\xi^1)}(P)$ as operator on $\mathcal{D}'(\mathbb{R}^{d-1})$.

Here we study the case where $\exp tH_p(x^0,\xi^0)$ tends to a hyperbolic fixed point $(0,0)$ as t tends to $+\infty$. To such a point associate the stable and unstable Lagrangian manifolds Λ_- and Λ_+, on which Hamilton flows tend to $(0,0)$ as t tends to $+\infty$ and $-\infty$ respectively. Moreover, any point close to Λ_+ comes from a point close to Λ_-. We expect, therefore, that a microlocal solution at a point on Λ_+ is determined by that on Λ_-.

The purpose of this report is to study this correspondence of microlocal solutions from Λ_- to Λ_+. After preparing the geometrical setting in section 2, we state a uniqueness theorem in section 3, which says that if a solution to (1) is microlocally exponentially small on Λ_-, it is also microlocally exponentially small on Λ_+ for E away from a discrete subset $\Gamma(h)$. In section 4, based on an idea in [2], we construct a solution with a given microlocal data at a point (x^0,ξ^0_-) on Λ_-, as superposition of time-dependent WKB solutions via Fourier transform with respect to E, and formally calculate its microlocal output at the corresponding point (x^0,ξ^0_+) on Λ_+. Section 5 is an appendix about the notion of *expandible symbol*, which is used repeatedly for the study of the large time behavior of both classical and quantum objects.

2 Symplectic geometry

Let $p(x,\xi) = \xi^2 + V(x)$ be the Hamitonian associated to the semiclassical Schrödinger operator $-h^2\Delta + V(x)$ in $\mathbb{R}^d$. Here, we use the following notations:

$$x = (x_1,\ldots,x_d), \quad \xi = (\xi_1,\ldots,\xi_d), \quad \xi^2 = \sum_{j=1}^{d} \xi_j^2, \quad \Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$$

Suppose that the potential $V(x)$ is real and analytic in a neighborhood of $x = 0$, and that $x = 0$ is a non-degenerate minimum of $V(x)$, so that $(x,\xi) = (0,0)$ is a saddle point of the Hamiltonian $p(x,\xi)$. After a change of variables, we can assume that $p(x,\xi)$ is of the form

$$p(x,\xi) = \xi^2 - \sum_{j=1}^{d} \frac{\lambda_j^2}{4} x_j^2 + O(|x|^3), \quad (x \to 0),$$

where $\{\lambda_j\}_{j=1}^d$ are positive numbers which we assume $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$.

Let H_p be the Hamilton vector field associated to p. In the (x,ξ) coordinates, the linearized vector field F_p of H_p at $(0,0)$ is simply

$$F_p = d_{(0,0)}H_p = \begin{pmatrix} 0 & I \\ L^2 & 0 \end{pmatrix}, \tag{3}$$

where L is the $d \times d$ matrix defined as $L = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$. The eigenvalues of F_p are the λ_j's and the $-\lambda_j$'s.

Associated to the hyperbolic fixed point, we have thus a natural decomposition of $T^*_{(0,0)}\mathbb{R}^d = \mathbb{R}^{2d}$ in a direct sum of two linear subspaces Λ^0_+ and Λ^0_-, of dimension d, associated respectively to the positive and negative eigenvalues of F_p. These spaces $\Lambda^0_\pm$ are given by

$$\Lambda^0_\pm = \{(x,\xi)\, ;\, \xi_j = \pm\frac{\lambda_j}{2}x_j,\ j = 1, \ldots, d\}. \tag{4}$$

The stable/unstable manifold theorem gives us the existence of two Lagrangian manifolds Λ_+ and Λ_-, defined in a vicinity Ω of $(0,0)$, which are stable under the H_p flow and whose tangent space at (0,0) are precisely Λ^0_+ and Λ^0_-. In particular, we see that these manifolds can be written as

$$\Lambda_\pm = \{(x,\xi)\, ;\, \xi = \nabla\phi_\pm(x)\}, \tag{5}$$

for some smooth functions ϕ_+ and ϕ_-, which can be chosen so that

$$\phi_\pm(x) = \pm\sum_{j=1}^d \frac{\lambda_j}{4}x_j^2 + o(x^2). \tag{6}$$

We shall say that Λ_+ is the outgoing Lagrangian manifold and Λ_- the incoming Lagrangian manifold associated to the hyperbolic fixed point. Indeed Λ_+ (resp. Λ_-) can be charactarized as the set of points $(x,\xi) \in \Omega$ such that $\exp tH_p(x,\xi) \to (0,0)$ as $t \to -\infty$ (resp. as $t \to +\infty$): Take a point $x^0 \in \mathbb{R}^d$ near 0. Then there exist unique $\xi^0_+ \in \mathbb{R}^d$ and $\xi^0_- \in \mathbb{R}^d$ such that $(x^0, \xi^0_\pm) \in \Lambda_\pm$. Let $\gamma_\pm(t) = \exp tH_p(x^0,\xi^0_\pm)$ be the Hamilton flow emanating from $(x^0,\xi^0_\pm)$. Then, we know from Proposition 10 in Appendix that $\gamma_\pm(t)$ are expandible, i.e.

$$\gamma_\pm(t) \sim \sum_{k=1}^\infty e^{\pm\mu_k t}\gamma_{\pm,k}(t), \qquad t \to \mp\infty, \tag{7}$$

where $\gamma_{\pm,k}(t)$ are vectors whose elements are polynomials in t ($\gamma_{\pm,1}$ is constant) and $0 < \mu_1 < \mu_2 < \cdots$ are the various non-vanishing linear combinations over $\mathbb{N}$ of the λ_j's. In particular, $\mu_1 = \lambda_1$. If we assume

(A1) $\lambda_1 < \lambda_2$,

then there exists a constant $\gamma_1 = \gamma_1(x^0)$ such that

$$\gamma_\pm(t) = \gamma_1 e^{\pm\lambda_1 t} \times {}^t(1, 0, \ldots, 0, \pm\lambda_1/2, 0, \ldots, 0) + O(e^{\pm\mu_2 t}), \quad (t \to \mp\infty). \tag{8}$$

We see that $\gamma_\pm(t)$ is tangential to the (x_1, ξ_1)-plane if $c \neq 0$.

3 Uniqueness

We begin this section by introducing the notion of *microsupport* of solutions.

For $u \in L^2(\mathbb{R}^n)$, the Bargman transform (or global FBI transform) is defined by

$$Tu(x, \xi; h) = c_d(h) \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi/h - (x-y)^2/2h} u(y; h) dy.$$

$Tu(x, \xi; h)$ belongs to $L^2(\mathbb{R}^{2d}_{x,\xi})$ and $c_d(h)$ is taken so that T be an isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{2d})$. It is seen that by this transform, the function u is localized in x by a Gaussian up to $O(\sqrt{h})$ when h is small. Moreover, it is localized also in ξ up to $O(\sqrt{h})$. Indeed we have an identity

$$Tu(x, \xi; h) = e^{ix\cdot\xi/h} T\hat{u}(\xi, -x; h),$$

where $\hat{u}$ is the semiclassical Fourier transform

$$\hat{u}(\xi) = (2\pi h)^{-d/2} \int_{\mathbb{R}^d} e^{-x\cdot\xi/h} u(x) dx. \tag{9}$$

A (h-dependent) function $u \in L^2$ is said to be *zero* at a point (x^0, ξ^0) in the phase space iff there exists a neighborhood U of (x^0, ξ^0) and a positive number ϵ such that

$$Tu(x, \xi; h) = O(e^{-\epsilon/h})$$

as $h \to 0$ uniformly in U. The complement of such points is called *microsupport* of u and denoted by $MS(u)$. Microsupport is a closed set. Two functions u and v are identified near (x^0, ξ^0) if $(x^0, \xi^0) \notin MS(u - v)$.

Microsupport has the following properties: Let u be a solution of $Pu = E(h)u$ in a domain $\Omega \subset \mathbb{R}^n$, where $E(h) = O(h)$, and assume that $||u||_{L^2(\Omega)} \leq 1$.

- The microsupport of u is included in the energy surface $p^{-1}(0)$.
- The microsupport of u propagates along a simple Hamilton flow in $p^{-1}(0)$.
- The microsupport of a WKB solution $u = e^{i\psi(x)/h}b(x,h)$, $b(x,h) = O(h^{-N})$ for some $N \in \mathbb{R}$ as h tends to 0, is included in the Lagrangian submanifold $\{(x,\xi);\xi = \partial_x\psi(x)\}$.

Now we come back to our problem near the hyperbolic fixed point. Let $\Gamma(h)$ be the discrete subset of $\mathbb{C}$ defined by

$$\Gamma(h) = \{E_\alpha = -ih\sum_{j=1}^{d}\lambda_j(\alpha_j + \frac{1}{2});\ \alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}^d\}. \qquad (10)$$

Notice that for $E = E_\alpha$, the functions

$$u_\alpha = \Pi_{j=1}^d H_{\alpha_j}\left(e^{-\pi i/4}\frac{\sqrt{\lambda_j}}{\sqrt{2h}}x_j\right)\exp\left(i\sum_{j=1}^{m}\frac{\lambda_j}{4h}x_j^2\right),$$

where H_n is the Hermite polynomial, satisfy the equation

$$-h^2\Delta u_\alpha - \sum_{j=1}^{m}\frac{\lambda_j^2}{4}x_j^2 u_\alpha = E_\alpha u_\alpha.$$

These functions are of WKB form and, by the above third property, the microsupport of u_α is Λ_+^0.

Let us assume

(A2) $|E(h)| \le Ch$ in $\mathbb{C}$ with $C > 0$, and there exists $\delta > 0$ such that $d(E(h),\Gamma(h)) > \delta h$ for all small h.

The following theorem says that the solution of (1) is uniquely determined microlocally in a neighborhood of $(0,0)$, modulo microlocally small functions, by its data on $\Lambda_-\backslash(0,0)$ if $E(h)$ is away from the exceptional set $\Gamma(h)$.

Theorem 1 *Assume (A2). If an h-dependent function $u \in L^2(\mathbb{R}^d)$ with $||u||_{L^2} \le 1$ satisfies*

$$MS((P-E(h))u) = \emptyset, \quad MS(u)\cap\{\Lambda_-\backslash(0,0)\} = \emptyset,$$

in a neighborhood of $(0,0)$, then $(0,0) \notin MS(u)$.

4 Integral representation of the solution

In order to study the correspondence of microlocal solutions from Λ_- to Λ_+, we fix a point (x^0, ξ_-^0) on Λ_- sufficiently close to the origin and consider solutions of (1) whose microsupport on Λ_- is included in a neighborhood of $\exp tH_p(x^0, \xi_-^0)$ (recall that the microsupport is invariant by the Hamilton flow). Then, under the assumption (A2), the solution u is uniquely determined in a full neighborhood of the origin, in particular on Λ_+, if a microlocal data u_0 is given at (x^0, ξ_-^0). We study in this section the map $\mathcal{I}_S$ which associates u_0 to the microlocal solution u at (x^0, ξ_+^0), which we call here *propagator* (it is called *singular part of the monodromy operator* in [3]).

The symbol p is of principal type at $(x^0, \xi_-^0) \in \Lambda_-$ and the space of microlocal solutions $\ker_{(x^0,\xi_-^0)}(P)$ is identified with $\mathcal{D}'(\mathbb{R}^{d-1})$. If we assume

(A3) $\gamma_1(x^0) \neq 0$,

where $\gamma_1(x^0)$ is defined in (8), a microlocal solution $u_0 \in \ker_{(x^0,\xi_-^0)}(P)$ can be considered as distribution on

$$H_0 = \{x \in \mathbb{R}^d; x_1 = x_1^0\},$$

since (the projection of) the Hamilton flows are tangential to the x_1 axis at the origin.

Let $u_0(x') \in \mathcal{D}'(\mathbb{R}^{d-1})$ be such that $\hat{u_0}(\eta)$, the semiclassical Fourier transform of u_0 (see (9)), is supported in a small neighborhood of ${\xi_0}'$.

Following an idea of Helffer and Sjöstrand [2], we write the solution u in the form

$$u(x,h) = \frac{1}{(2\pi h)^{d/2}} \int_{\mathbb{R}^{d-1}} \int_0^{+\infty} e^{i\phi(t,x,\eta)/h} a(t,x,\eta,h)\hat{u}_0(\eta) dt d\eta, \tag{11}$$

with

$$\{\frac{h}{i}\frac{\partial}{\partial t} + P(x,hD) - E(h)\}(e^{i\phi/h}a) = O(h^\infty).$$

If a and the energy $E(h)$ have classical asymptotic expansions with respect to h:

$$a(t,x,\eta,h) \sim \sum_{l=0}^{\infty} a_l(t,x,\eta)h^l, \quad E(h) \sim \sum_{l=0}^{\infty} E_l h^{l+1},$$

(recall here that $E(h)$ is assumed to be of $O(h)$ in (A2)) then ϕ and a should satisfy the eikonal and transport equations respectively:

$$\partial_t \phi + p(x, \nabla_x \phi) = 0, \tag{12}$$

$$\partial_t a_0 + 2\nabla_x \phi \cdot \nabla_x a_0 + (\Delta\phi - iE_0)a_0 = 0, \tag{13}$$

$$\partial_t a_l + 2\nabla_x \phi \cdot \nabla_x a_l + (\Delta\phi - iE_0)a_l = i\Delta a_{l-1} + i\sum_{m=1}^{l} E_m a_{l-m} \quad (l \geq 1). \tag{14}$$

The phase function ϕ will be constructed as generating function of the evolution $\Lambda_t^\eta = \exp tH_p(\Lambda_0^\eta)$ of a suitably chosen Lagrangian manifold Λ_0^η transverse to Λ_- at (x^0, ξ_-^0). Let us fix η sufficiently close to ${\xi_-^0}'$, and look at the integral in (11) with respect to t. It will be shown that, for x close to x^0, there exsists a unique critical point $t = t(x, \eta)$. On the other hand, the Lagrangian manifold Λ_t^η tends to Λ_+ as $t \to +\infty$, which means that $\partial_t \phi$ tends to ϕ_+. Thus we will have microlocally

$$\int_0^{+\infty} e^{i\phi(t,x,\eta)/h} a(t, x, \eta, h)dt \sim \begin{cases} e^{i\psi(x,\eta)} b(x, \eta, h) & \text{near } (x, \xi) = (x^0, \xi_-^0), \\ e^{i\theta(x,\eta)} c(x, \eta, h) & \text{near } (x, \xi) = (x^0, \xi_+^0), \end{cases}$$

with $\psi(x, \eta) = \phi(t(x, \eta), x, \eta)$ and $\theta(x, \eta) = \phi_+(x) + \tilde{\psi}(\eta)$ for some $\tilde{\psi}$.

We require that u is equal to u_0 on H_0 microlocally near (x^0, ξ^0), which is satisfied if

$$\psi(x, \eta) = x' \cdot \eta, \quad b(x, \eta, h) = 1 \quad \text{on } H_0. \tag{15}$$

We will see in the following that it is possible to construct ϕ and a so that ψ and b satisfy the condition (15) and to calculate θ and c. Then we will write $\mathcal{I}_S$ as Fourier integral operator.

4.1 The phase function

Since γ_- is a simple characteristic for the operator p, by the usual Hamilton-Jacobi theory we have first the

Lemma 2 *For all $\eta \in \mathbb{R}^{d-1}$ close enough to ${\xi^0}'$, there is a unique function $\psi_\eta = \psi(x, \eta)$, defined in a neighborhood ω_0 of x_0 such that*

$$\begin{cases} p(x, \nabla\psi_\eta(x)) = 0 & \text{in} \quad \omega_0, \\ \psi_\eta(x) = x' \cdot \eta & \text{on} \quad H_0 \cap \omega_0. \end{cases}$$

We denote by Λ_ψ^η the corresponding Lagrangian manifold

$$\Lambda_\psi^\eta = \{(x,\xi) \in T^*\mathbb{R}^d,\ x \in \omega_0,\ \xi = \nabla\psi_\eta(x)\}. \tag{16}$$

Lemma 3 *The Lagrangian manifolds Λ_- and Λ_ψ^η intersect along an intergral curve γ^η for H_p, and the intersection is clean. In particular, $\gamma^{\xi^{0\prime}} = \gamma_-$.*

Let $(x^0(\eta), \xi^0(\eta))$ be the intersection of γ^η and $H_0 \times \mathbb{R}_\xi^d$. The curve γ^η is parametrized as $\gamma^\eta(t) = \exp tH_p(x^0(\eta), \xi^0(\eta))$, and it has the asymptotic property like (8);

$$\gamma^\eta(t) \sim \gamma_1(\eta)e^{-\lambda_1 t} \times {}^t(1,0,\ldots,0,-\lambda_1/2,0,\ldots,0) \quad (t \to +\infty), \tag{17}$$

with a non vanishing constant $\gamma_1(\eta)$ for η close to $\xi^{0\prime}$.

Let Γ_0^η be the level set of ψ_η passing by $x^0(\eta)$:

$$\Gamma_0^\eta = \{(x,\xi) \in \Lambda_\psi^\eta, \psi_\eta(x) = \psi_\eta(x^0(\eta))\}. \tag{18}$$

Lemma 4 *For any η close enough to $\xi^{0\prime}$, one can find a Lagrangian manifold Λ_0^η such that*

1. *Λ_0^η intersects cleanly with Λ_ψ^η along Γ_0^η,*
2. *for any $t \geq 0$, the projection $\Pi : \Lambda_t^\eta = \exp(tH_p)(\Lambda_0^\eta) \to \mathbb{R}_x^d$ is a diffeomorphism in a neighborhood of $\gamma^\eta(t) \in \Lambda_t^\eta$.*

The Lagrangian manifold $\Lambda_t^\eta = \exp(tH_p)(\Lambda_0^\eta)$ is then represented by a generating function $\phi(t,x,\eta)$:

$$\Lambda_t^\eta = \{(x,\xi); \xi = \nabla_x\phi(t,x,\eta)\}. \tag{19}$$

and $\phi(t,x,\eta)$ satisfies the eikonal equation (12) for every η.

Now we fix η and define

$$\Gamma_t^\eta = \Lambda_t^\eta \cap \Lambda_\psi^\eta \quad (= \exp(tH_p)\Gamma_0^\eta). \tag{20}$$

If $(x,\xi) \in \Gamma_t^\eta$, then $\xi = \nabla_x\phi(t,x,\eta)$ and $p(x,\xi) = 0$ $(\Lambda_\psi^\eta \subset p^{-1}(0))$. Together with (12), we get that t is a critical point for the function $t \mapsto \phi(t,x,\eta)$ if and only if $x \in \Pi_x\Gamma_t^\eta$. More precisely, we have

Proposition 5 *For each x close enough to γ^η, there is a unique time $t = t(x,\eta)$ such that $x \in \Pi_x \Gamma_t^\eta$. Moreover, it is the only critical point for the function $t \mapsto \phi(t,x,\eta)$ and it is non-degenerate, $\partial_t^2 \phi(t(x,\eta),x,\eta) > 0$.*

As a consequence, we obtain

$$\nabla_x \psi_\eta(x) = \nabla_x(\phi(t(x,\eta),x)), \tag{21}$$

so that $x \mapsto \psi_\eta(x)$ and $x \mapsto \phi(t(x),x)$ are equal up to constant. We choose ϕ so that

$$\phi(t(x,\eta),x,\eta) = \psi_\eta(x). \tag{22}$$

Finally we observe the asymptotic behavior of the phase function $\phi(t,x,\eta)$ when t tends to $+\infty$.

Proposition 6 *The phase function $(t,x) \mapsto \phi(t,x,\eta)$ is expandible uniformly with respect to η:*

$$\phi(t,x,\eta) - (\phi_+(x) + \tilde{\psi}(\eta)) \sim \sum_{j\geq 1} e^{-\mu_j t}\phi_j(t,x,\eta). \tag{23}$$

Here $\tilde{\psi}$ is a generating function of the $d-1$ dimensional Lagrangian submanifold $\Lambda_- \cap (H_0 \times \mathbb{R}_\xi^d)$, i.e.

$$\{(y',\eta) \in T^*\mathbb{R}^{d-1}; \eta = \nabla_{y'}\phi_-(x_1^0,y')\} = \{(y',\eta) \in T^*\mathbb{R}^{d-1}; y' = \nabla_\eta \tilde{\psi}(\eta)\},$$

and so

$$\tilde{\psi}(\eta) \sim -\sum_{j=2}^{d} \frac{1}{\lambda_j}\eta_j^2, \quad (\eta \to 0).$$

Moreover, the function ϕ_1 does not depend on t, and

$$\phi_1(x,\eta) = -2\lambda_1\gamma_1(\eta)x_1 + O(x^2), \tag{24}$$

where $\gamma_1(\eta)$ is defined in (17).

4.2 Transport equations

We study the transport equations (13), (14), using the informations about the phase function $\phi(t,x,\eta)$ obtained in the previous subsection. We want to solve these equations under the condition

$$a(t(x,\eta),x,\eta,h)|_{H_0} = e^{-\pi i/4}\sqrt{\partial_t^2\phi(t(x,\eta),x,\eta)}, \tag{25}$$

so that the right hand side of (11), after the stationary phase method applied to the integration with respect to t at the critical point $t = t(x,\eta)$, reduces to u_0 on H_0. Notice that the initial condition (25) determines uniquely the solutions of (13), (14) on the hypersurface $\{(t,x); t = t(x,\eta)\}$, since this hypersurface is invariant under the flow of the vector field $\partial_t + 2\nabla_x\phi\cdot\nabla_x$.

As for the asymptotic behavior as $t \to +\infty$, we recall that ϕ is expandible and

$$\nabla_x\phi\cdot\nabla_x = \sum_{j=1}^{d}\left(\frac{\lambda_j}{2}x_j + O(x^2)\right)\frac{\partial}{\partial x_j}, \quad \Delta\phi = \sum_{j=1}^{d}\frac{\lambda_j}{2} + O(x) \quad (x\to 0).$$

Then again by Proposition 10 applied to $e^{St}a_j$, where

$$S = \frac{1}{2}\sum_{j=1}^{d}\lambda_j - iE_0,$$

we have the following asymptotic expansion.

Proposition 7 *For each l, $a_l(t,x,\eta)$ is expandible and has an asymptotic expansion as $t\to\infty$*

$$a_l(t,x,\eta) \sim e^{-St}\sum_{k=0}^{\infty} a_{l,k}(t,x,\eta)e^{-\mu_k t}, \tag{26}$$

which is uniform with respect to η. Here μ_0 is defined to be 0, and $a_{0,0}$ is independent of t.

4.3 Asymptotics of the propagator

Let us fix η close to $\xi^{0'}$ and x close to γ_η. Then there are two t's which contribute in the semiclassical limit to the integration with respect to t of the expression (11). One is $t = t(x,\eta)$, which is the unique critical point, and the other is $t = +\infty$. They correspond to the Lagrangian manifolds $\Lambda^{\eta}_{t(x,\eta)}$ and Λ_+ respectively.

Since the contribution from $t = t(x,\eta)$ reproduces the given data $u_0(x')$ on H_0 after integration with respect to η, we will obtain the propagator $\mathcal{I}_S$ in the form of Fourier integral operator after calculating the contribution from $t = +\infty$.

Lemma 8 *Suppose $b \in \mathbb{R}$, $\lambda > 0$ and $\rho > 0$. Then as $h \to 0$, we have*

$$\int_0^\infty \exp\{ibe^{-\lambda t}/h - \rho t\}dt - \frac{1}{\lambda}\left(\frac{ih}{b}\right)^{\rho/\lambda}\Gamma\left(\frac{\rho}{\lambda}\right)$$

$$\sim \frac{e^{ib/h}}{\lambda}\sum_{n=0}^{\infty}\binom{\rho/\lambda - 1}{n} n! \left(\frac{ih}{b}\right)^{n+1}$$

Let us compute the contribution from $t = \infty$ of the integral

$$\int_0^\infty e^{i\phi(t,x,\eta)/h}a(t,x,\eta,h)dt.$$

If we substitute $\phi_+(x) + \tilde{\psi}(\eta) + e^{-\lambda_1 t}\phi_1(x,\eta)$ to $\phi(t,x,\eta)$ and $a_{0,0}(x,\eta)e^{-St}$ to $a(t,x,\eta,h)$ according to (23), (26), we get

$$\int_0^\infty e^{i\phi/h}adt = e^{i(\phi_+ + \tilde{\psi})/h}a_{0,0}\int_0^\infty \exp\{i\phi_1 e^{-\lambda_1 t}/h - St\}dt$$

Applying Lemma 8 with $b = \phi_1$, $\lambda = \lambda_1$ and $\rho = S$, we get

$$\int_0^\infty e^{i\phi/h}adt \sim e^{i(\phi_+ + \tilde{\psi})/h}a_{0,0}$$

$$\times\left\{\frac{1}{\lambda_1}\Gamma\left(\frac{S}{\lambda_1}\right)\left(\frac{ih}{\phi_1}\right)^{S/\lambda_1} + \frac{e^{i\phi_1/h}}{\lambda_1}\frac{ih}{\phi_1} + O(h^2)\right\} \quad (h \to 0).$$

The leading term of the left hand side changes according to the real part of S/λ_1:

$$\operatorname{Re} S/\lambda_1 > 1 \;\Leftrightarrow\; \operatorname{Im} E_0 > \left(\lambda_1 - \sum_{j=2}^{d}\lambda_j\right)/2.$$

Theorem 9 *The propagator $\mathcal{I}_S$ can be written in the form*

$$\frac{1}{\sqrt{2\pi h}^{d-1}} \int_{\mathbb{R}^{d-1}} e^{i\theta(x,\eta)} c(x,\eta,h) \hat{u_0}(\eta) d\eta,$$

microlocally near (x^0, ξ^0_+) with

$$\theta(x,\eta) = \phi_+(x) + \tilde{\psi}(\eta),$$

and if $\operatorname{Im} E_0 < (\lambda_1 - \sum_2^d \lambda_j)/2$

$$c(x,\eta,h) \sim \frac{1}{\sqrt{2\pi h}\lambda_1} \Gamma\left(\frac{S}{\lambda_1}\right) \left(\frac{ih}{\phi_1(x)}\right)^{S/\lambda_1} a_{0,0}(x,\eta),$$

if $\operatorname{Im} E_0 > (\lambda_1 - \sum_2^d \lambda_j)/2$

$$c(x,\eta,h) \sim \frac{1}{\sqrt{2\pi h}\lambda_1} e^{i\phi_1(x)/h} \frac{ih}{\phi_1(x)} a_{0,0}(x,\eta),$$

and if $\operatorname{Im} E_0 = (\lambda_1 - \sum_2^d \lambda_j)/2$

$$c(x,\eta,h) \sim \frac{1}{\sqrt{2\pi h}\lambda_1} \left(\Gamma\left(\frac{S}{\lambda_1}\right) \left(\frac{ih}{\phi_1(x)}\right)^{S/\lambda_1} + e^{i\phi_1(x)/h} \frac{ih}{\phi_1(x)} \right) a_{0,0}(x,\eta),$$

where $\tilde{\psi}(\eta)$ and $\phi_1(x)$ are given in Proposition 6 and $a_{0,0}$ is given in Proposition 7.

5 Appendix - Expandible symbols

Here we recall from [2] the notion of expandible symbol.

We denote by $(\mu_j)_{j\geq 0}$ the strictly growing sequence of linear combinations over $\mathbb{N}$ of the λ_j's. We have for example $\mu_0 = 0$, $\mu_1 = \lambda_1$ and $\mu_2 = 2\lambda_1$ or $\mu_2 = \lambda_2$, whether $2\lambda_1 < \lambda_2$ or not.

First we introduce a convenient notation for error terms. We shall write, with $\mu \in \mathbb{R}^+$, $M \in \mathbb{N}$,

$$w(t,x) = \tilde{O}(e^{-\mu t}|x|^M) \tag{27}$$

if, for every $\epsilon > 0$, we have

$$w(t,x) = O(e^{-(\mu-\epsilon)t}|x|^M). \tag{28}$$

Definition 1 *([2], Definition 3.1) Let ω be a neighborhood of 0 in $\mathbb{R}^d$. A smooth function $u : [0,+\infty[\times\omega \to \mathbb{R}$ is expandible if there exists a sequence (u_k) of smooth functions on $[0,+\infty[\times\omega$, which are polynomials in t, such that for any $n, N \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$*

$$\partial_t^n \partial_x^\alpha \left(u(t,x) - \sum_{j=0}^{N} u_k(t,x) e^{-\mu_k t} \right) = \tilde{O}(e^{-\mu_{N+1} t}) \tag{29}$$

If (29) holds, we write simply

$$u(t,x) \sim \sum_{k\geq 0} u_k(t,x) e^{-\mu_k t}. \tag{30}$$

Proposition 10 *([2], Theorem 3.8) Let $A(t,x)$ be a real smooth expandible matrix with $A(0,0) = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$. Then, if $v(t,x)$ is expandible, the solution $u(t,x)$ to the problem*

$$\begin{cases} \partial_t u + A(t,x)x \cdot \partial_x u = v, \ \ t \geq 0, x \in \omega, \\ u_{|t=0} = 0, \end{cases} \tag{31}$$

is expandible.

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