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THE NAVIER-STOKES EQUATIONS IN $\mathbb{R}^n$ WITH LINEARLY GROWING INITIAL DATA[1]

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ABSTRACT. The local-in-time mild solutions to the Navier-Stokes equations with the initial velocity U_0 of the form $U_0(x) = -Mx + u_0(x)$ is constructed, where M is an $n \times n$ constant matrix with $\operatorname{tr} M = 0$ and $u_0 \in L^p_\sigma(\mathbb{R}^n)$. Key method is to establish Ornstein-Uhlenbeck semigroup and studying its property, for example, to establish the $L^p - L^q$ estimates. The solution is smooth in x, but no differentiable in t. Moreover, if $\|e^{tM}\| \leq 1$ for all $t \geq 0$, then this mild solution is even analytic in x. Also, some results related to main theorem are mentioned.

This paper is essentially based on the results in [21] with Matthias Hieber (in Technische Universität Darmstadt, Germany).

1. INTRODUCTION.

We consider the Navier-Stokes equations in the whole space $\mathbb{R}^n$ ($n \geq 2$):

$$\text{(NS)} \qquad \begin{cases} U_t - \Delta U + (U, \nabla)U + \nabla P = 0, \quad \nabla \cdot U = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ U|_{t=0} = U_0 \quad \text{with} \quad \nabla \cdot U_0 = 0 & \text{in } \mathbb{R}^n. \end{cases}$$

Here, $U = (U^1(x,t), \dots, U^n(x,t))$ and $P(x,t)$ stand for the unknown velocity and the unknown pressure of the viscous fluid at $x \in \mathbb{R}^n$ and $t > 0$; $U_0 = (U_0^1(x), \dots, U_0^n(x))$ is the given initial velocity. The notations of differentiations are denoted as following: $U_t := \partial U/\partial t$, $\partial_i := \partial/\partial x_i$, $\Delta := \sum_{i=1}^n \partial_i^2$, $(U, \nabla) := \sum_{i=1}^n U^i \partial_i$, $\nabla P := (\partial_1 P, \dots, \partial_n P)$, $\nabla \cdot U := \sum_{i=1}^n \partial_i U^i$.

Our purpose of this paper is to construct the mild solution of (NS), when the initial velocity may grow linearly at space infinity. So, we select the initial velocity is of the form

$$U_0(x) = -Mx + u_0(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where M is a real valued $n \times n$ constant matrix with $\operatorname{tr} M = 0$, and $u_0 \in L^p_\sigma(\mathbb{R}^n)$. Here, we denote $L^p(\mathbb{R}^n)$ by the usual Lebesgue space, and $L^p_\sigma(\mathbb{R}^n)$ by its solenoidal subspace; $H^s_p(\mathbb{R}^n) := (I - \Delta)^{-s/2} L^p(\mathbb{R}^n)$ stands for Sobolov space, and $H^s(\mathbb{R}^n) := H^s_2(\mathbb{R}^n)$ for simplicity. Throughout of this paper we do not distinguish the vector

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valued functions and scalar as well as function spaces. Also, we sometimes omit $(\mathbb{R}^n)$ as $L^p := L^p(\mathbb{R}^n)$, if no confusion occurs likely.

If the case $M = 0$, it is well known that (NS) admits a local-in-time smooth solution, provided that the initial velocity U_0 belongs to $H^{n/2-1}(\mathbb{R}^n)$ (by [9, 28]), $L^p_\sigma(\mathbb{R}^n)$ for $n \le p \le \infty$ (see e.g. [12, 14, 17, 27]). Some researchers tried (and still try) to construct the mild solution in several functions spaces (see e.g. [1, 6, 7, 30, 31, 32, 33, 34, 42]). However, the author has never seen yet that one can succeed it in the function space which contains some growing functions up to now.

In the case of $M \neq 0$, the situation is more complicated, in general. Once we choose M so that $Mx = (x_2, -x_1, 0)$, we can easily get a unique classical solution to (NS) with initial data given by (1.1), using *rotating coordinate*; see e.g. [3, 22]. However, we now impose M satisfying $\operatorname{tr} M = 0$ only. We thus cannot expect to apply this method, directly.

On the other hand, we consider the substitution

$$u := U - \bar{U} \qquad \text{and} \qquad \tilde{P} := P - \bar{P},$$

where $\bar{U} := -Mx$, $\bar{P} := (\Pi x, x)$, $\Pi := \frac{1}{2}((M^{sym})^2 + (M^{ssym})^2)$ and $M^{sym} := \frac{1}{2}(M + M^T)$ and $M^{ssym} := \frac{1}{2}(M - M^T)$. Here M^T denotes the transposed matrix of M. At that time we notice that the pair (U, P) satisfies (NS) in classical sense if and only if $(u, \tilde{P})$ solves

$$\text{(NS2)} \qquad \begin{cases} u_t - \Delta u + (u, \nabla)u - (Mx, \nabla)u - Mu + \nabla\tilde{P} = 0, \\ \nabla \cdot u = 0 \qquad \text{in } \mathbb{R}^n \times (0, T), \\ u|_{t=0} = u_0 \qquad \text{in } \mathbb{R}^n. \end{cases}$$

Look at that $(\bar{U}, \bar{P})$ is a solution of not only (NS) but also the stationary Euler equations; this fact was firstly shown by Majda in [35]. Then $(u, \tilde{P})$ can be regarded as a perturbation between the solution to (NS) and Majda's stationary solution. One of our motivations is to observe the stability and uniqueness of Majda's solution.

A typical example of M is $M = R + J$, where

$$R = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad J = \begin{pmatrix} -b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 2b \end{pmatrix}$$

for $a, b \in \mathbb{R}$. Note that R corresponds to pure rotation, and describes the Coriolis force. As we mentioned before, in the case of $M = R$, the problem (NS2) was investigated by Hishida [22, 23, 24] and by Babin, Mahalov and Nicolaenko [3, 4].

Indeed, Hishida considered (NS2) with $M = R$ in an exterior domain $\Omega \subset \mathbb{R}^3$ and constructed a local-in-time mild solution, when the initial data u_0 belongs to $H^s(\Omega)$ for $s \geq 1/2$. Babin, Mahalov and Nicolaenko also showed that (NS) with $U_0(x) = -Rx + u_0(x)$ has a unique classical solution, provided that u_0 is in $L^p_\sigma(\mathbb{R}^n)$ or u_0 is a smooth periodic function. In [44], the author of this paper proved the existence of a unique classical solution, still for $M = R$, provided that u_0 belongs to the Besov space $\dot{B}^0_{\infty,1}$. Note that $\dot{B}^0_{\infty,1} \subset L^\infty$, and contains some almost periodic functions. In addition, the advantage of using $\dot{B}^0_{\infty,1}$ is the boundedness of the Riesz transform in $\dot{B}^0_{\infty,1}$. The definitions and properties of the homogeneous Besov spaces are found in e.g. [5, 47, 48]. In particular, $\dot{B}^0_{\infty,1}$ is investigated in [44, 45], more precise.

On the other hand, according to Majda in [35], $M = J$ illustrates the jet flows of the fluid. In fact, Jx corresponds to the drain along to x_1 and x_2-axises and to the outgoing to infinity along to x_3-axis. Giga and Kambe [15] also investigated the axisymmetric irrotational flow and studied the stability of the vortex, when the velocity field of the fluid U is expressed as $U = Jx + V$, where V is a two-dimensional velocity field $V = (V^1, V^2, 0)$.

In the back groud of this works, the author consideres the following problem:

What is the boarder case between the well-posed and ill-posed of (NS)?

Here the (time-local) well-posed means that one can construct a local-in-time unique classical solution to (NS) with value continious up to initial time. The author guesses that the boarder is just when the initial velocity grows linear order at space infinity. To consider the 1-dimensional Burgers equaiton $U_t - U_{xx} + UU_x = 0$, $U(0) = U_0$, which seems to be a model case of 1-dimensional Navier-Stokes equation, we know the answer: let $|U_0(x)| \sim |x|^s$ as $x \to \infty$,

$$\begin{aligned}
&(1)\ \ if\ s < 1,\ then\ time-global\ well-posed,\\
&(2)\ \ if\ s = 1,\ then\ time-local\ well-posed,\\
&(3)\ \ if\ s > 1,\ then\ ill-posed\ for\ any\ time.
\end{aligned}$$

Using the Cole-Hopf transform, we apply the classical results by Tychonoff [49] to know above. On the multi-dimensional Burgers-like equation, similar results were also obtained by Giga and Yamada [20, 50]. Maybe, the structure of Burgers equation is far form that of Navier-Stokes, but the author still believes to obtain similar results on (NS).

On the other hand, Okamoto [40] (and see also Kim and Chae [29]) studied the uniqueness of (NS), when the velocity behaves $|x|$:

Theorem. *Let $n = 2, 3$. If two pairs (U, P) and $(\widehat{U}, \widehat{P})$ are classical solutions to* (NS) *with same initial velocity, satisfying $|U| = O(|x|)$, $|\nabla U| = O(1)$, $|P| = O(|x|^{1-n/2})$ as $|x| \to \infty$, then $U(x,t) \equiv \widehat{U}(x,t)$ for $x \in \mathbb{R}^n$ and $t > 0$.*

Nobody knows there is a solution satisfying above condition. One of our motivations is to give such a solution, ignoring the pressure condition.

This paper is organized as follows. In section 2 we shall state the main results on this paper, and refer to related results. In section 3 we prepare the tools. In particular, we establish several estimates for the semigroup. In section 4 we shall give the proofs of our main theorems, breifly.

2. Main Results.

Before mentioning the main results on this paper, we now define the operator A in $L^p_\sigma(\mathbb{R}^n)$ for $p \in [1, \infty]$ as

$$Au := -\Delta u - (Mx, \nabla)u + Mu$$

with domain $D(A) := \{u \in H^2_p \cap L^p_\sigma; (Mx, \nabla)u \in L^p\}$. We may prove that $-A$ generates a C_0-semigroup e^{-tA} on L^p_σ for $p \in [1, \infty)$; see e.g. [37, 38]. For $p = \infty$, $-A$ also generates a semigroup on L^∞_σ, but there is a lack of the strong continuity at $t = 0$. Remark that the semigroup e^{-tA} is not analytic, see [22]. In the next section the detail of properties of this semigroup will be observed.

Applying the projection $\mathbb{P}$ to (NS2), formally, we have the abstract equation:

$$\text{(ABS)} \qquad u_t + Au + \mathbb{P}(u, \nabla)u - 2\mathbb{P}Mu = 0, \quad u(0) = u_0.$$

We now deal with the whole space problem, the projection $\mathbb{P}$ can be written explicitly by $\mathbb{P} := (\delta_{ij} + R_i R_j)_{1 \le i,j \le n}$, where δ_{ij} denotes the Kronecker's delta, and R_i is the Riesz transform defined by $R_i := \partial_i(-\Delta)^{-1/2}$. Note that A and $\mathbb{P}$ commute, since $\nabla \cdot Au = 0$ if $\nabla \cdot u = 0$. Then, it is straightforward to get the integral equation:

$$\text{(INT)} \qquad u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}\mathbb{P}u(s) \cdot \nabla u(s) ds + 2\int_0^t e^{-(t-s)A}\mathbb{P}Mu(s) ds$$

for $t \in (0, T)$ with $u(0) = u_0$, integrating (ABS) in time. For $T > 0$ we call a function $u \in C([0, T); L^p_\sigma(\mathbb{R}^n))$ *a mild solution*, if u satisfies (INT).

We are now in position to state the local-in-time existence and uniqueness results for mild solutions in L^p_σ spaces.

2.1. **Theorem.** *Let $n \geq 2$, $p \in [n, \infty)$ and $q \in [p, \infty]$. Let M be a real valued $n \times n$ constant matrix with $\operatorname{tr} M = 0$, and assume that $u_0 \in L^p_\sigma(\mathbb{R}^n)$. Then there exist $T_0 > 0$ and a unique mild solution u such that*

$$t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} u \in C([0,T_0); L^q_\sigma(\mathbb{R}^n)), \tag{2.1}$$

$$t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{2}} \nabla u \in C([0,T_0); L^q(\mathbb{R}^n)). \tag{2.2}$$

2.2. **Remark.** (i) *The functions defined in* (2.1) *and* (2.2) *are continuous in t up to initial time, moreover, they vanish at $t = 0$ provided $q \neq p$ in* (2.1).
(ii) *The case $p = \infty$. It seems to be difficult to obtain the solvability in L^∞ or BUC. This difficulty comes form unboundedness of the Riesz transform onto L^∞. Therefore, if we choose the initial data u_0 in $\dot{B}^0_{\infty,1}$, we can show the local existence of mild solution in $C([0,T_0); \dot{B}^0_{\infty,1})$.*

In order to prove Theorem 2.1 we derive the benefit estimates (for example, $L^p - L^q$ estimates) for the semigroup e^{-tA} as well as heat semigroup. Nevertheless the semigroup e^{-tA} is not analytic, thanks to the explicit formula of the semigroup, we can derive them by direct calculations of the kernel; see Lemma 3.2. To construct the mild solution we use a standard iteration scheme.

From similar argument of the proof of Theorem 2.1 we are able to derive uniform bounds for $\nabla^k u(t)$ for any $k \in \mathbb{N}$, if $t \leq T_k$ for some $T_k \sim k^{-k}$. This implies evidently that $u(t) \in C^\infty(\mathbb{R}^n)$ as long as mild solution exists.

Conversely, we cannot control the time-differentiation of u, even if the initial data belongs to $D(A)$, in general. Because, it cannot be expected that the solution is in $D(A)$. This means, we do not know our mild solution is a strong solution, i.e.,

$$?? \quad u \in C([0,T); D(A)) \cap C^1([0,T); L^p_\sigma) \quad ??$$

Of course, this difficulty comes from non analyticity of the semigroup. Therefore, we do not know whether or not the mild solution satisfies (ABS), and (NS) with some pressure. Once the mild solution u solves (ABS), we show that the pair $(u, \nabla\tilde{P})$ fulfilles (NS2), provided that

$$\partial_l \tilde{P} := \sum_{i,j=1}^{n} \partial_l R_i R_j u^i u^j + 2 \sum_{i,j=1}^{n} m_{ij} R_l R_i u^j.$$

We thus get the solution to (NS) as $(U, P) := (u + Mx, \tilde{P} + (\Pi x, x))$, formally. The uniqueness theorem related this situation was obtained in [44].

The estimates for the semigroup show that the linear term of (INT) grows at $t \to \infty$ exponentially, in general. Furthermore, the linear remainders, which is the last term of (INT), prevents Kato's argument in [27] (time-global well-posedness for small data). Hence, it seems to be difficult to obtain results on global existence of mild solutions, even if we solve it in scaling invariant space (e.g. $L^n(\mathbb{R}^n)$).

In 2-dimensional case, we can apply the maximum principle for the vorticity, at least when $M = 0$, see e.g. [11, 13]. Once we obtain the uniform bound for vorticity, we can get global solution, see [16]. However, in our situation we need some new idea. Indeed, taking rot into (NS2), for general M we have the vorticity equation on the scalar function $\omega := \operatorname{rot} u$:

$$\text{(VOR)} \qquad \omega_t - \Delta\omega - (Mx, \nabla)\omega + \operatorname{tr} M\omega + (u, \nabla)\omega = 0$$

with $\omega(0) = \omega_0 := \operatorname{rot} u_0$; under our assumption we suppose $\operatorname{tr} M = 0$. At least we may not apply the maximum principle for (VOR) directly, so it is not known how to get the estimate like $\|\omega(t)\|_q \le \|\omega_0\|_q$ for $t > 0$ with some q. In [44, Lemma 3.3] we have the following estimates:

$$\|\omega(t)\|_{\dot{B}^0_{\infty,1}} \le C\|\omega_0\|_{\dot{B}^0_{\infty,1}} \exp\{C\sum_{k=0}^{2}\int_0^t \|\nabla^k u(s)\|_{\dot{B}^0_{\infty,1}} ds\}.$$

But this is very far from what we desire, this does not help us.

It is a natural question to consider the exterior domains Ω, instead of $\mathbb{R}^n$. This initial-boundary value problem leads us to interesting applications such as spin-coating of fluids. This will be the content of a forthcoming publication; in the future we will prove that $-A$ generates a C_0-semigroup on $L^p_\sigma(\Omega)$ for $1 < p < \infty$.

We are forced to derive the estimates T_k independent of k under some condition on M. In fact, if we select M so that $\|e^{tM}\| \le 1$ for all $t \le 0$, then we take T_k uniformly in k; involving the iteration scheme, we can control $\|\nabla^k u(t)\|_q$ for all k, simultaneously. It is easy to verify that $M = R$ should satisfy $\|e^{tR}\| = 1$. Once we obtain it, the analyticity in x of $u(t)$ can be shown. Actually, spatial-analyticity is deduced form the following estimates of regularizing rates for higher order derivatives of u:

2.3. **Theorem.** *Let $n \ge 2$, $u_0 \in L^n_\sigma(\mathbb{R}^n)$. Assume that $\|e^{tM}\| \le 1$ for all $t \ge 0$. Let u be the local-in-time mild solution obtained by Theorem 2.1 in the class of*

$$u \in C([0,T]; L^n_\sigma(\mathbb{R}^n)) \cap C((0,T]; L^r_\sigma(\mathbb{R}^n))$$

for some $r \in (n, \infty]$ *and* $T > 0$. *Assume further that there exist positive constants* M_1, M_2 *such that*

$$\sup_{0<t<T} \|u(t)\|_n \leq M_1 \qquad \text{and} \qquad \sup_{0<t<T} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|u(t)\|_r \leq M_2.$$

Then there exist constants K_1 *and* K_2 *(depending only on* n, M, r, T, M_1, M_2*) such that*

$$\|\nabla^m u(t)\|_q \leq K_1(K_2 m)^m t^{-\frac{m}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})} \tag{2.3}$$

for all $t \in (0, T]$, $q \in [n, \infty]$ *and* $m \in \mathbb{N}_0$. *Here* $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

It is easy to see that from Theorem 2.3 the mild solution $u(t)$ is analytic in x. More precisely, we get the estimate for the size of the radius of convergence of Taylor expansion (=: $\rho(t)$) from below:

$$\rho(t) \geq \limsup_{m\to\infty} \left(\frac{\|\nabla^m u(t)\|_\infty}{m!}\right)^{-1/m} \geq C\sqrt{t}$$

for $t \in (0, T]$. This estimate comes from Cauchy's criterion and Stirling's formula.

To get (2.3) we prove an equivalent estimate

$$\|\partial_x^\alpha u(t)\|_q \leq K_1(K_2|\alpha|)^{|\alpha|-\delta} t^{-\frac{|\alpha|}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})}$$

for all $t \in (0, T]$, $q \in [n, \infty]$ and $\alpha \in \mathbb{N}_0^n$ with some $\delta \in (1/2, 1]$. Here the constant K_1 and K_2 may depend on δ, but independent of α and t. We differentiate the both hand sides of (INT) and take L^q-norm. We notice that e^{-tA} and ∇ do not commute, in general, we actually obtain that

$$\nabla e^{-tA} f = e^{tM} e^{-tA} \nabla f. \tag{2.4}$$

(The meaning of the assumption on M is for the uniform bound of shifting the derivatives over semigroup as well as we like.) We divide the integral $\int_0^t$ into two parts as $\int_0^{(1-\varepsilon)t} + \int_{(1-\varepsilon)t}^t$ in order to distribute the singularity, and apply the Gronwall type inequality (see [19, Lemma 2.4]). Finally, ε is taken small enough such that $\varepsilon \sim 1/|\alpha|$ with induction on $|\alpha|$ to get (2.3). This is essentially same strategy in [19], they also prove the analyticity in x for the mild solution in the case $M = 0$.

As the author mentioned before, due to the unbounded coefficient in the drift term, e^{-tA} is not analytic. Hence the estimate for $\|\nabla^m e^{-tA}\|$ does not follow automatically as the classical Stokes semigroup from the analytic semigroup theory. Therefore, we must establish the $L^p - L^q$ estimates with higher order differentials, see Lemma 3.3 in the next section.

The author does not know whether one can still show (2.3), when we relax the assumption on M, for example, $\|e^{tM}\| \leq C_*$ with some $C_* > 1$. In our proof we need $C_* = 1$ to choose the constants K_1 and K_2 independently in m. We only obtain the spatial-analyticity, since the time-analyticity of u does not follow from our method directly. Probably, the mild solution should not be analytic in time! The author also guesses that this method is not applicable for the boundary value problem, since we need suitable commutativity between the semigroup and differential.

3. Estimates for the semigroup e^{-tA}.

In this section we establish the semigroup theory and research its properties. In the next section we use these tools for proofs of main theorems.

Let M be an $n \times n$ matrix of real valued constants; it is not necessary to impose $\operatorname{tr} M = 0$ troughout this section. We now introduce the operator $\mathcal{A}$ by

$$\mathcal{A}u := -\Delta u - (Mx, \nabla)u + Mu,$$

where $u := (u_1, \dots, u_n) \in L^p(\mathbb{R}^n)$ for $p \in [1, \infty]$ and $\mathcal{A}$ is an $n \times n$ matrix operator. Observe that by simple calculation

$$\nabla \cdot \{-(Mx, \nabla)u + Mu\} = 0, \qquad \text{provided} \quad \nabla \cdot u = 0.$$

We thus define A as the realization of $\mathcal{A}$ in $L^p_\sigma(\mathbb{R}^n)$

$$\text{(3.1)} \qquad \left\{ \begin{array}{rcl} Au & := & \mathcal{A}u \\ D(A) & := & \{u \in H^2_p \cap L^p_\sigma; (Mx, \nabla)u \in L^p\}. \end{array} \right.$$

By standard perturbation theory it follows that

3.1. **Lemma.** *The operator $-A$ generates a C_0-semigroup on $L^p_\sigma(\mathbb{R}^n)$ for $p \in [1, \infty)$. The semiroup $\{e^{-tA}\}_{t\geq 0}$ has an explicit formula by*

$$\text{(3.2)} \qquad (e^{-tA}u)(x) := \frac{e^{-tM}}{(4\pi)^{n/2}(detQ_t)^{1/2}} \int_{\mathbb{R}^n} u(e^{tM}x - y)e^{-\frac{1}{4}(Q_t^{-1}y, y)}dy,$$

where $Q_t := \int_0^t e^{sM}e^{sM^T}ds$.

Notice that in the case $M = 0$ the semigroup e^{-tA} coincides with the heat semigroup, since $t^{-1}Q_t = Id$. The proof of Lemma 3.1 was shown by e.g. Metafune and his collaborators [37, 38]. Note that the semigroup e^{-tA} is not analytic, In fact, if we intend to show that e^{-tA} is analytic semigroup, we may construct the counter example by using (Mx, ∇) term; see [22]. The operator $-A$ also generates a semigroup on $L^\infty_\sigma(\mathbb{R}^n)$. But, as same as heat semigroup, there is a lack of strong continuity at $t=0$ in L^∞.

We now turn to L^p-L^q smoothing properties as well as gradient estimates for e^{-tA}. Due to the non analyticity of e^{-tA}, gradient estimates do not follow from the general theory of analytic semigroups.

3.2. **Lemma.** *Let $n\geq 1$ and $1\leq p\leq q\leq\infty$. Then there exist constants $\tilde{C}_0>0$ and $\omega_0\geq 0$ such that*

$$\begin{aligned} &(3.3) && \|e^{-tA}f\|_q \;\leq\; \tilde{C}_0 t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}e^{\omega_0 t}\|f\|_p, \qquad t>0,\\ &(3.4) && \|\nabla e^{-tA}f\|_p \;\leq\; \tilde{C}_0 t^{-\frac{1}{2}}e^{\omega_0 t}\|f\|_p, \qquad t>0. \end{aligned}$$

Moreover, for $p<q$ and $f\in L^p(\mathbb{R}^n)$ we have

$$\begin{aligned} &(3.5) && t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|e^{-tA}f\|_q \;\to\; 0 \qquad \text{as } t\to 0,\\ &(3.6) && t^{\frac{1}{2}}\|\nabla e^{-tA}f\|_p \;\to\; 0 \qquad \text{as } t\to 0. \end{aligned}$$

We can prove (3.3) and (3.4) by direct calculations of the kernel of explicit formula and Young's inequality. In the proofs of (3.5) and (3.6) we use triangle inequality, (3.3), (3.4) and the density $C_0^\infty\subset L^p$ for $p<\infty$. We skip the proof of Lemma 3.2 in this paper, because one can find it in [21]. Note also that if M satisfies $\|e^{-tM}\|\leq C$ for all $t>0$ with some constant C, we may take $\omega_0=0$. In the special case $M=Id$, L^p-L^q estimates for e^{-tA} were obtained by Gallay and Wayne [10].

To next we estimate for higher order derivatives of semigroup, i.e., for $\nabla^m e^{-tA}f$, which are very useful to consider smoothing properties of mild solutions. The main difficulty is that the semigroup e^{-tA} and tdifferential ∇ do not commute, in general. Nevertheless, we obtain following estimates similar to those of the heat semigroup.

3.3. **Lemma.** *Let $n\geq 1$ and $1\leq p\leq q\leq\infty$. Then there exist constants $\tilde{C}_1,\tilde{C}_2,\tilde{C}_3>0$, $\omega_1,\omega_2,\omega_3,\omega_4\geq 0$ (depending only on n, p, q and M) such that*

$$(3.7)\qquad \|\nabla^m e^{-tA}f\|_q\leq \tilde{C}_1 e^{(\omega_1+\omega_2 m)t}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|\nabla^m f\|_p$$

for $t>0$, $m\in\mathbb{N}$ and $f\in H^m_p(\mathbb{R}^n)$, and

$$(3.8)\qquad \|\nabla^m e^{-tA}f\|_q\leq \tilde{C}_2(\tilde{C}_3 m)^{m/2}e^{(\omega_3+\omega_4 m)t}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}}\|f\|_p$$

for $t>0$, $m\in\mathbb{N}$ and $f\in L^p(\mathbb{R}^n)$.

It is evident to get (3.7) by (2.4) m-th times. So, it is clear to see that the assertion (3.7) holds true with $\omega_2=0$, provided that $\|e^{tM}\|\leq 1$ for all $t>0$. To obtain (3.8), we split e^{-tA} into $m+1$ parts, and use (2.4) m-th times. Then we have

$$\|\nabla^m e^{-tA} f\|_q \leq C\tilde{C}^m \|(\nabla e^{-\frac{t}{m+1}A})^m e^{-\frac{t}{m+1}A} f\|_q$$

with some constants C and $\tilde{C}$. For each terms we apply (3.3) and (3.4), and sum up with them to show (3.8). In (3.8) the top order of dependence of m is $m^{m/2}$, which is natural in the sense that this order is same as that of heat semigroup.

As shown by Theorem 2.3 and its remarks, it is important to derive $\|\nabla^m u\|_\infty$ for proving the spatial-analyticity. In the following lemma, the estimate of the operator norm of $\nabla e^{-tA}\mathbb{P}$ into L^p for all $p \in [1, \infty]$ will be done:

3.4. **Lemma.** *Let $n \geq 1$, $1 \leq p \leq \infty$ and let A and $\mathbb{P}$ be as above. Then there exist constants $C > 0$ and $\omega \geq 0$ such that*

$$\|\nabla e^{-tA}\mathbb{P}\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \leq Ct^{-1/2}e^{\omega t}, \qquad t > 0.$$

The proof is based on [2, Proposition 8.2.3, Lemma 8.2.2]. In the case $M = 0$, we find it in [14]. We omit its detail to make this paper short.

4. Proofs of Theorems.

We are now in position to show that (NS2) admits a local-in-time mild solution, and to investigate its properties. Fistly, we give the proof of Theorem 2.1 briefly, in the case $p = n$, although that is standard argument by Kato [27].

Proof of Theorem 2.1. Let $n \geq 2$ and $u_0 \in L^n_\sigma(\mathbb{R}^n)$. For $j \geq 1$ and $t > 0$ we define functions u_{j+1} by

$$u_{j+1}(t) := e^{-tA}u_0 - \int_0^t e^{-(t-s)A}\mathbb{P}(u_j(s), \nabla)u_j(s)ds + 2\int_0^t e^{-(t-s)A}\mathbb{P}Mu_j(s)ds,$$

and strated at $u_1(t) := e^{-tA}u_0$. Note that $u_j(t)$ keeps divergence-free for all $t > 0$ and j. For $T \in (0, 1]$ and $\delta \in (0, 1)$ we define

$$A_0 := \sup_{0<t\leq T} t^{\frac{1-\delta}{2}}\|e^{-tA}u_0\|_{n/\delta} \quad \text{and} \quad A_0' := \sup_{0<t\leq T} t^{\frac{1}{2}}\|\nabla e^{-tA}u_0\|_n$$

as well as $A_j := A_j(T)$ and $A_j' := A_j'(T)$, where

$$A_j(T) := \sup_{0<t\leq T} t^{\frac{1-\delta}{2}}\|u_j(t)\|_{n/\delta} \quad \text{and} \quad A_j'(T) := \sup_{0<t\leq T} t^{1/2}\|\nabla u_j(t)\|_n, \qquad j \geq 1.$$

We thus obtain that

$$
\begin{aligned}
&\|u_{j+1}(t)\|_{n/\delta} \\
&\leq \|e^{t\Delta}u_0\|_{n/\delta} + \int_0^t \|e^{-(t-s)A}\mathbb{P}u_j(s)\cdot\nabla u_j(s)\|_{n/\delta}ds + 2\int_0^t \|e^{-(t-s)A}\mathbb{P}Mu_j(s)\|_{n/\delta}ds \\
&\leq t^{-\frac{1-\delta}{2}}A_0 + C\int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{\delta}{n})}\|u_j(s)\cdot\nabla u_j(s)\|_r ds + C\int_0^t \|u_j(s)\|_{n/\delta}ds,
\end{aligned}
$$

where $r := \frac{n}{1+\delta}$. In order to estimate the second term on the right hand side of last inequality, we now apply Hölder's inequality to conclude that

$$\|u_j(s)\cdot\nabla u_j(s)\|_r \leq \|u_j(s)\|_{n/\delta}\|\nabla u_j(s)\|_n \leq A_jA_j's^{-\frac{1-\delta}{2}-\frac{1}{2}}.$$

Multiplying with $t^{\frac{1-\delta}{2}}$ and taking $\sup_{0<t\leq T}$ on both sides we obtain

$$A_{j+1} \leq A_0 + C_1A_jA_j' + C_2TA_j \tag{4.1}$$

with some positive constants C_1, C_2 independent of j and T.

Similarly, taking ∇ into approximations, and estimating it in the L^n-norm, by (3.4) we obtain

$$A_{j+1}' \leq A_0' + C_3A_jA_j' + C_4TA_j \tag{4.2}$$

with some positive constants C_3 and C_4. The estimates (3.5) and (3.6) imply that for any $\lambda > 0$, there exists $\tilde{T}_0 > 0$ such that $A_0, A_0' \leq \lambda$ for all $T \leq \tilde{T}_0$. More precisely, we may choose $\tilde{T}_0 \leq \min(1, \frac{1}{3C_2}, \frac{1}{3C_4})$ provided $\lambda \leq \min(\frac{1}{9C_1}, \frac{1}{9C_3})$. Therefore, we obtain bounds for $A_j(T)$ and $A_j'(T)$ for any $T \leq \tilde{T}_0$ uniformly in j provided that $\tilde{T}_0$ is small enough.

Using the uniform bounds of A_j and A_j' we obtained, it follows that $t^{\frac{1}{2}-\frac{n}{2q}}\|u_j(t)\|_q$ as well as $t^{1-\frac{n}{2q}}\|\nabla u_j(t)\|_q$ are bounded for $q \in [n,\infty)$, $t \leq \tilde{T}_0$ and all $j \in \mathbb{N}$. The continuity of the above functions also follows from similar calculations and (3.5). We can derive estimates for the differences $u_{j+1} - u_j$ vanish as $j \to \infty$ on $[0, T_0]$ by similar way, provided that we take suitable $T_0 \leq \tilde{T}_0$.

It thus follows that the above sequences are Cauchy sequences and we conclude that there are unique limit functions

$$t^{\frac{1}{2}-\frac{n}{2q}}u(t) \in C([0,T_0];L^q), \qquad t^{1-\frac{n}{2q}}v(t) \in C([0,T_0];L^q),$$

of the sequences $(t^{\frac{1}{2}-\frac{n}{2q}}u_j(t))_{j\geq1}$ and $(t^{1-\frac{n}{2q}}\nabla u_j(t))_{j\geq1}$. Finally, note that $v(t) = t^{1/2}\nabla u(t)$ and that u is a mild solution on $[0, T_0]$. Uniqueness of mild solutions follows from Gronwall inequality. This completes the proof of Theorem 2.1. $\square$

We now turn to the proof of Theorem 2.3. In the case $M = 0$, recently Giga and author [19] proved that mild solutions are analytic in x. The following proof is a modification of their proof to our situation. So, we only give the outline of the proof, briefly. Of course, the reader can find the precise proof in [21].

Proof of Theorem 2.3. We start by proving the assertion under the additional assumption that mild solution is already smooth. Because, we may show this property using same argument as below. We derive first an equivalent estimates to (2.3):

For $\delta \in (1/2, 1]$ there exist constants $K_1 > 0, K_2 > 0$ (depending only on n, r, M, M_1, M_2, T and δ) such that

$$\|\nabla^m u(t)\|_q \leq K_1 (K_2 m)^{m-\delta} t^{-\frac{m}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})} \tag{4.3}$$

for all $t \in (0, T)$, $q \in [n, \infty]$ and $m \in \mathbb{N}_0$.

To get (4.3), we use an induction with respect to m. One may suppose $\nabla^m u$ is continuous up to $t = 0$ with value in $L^q(\mathbb{R}^n)$ by considering $u(\eta)$ for $\eta > 0$ as initial data and sending $\eta \to 0$. To this end, let $k_0 \geq 2$ (depending only on n and M). Then (4.3) follows for all $m \leq k_0$, provided K_1 is chosen large enough. Assume hence that $k \geq k_0$, and that (4.3) holds for all $q \in [n, \infty]$ and all $m \leq k - 1$. We claim that (4.3) holds for $m = k$.

For simplicity, we first prove the assertion under the additional assumptions that $T \leq 1$, $n \geq 3$ and $q < \infty$. The claim then follows by minor modifications of the proof given below. We start by noticing that for $q \in [n, \infty)$ and $\varepsilon \in (0, 1)$

$$\begin{aligned}
\|\nabla^k u(t)\|_q \leq \|\nabla^k e^{-tA} u_0\|_q &+ \left(\int_0^{(1-\varepsilon)t} + \int_{(1-\varepsilon)t}^t \right) \|\nabla^k e^{-(t-s)A} \mathbb{P} u \cdot \nabla u(s)\|_q ds \\
&+ 2 \left(\int_0^{(1-\varepsilon)t} + \int_{(1-\varepsilon)t}^t \right) \|\nabla^k e^{-(t-s)A} \mathbb{P} M u(s)\|_q ds \\
&=: B_1 + B_2 + B_3 + B_4 + B_5.
\end{aligned}$$

We shall estimate each the above terms $B_1 - B_5$ separately.

The estimates for B_1 are derived from (3.8) as follows:

$$B_1 \leq \tilde{C}_2 (\tilde{C}_3 k)^{k/2} e^{\omega_3 k t} \|u_0\|_n t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}} \leq C_5 (C_6 k)^{k-\delta} t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}}, \qquad t \in (0, T)$$

with constants $C_5 := \tilde{C}_2 \|u_0\|_n \leq \tilde{C}_2 M_1$ and $C_6 := \tilde{C}_3 e^{\omega_3}$. Similarly, we also have the estimates for B_2, B_4 and B_5.

The main part is B_3. We now calculate $\nabla^k(u \otimes u)$ by Leibniz's rule. We divide the sum into two parts:

$$\begin{aligned}
B_3 &\leq C_7 \int_{(1-\varepsilon)t}^{t} (t-s)^{-1/2} \|\nabla^k u(s)\|_q \|u(s)\|_\infty ds \\
&\quad + C_7 \int_{(1-\varepsilon)t}^{t} (t-s)^{-1/2} \max_{|\beta|=k} \sum_{0<\gamma<\beta} \binom{\beta}{\gamma} \|\partial_x^\gamma u(s)\|_q \|\partial_x^{\beta-\gamma} u(s)\|_\infty ds \\
&=: B_{3a} + B_{3b}
\end{aligned}$$

with constant $C_7 = 2\tilde{C}_1 e^{\omega_1}$; note that C_7 does not depend on k, since we assumed that $\|e^{tM}\| \leq 1$ and $T \leq 1$. Here $\gamma < \beta$ means $\gamma_i \leq \beta_i$ for all i and $|\gamma| < |\beta|$ for multi-indices β and γ.

Consider B_{3a}. Then there exists $C > 0$ (depending only on n, p, M, M_1, M_2 such that $\|u(s)\|_\infty \leq C s^{-1/2}$; see Step 1 of the proof of Proposition 3.1 in [19]. Thus

$$B_{3a} \leq C_8 \int_{(1-\varepsilon)t}^{t} (t-s)^{-1/2} s^{-1/2} \|\nabla^k u(s)\|_q ds$$

with some constant $C_8 := C_8(n, p, q, M, M_1, M_2)$. We next estimate B_{3b}. By assumption of induction we obtain that

$$\begin{aligned}
B_{3b} &\leq C_7 \int_{(1-\varepsilon)t}^{t} (t-s)^{-\frac{1}{2}} \max_{|\beta|=k} \sum_{0<\gamma<\beta} \binom{\beta}{\gamma} K_1 (K_2|\gamma|)^{|\gamma|-\delta} s^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{|\gamma|}{2}} \\
&\quad \times K_1 (K_2|\beta-\gamma|)^{|\beta-\gamma|-\delta} s^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q}-\frac{|\beta-\gamma|}{2}} ds \\
&\leq C_7 K_1^2 K_2^{k-2\delta} \sum_{0<\gamma<\beta} \binom{\beta}{\gamma} |\gamma|^{|\gamma|-\delta} |\beta-\gamma|^{|\beta-\gamma|-\delta} \int_{(1-\varepsilon)t}^{t} (t-s)^{-\frac{1}{2}} s^{-1-\frac{n}{2q}-\frac{k}{2}} ds.
\end{aligned}$$

For the multiplication of multi-sequences we apply Kahane's lemma [25, Lemma 2.1] and obtain

$$B_{3b} \leq C_9 K_1^2 K_2^{k-2\delta} k^{k-\delta} t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}} I(\varepsilon)$$

where $I(\varepsilon) := \int_{1-\varepsilon}^{1} (1-\tau)^{-\frac{1}{2}} \tau^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}-\frac{1}{2}} d\tau$ and C_9 depends only on C_7 and δ; so C_9 is indenpendent of k and $C_9 \sim \sum_{j=1}^{\infty} j^{-1/2-\delta/2}$.

We now put b_ε by

$$b_\varepsilon := \tilde{C}_5 (\tilde{C}_6 k/\varepsilon)^{k/2} + C_9 K_1^2 K_2^{k-2\delta} k^{k-\delta} I(\varepsilon)$$

with some $\tilde{C}_5$ and $\tilde{C}_6$. Combining the estimates for B_1-B_5, we thus obtain

$$\|\nabla^k u(t)\|_q \leq b_\varepsilon t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}} + \tilde{C}_8 \int_{(1-\varepsilon)t}^{t} (t-s)^{-1/2} s^{-1/2} \|\nabla^k u(s)\|_q ds$$

with some $\tilde{C}_8$ independent of k. Applying a Gronwall's type inequality (see [19, Lemma 2.4]), there exists $\varepsilon_k \in (0,1)$ such that

$$\|\nabla^k u(t)\|_q \le 2b_{\varepsilon_k} t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}}, \qquad t \in (0,T). \tag{4.4}$$

If $\varepsilon_k := 1/k$ then $I(1/k) \le \frac{1}{2(\tilde{C}_8)}$ for sufficiently large k, say $k \ge k_0 := k_0(n,p,M,M_1,M_2)$. Finally, we show $2b_{1/k} \le K_1(K_2 k)^{k-\delta}$ for any k with suitable constants K_1 and K_2. Choosing K_1 large enough (4.3) holds for $k \le k_0$, i.e., there exists a constant $K_0 > 0$ (depending only on n, p, M, M_1 and M_2) such that $\|\nabla^k u(t)\|_q \le K_0$ for $k \le k_0$. Since $I(1/k) \le 2$ for all $k \ge 2$,

$$2b_{1/k} \le 2\{\tilde{C}_5 \tilde{C}_6^{k-\delta} + 2C_9 K_1^2 K_2^{k-2\delta}\} k^{k-\delta}.$$

Choosing the constants K_1 and K_2,

$$K_1 := \max\left(K_0, 4\tilde{C}_5\right) \quad \text{and} \quad K_2 := \max\left(\tilde{C}_6, (4C_9K_1)^\delta\right),$$

we obtain (4.3) for all k. The proof is complete. □

REFERENCES

[1] H. Amann, *On the strong solvability of the Navier-Stokes equations.* J. Math. Fluid Mech., **2** (2000), 16-98.

[2] W. Arendt, Ch. Batty, M. Hieber and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems.* Birkhäuser, Basel, 2001.

[3] A. Banin, A. Mahalov and B. Nicolaenko, *Global regularity of 3D rotating Navier-Stokes equations for resonant domains.* Indiana Univ. Math. J. **48** (1999), 1133-1176.

[4] A. Banin, A. Mahalov and B. Nicolaenko, *3D Navier-Stokes and Euler equations with initial data characterized by uniformly large vorticity.* Indiana Univ. Math. J. **50** (2001), 1-35.

[5] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction.* Springer, Berlin, (1976).

[6] M. Cannone, *Harmonic analysis tools for solving the incompressible Navier-Stokes equations.* to appear in Handbook of Mathematical Fluid Dynamics, vol 3 (eds. S. Friedlander, D. Serre), Elsevier.

[7] M. Cannone and Y. Meyer, *Littlewood-Paley decomposition and Navier-Stokes equations.* Methods and Applications in Analysis, **2** (1995), 307-319.

[8] E.B. Fabes, B.F. Jones and N.M. Riviere, *The initial value problem for the Navier-Stokes equations with data in L^p.* Arch. Rational Mech. Anal., **45** (1972), 222-240.

[9] H. Fujita and T. Kato, *On the Navier-Stokes initial value problem I.* Arch. Rat. Mech. Anal., **16** (1964), 269-315.

[10] Th. Gallay and E. Wayne, *Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbb{R}^2$.* Arch. Rational Mech. Anal., **163** (2002), 209-258.

[11] M-H. Giga and Y. Giga, *Nonlinear Partial Differential Equations.* Kyouritsu Shuppan, 1999 (in Japanese).

[12] Y. Giga, *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system.* J. Differential Equations, **62** (1986), 186-212.

[13] Y. Giga, *On the two-dimensional nonstationary vorticity equations.* In 'Tosio Kato's Method and Principle for Evolution Equations in Mathematical Physics' (eds. H. Fujita et al.), pp. 86-97, University of Tokyo Press, Tokyo, (2002).

[14] Y. Giga, K. Inui and S. Matsui, *On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data.* Quaderni di Matematica, **4** (1999), 28-68.

[15] Y. Giga and T. Kambe, *Large time behaviour of the vorticity of two-dimensional viscous flow and its application to vortex formation.* Commun. Math. Phys., **117** (1988), 549-568.

[16] Y. Giga, S. Matsui and O. Sawada, *Global existence of two-dimensional Navier-Stokes flow with nondecaying initial velocity.* J. Math. Fluid Mech., **3** (2001), 302-315.

[17] Y. Giga and T. Miyakawa, *Solutions in L^r of the Navier-Stokes initial value problem.* Arch. Rational Mech. Anal., **89** (1985), 267-281.

[18] Y. Giga, T. Miyakawa and H. Osada, *Two-dimensional Navier-Stokes flow with measures as initial vorticity.* Arch. Rational Mech. Anal., **104** (1988), 223-250.

[19] Y. Giga and O. Sawada, *On regularizing-decay rate estimate for solutions to the Navier-Stokes initial value problem.* pp. 549-562, Nonlinear Anal. and Appl., Kluwer, Dordrecht, (2003).

[20] Y. Giga and K. Yamada, *On viscous Burgers-like equations with linearly growing initial data.* Bol. Soc. Parana. Math. (3), **20** (2003), 29-49.

[21] M. Hieber and O. Sawada, *The Ornstein-Uhlenbeck semigroup on exterior domains.* Preprint (2004).

[22] T. Hishida, *An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle.* Arch. Rat. Mech. Anal., **150** (1999), 307-348.

[23] T. Hishida, *The Stokes operator with rotation effect in exterior domains.* Analysis, **19** (1999), 51-67.

[24] T. Hishida, *L^2 theory for the operator $\Delta + (k \times x) \cdot \nabla$ in exterior domains.* Nihonkai Math J., **11** (2000), 103-135.

[25] C. Kahane, *On the spatial analyticity of solutions of the Navier-Stokes equations.* Arch. Rational Mech. Anal., **33** (1969), 386-405.

[26] J. Kato, *The uniqueness of nondecaying solutions for the Navier-Stokes equations.* Arch. Rat. Mech. Anal., **169** (2003), 159-175.

[27] T. Kato, *Strong L^p-solutions of Navier-Stokes equations in $\mathbb{R}^n$ with applications to weak solutions.* Math. Z., **187** (1984), 471-480.

[28] T. Kato and H. Fujita, *On the nonstationary Navier-Stokes system.* Rend. Sem. Mat. Univ. Padova, **32** (1962), 243-260.

[29] N. Kim and D. Chae, *On the uniqueness of the unbounded classical solutions of the Navier-Stokes and associated equations.* J. Math. Anal. Appl., **186** (1994), 91-96.

[30] T. Kobayashi and T. Muramatu, *Abstract Besov space approach to the nonstationary Navier-Stokes equations.* Math. Methods Appl. Sci., **15** (1992), 599-620.

[31] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes equations.* Adv. Math., **157** (2001), 22-35.

[32] H. Kozono, T. Ogawa and Y. Taniuchi, *Navier-Stokes equations in Besov spaces near L^∞ or BMO.* (preprint).

[33] H. Kozono and M. Yamazaki, *Semilenear heat equations and the Navier-Stokes equation with disributions in new function spaces as initial data.* Comm. Partial Differential Equations, **19** (1994), 959-1014.

[34] P. G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem.* A CRC Press, (2002).

[35] A. Majda, *Vorticity and the mathematical theory of incompressible fluid flow.* Comm. Pure Appl. Math., **34** (1986), 187-220.

[36] K. Masuda, *Weak solutions of Navier-Stokes equations.* Tohoku Math. J., **36** (1984), 623-646.

[37] G. Metafune, D. Pallara, E. Priola, *Spectrum of Ornstein-Uhlenbeck operators in L^p spaces with respect to invariant measures.* J. Funct. Anal., **196** (2002), 40-60.

[38] G. Metafune, J. Prüss, A. Rhandi, R. Schnaubelt, *The domain of the Ornstein-Uhlenbeck operator on an L^p-space with invariant measure.* Ann. Sc. Norm. Super. Pisa Cl. Sci., **1** (2002), 471-485.

[39] K. Ohkitani and H. Okamoto, *Blow-up problem modelled from the Strain-Vorticity dynamics.* In 'Tosio Kato's Method and Principle for Evolution Equations in Mathematical Physics' (eds. H. Fujita et al.), pp. 239-249, Yurinsha, Tokyo, (2002).

[40] H. Okamoto, *A uniqueness theorem for the unbounded classical solution of the nonstationary Navier-Stokes equations in $\mathbb{R}^3$.* J. Math. Anal. Appl., **181** (1994), 191-218.

[41] H. Okamoto, *Exact solutions of the Navier-Stokes equations via Leray's scheme.* Japan J. Indust. Appl. Math., **14** (1997), 169-197.

[42] O. Sawada, *On time-local solvability of the Navier-Stokes equations in Besov spaces.* Adv. Differential Equations, **8** (2003), 385-412.

[43] O. Sawada, *On analyticity rate estimates of the solutions to the Navier-Stokes equations in Bessel-potential spaces.* (preprint), 2003.

[44] O. Sawada, *The Navier-Stokes flow with linearly growing initial velocity in the whole space.* Bol. Soc. Paran. Mat., (to appear).

[45] O. Sawada and Y. Taniuchi, *On the Boussinesq flow with nondecaying initial data.* Funkcial. Ekvac., **47** (2004), 225-250.

[46] E.M. Stein, *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton Univ. Press, Princeton, 1993.

[47] H. Triebel, *Theory of Function Spaces.* Birkhäuser, Basel-Boston-Stuttgart, (1983).

[48] H. Triebel, *Theory of Function Spaces II.* Birkhäuser, Basel-Boston-Stuttgart, (1992).

[49] A. Tychonoff, *Théorèmes d'unicité pour l'équations de la chaleur.* Mat. Sbron., **42** (1935), 199-216.

[50] K. Yamada, *On viscous conservation laws with growing initial data.* Hokkaido University preprint series in Mathematics, (preprint).

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