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On the isomorphism problem for a one-parameter family of infinite measure preserving transformations

by
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1 Introduction and basic definitions

In this manuscript, we consider the isomorphism problem for a one-parameter family of non-invertible infinite measure preserving transformations, which we call α-Farey maps, $\frac{1}{2} \leq \alpha \leq 1$, together with the same problem for their natural extensions, based on [8]. The author has introduced the notion of the α-Farey maps as a generalization of the Farey map in [7]. The notion of the Farey map originally arose from the mediant convergents of the regular continued fractions, see [3], and this map induces the Gauss map as a jump transformation or an induced transformation, see [2]. Here, the Farey map F and the Gauss map T are defined by the following, respectively :

$$F(x) = \begin{cases} \frac{x}{1-x} & \text{if} \quad x \in [0, \frac{1}{2}), \\ \frac{1-x}{x} & \text{if} \quad x \in [\frac{1}{2}, 1], \end{cases}$$

$$T(x) = \begin{cases} \frac{1}{x} - \left[\frac{1}{x}\right] & \text{if} \quad x \in (0, 1], \\ 0 & \text{if} \quad x = 0, \end{cases}$$

where $[y]$ denotes the integer part of a real number y. If we put $\tau(x) = \min\{n \geq 0 : F^n(x) \in [\frac{1}{2}, 1]\}$, then we see that $T(x) = F^{\tau(x)+1}(x)$. It is well-known that T preserves the Gauss measure, which is given by $\frac{1}{\log 2} \cdot \frac{1}{1+x} dx$ and F preserves the infinite measure given by $\frac{1}{x} dx$. The α-Farey maps F_α are related in a similar way to the α-Gauss maps T_α, which are generalization of the Gauss map. Their maps are defined as follows explicitly. For $\frac{1}{2} \leq \alpha \leq 1$, we put $\mathbf{I}_\alpha = [\alpha - 1, \alpha]$.

Define the map T_α of $\mathbf{I}_\alpha$, which we call the α-Gauss map, by the following :

$$T_\alpha(x) = \begin{cases} \left|\frac{1}{x}\right| - \left[\left|\frac{1}{x}\right|\right]_\alpha & \text{if } \ x \in \mathbf{I}_\alpha \setminus \{0\}, \\ 0 & \text{if } \ x = 0, \end{cases}$$

here $[y]_\alpha = n$ if $y \in [n-1+\alpha,\, n+\alpha)$. We note that T_1 is the Gauss map and $T_{\frac{1}{2}}$ is the nearest integer continued fraction transformation. The fundamental properties of T_α were discussed in [6] together with some ergodic properties. In particular, it was shown that there exists an absolutely continuous ergodic invariant finite measure μ_α for T_α, which is given by

$$\mu_\alpha(A) = \int_A h_\alpha(x)\,dx \quad \text{for any measurable subset } A \subset \mathbf{I}_\alpha$$

with the following :

Case(i) $\frac{1}{2} \le \alpha < \frac{\sqrt{5}-1}{2}$

$$h_\alpha(x) = \begin{cases} \frac{1}{x+G+1} & \text{if } \ x \in [\alpha-1,\, \frac{1-2\alpha}{\alpha}] \\ \frac{1}{x+2} & \text{if } \ x \in (\frac{1-2\alpha}{\alpha},\, \frac{2\alpha-1}{1-\alpha}) \\ \frac{1}{x+G} & \text{if } \ x \in [\frac{2\alpha-1}{1-\alpha},\, \alpha) \end{cases} \tag{1.1}$$

with $G = \frac{\sqrt{5}+1}{2}$.
Case(ii) $\frac{\sqrt{5}-1}{2} \le \alpha \le 1$

$$h_\alpha(x) = \begin{cases} \frac{1}{x+2} & \text{if } \ x \in [\alpha-1,\, \frac{1-\alpha}{\alpha}] \\ \frac{1}{x+1} & \text{if } \ x \in (\frac{1-\alpha}{\alpha},\, \alpha). \end{cases} \tag{1.2}$$

Next, we put $\mathcal{J}_\alpha = [\alpha-1, 1]$ and define F_α of $\mathcal{J}_\alpha$, which we call the α-Farey map, by

$$F_\alpha(x) = \begin{cases} -\frac{x}{1+x} & \text{if } \ x \in [\alpha-1, 0) =: \mathcal{J}_{\alpha,1}, \\ \frac{x}{1-x} & \text{if } \ x \in [0, \frac{1}{2}) =: \mathcal{J}_{\alpha,2}, \\ \frac{1-2x}{x} & \text{if } \ x \in [\frac{1}{2}, \frac{1}{1+\alpha}] =: \mathcal{J}_{\alpha,3}, \\ \frac{1-x}{x} & \text{if } \ x \in (\frac{1}{1+\alpha}, 1] =: \mathcal{J}_{\alpha,4}. \end{cases}$$

As for the case between the Gauss map and the Farey map, we see that F_α induces T_α as a jump transformation for each α, $\frac{1}{2} \le \alpha \le 1$.

Proposition 1. *For $x \in \mathbf{I}_\alpha$, put*

$$\tau_\alpha(x) = \min_{n \ge 0}\{n \,:\, F_\alpha^n(x) \in \mathcal{J}_{\alpha,3} \cup \mathcal{J}_{\alpha,4}\}.$$

Then, we have

$$T_\alpha(x) = F_\alpha^{\tau_\alpha(x)+1}(x).$$

Corollary 1. *There exists an absolutely continuous invariant infinite measure ν_α for F_α. Moreover F_α is ergodic with respect to ν_α.*

The main claim of this manuscript is the following:

Main Result.
For any α and α', $\frac{1}{2} \leq \alpha \neq \alpha' \leq 1$, F_α and $F_{\alpha'}$ are not isomorphic, on the other hand, their natural extensions are isomorphic.

In the next section, we give the standard representation $\tilde{F}_\alpha$ of the natural extension of F_α. However, for the proof of the main result, a different representation $\hat{F}_\alpha$ of the natural extension is useful. We construct an isomorphism between these two representations $\tilde{F}_\alpha$ and $\hat{F}_\alpha$, and claim the fact that $\hat{F}_\alpha$ is also the natural extension of F_α. Then it is easy to see that $\hat{F}_\alpha$ is always isomorphic to $\hat{F}_1$ for any $\alpha, \frac{1}{2} \leq \alpha \leq 1$. In the following, we show that F_α and $F_{\alpha'}$ are not isomorphic whenever $\alpha \neq \alpha'$. In general, if two ergodic probability measure preserving transformations are isomorphic, then the measure of a measurable subset and that of its image by the isomorphism have to be the same. We may use this fact to prove the non-isomorphy of two transformations. In the case of infinite measure preserving transformations, these measures are not necessarily the same but they must be constant multiples of each other, see later. For the α-Farey maps with the invariant measures ν_α given in §2, we see that the multiplicative constant is always equal to 1, which is shown in Lemma 1, §2. With this fact, we prove the non-isomorphy of $\{F_\alpha : \frac{1}{2} \leq \alpha \leq 1\}$ by using the above idea.

In the sequel, we give basic definitions and some facts on the natural extension. The construction of the natural extension, which is stated later, will be used for T_α and F_α in the next section. Let T_i be ergodic σ-finite measure preserving transformations defined on the standard measure spaces $(X_i, \mathbf{B}_i, m_i)$, $i = 1, 2$. A map π from X_1 onto X_2 is said to be a factor map from T_1 to T_2 if $\pi \circ T_1 = T_2 \circ \pi$ (m_1-a.e.), $\pi^{-1}\mathbf{B}_2 \subset \mathbf{B}_1$ (mod.0) and there exists a positive constant c such that

$$m_1 \circ \pi^{-1}(A) = c \cdot m_2(A) \quad \text{for any } A \in \mathbf{B}_2.$$

If a map $\pi : X_1 \to X_2$ is a factor map from $(X_1, \mathbf{B}_1, m_1, T_1)$ to $(X_2, \mathbf{B}_2, m_2, T_2)$, then we write $\pi : T_1 \to T_2$ for brevity.

Definition 1 (Factor, Extension). *If there exists a factor map from T_1 to T_2, then T_1 is said to be an extension of T_2 and T_2 is said to be a factor of T_1.*

Definition 2 (Isomorphism, Isomorphic). *If there exists a factor map $\pi : T_1 \to T_2$ with the constant c such that π is a one-to-one onto map and $\pi^{-1} : T_2 \to T_1$ is also a factor map with the constant $1/c$, then T_1 and T_2 are said to be isomorphic and π is said to be an isomorphism.*

Definition 3 (Natural extension). *A measure preserving transformation $\tilde{T}$ is said to be a natural extension of a measure preserving transformation T if*

$\tilde{T}$ is a minimal invertible extension of T, that is, if S is invertible and is an extension of T, then S is an extension of $\tilde{T}$.

About the existence and the uniqueness of the natural extension, we quote the following fact, see Theorem 3.1.5 and 3.1.6, J. Aaronson [1] :

Theorem. *For any ergodic measure preserving transformation T of a standard, σ-finite measure space, there exists a natural extension $\tilde{T}$ on a standard space. Moreover, if $\tilde{T}$ and $\bar{T}$ are natural extensions of T, $\tilde{T}$ and $\bar{T}$ are isomorphic.*

One standard way of constructing the natural extension is the following. Let T be an ergodic σ-finite measure preserving transformation of a σ-finite standard measure space $(X, \mathbf{B}, m)$. We put

$$\mathbf{X} = \prod_0^\infty X = \{(x_0, x_1, x_2, \dots) : x_i \in X,\ i \geq 0\}.$$

Define

$$\tilde{\mathbf{X}} = \{(x_0, x_1, x_2, \dots) \in \mathbf{X} : x_i = T(x_{i+1}),\ i \geq 0\} \tag{1.3}$$

and

$$\tilde{T}(x_0, x_1, x_2, \dots) = (Tx_0, x_0, x_1, \dots) \quad \text{for } (x_0, x_1, x_2, \dots) \in \tilde{\mathbf{X}}. \tag{1.4}$$

Let $\tilde{\mathbf{B}}$ be the σ-algebra of $\tilde{\mathbf{X}}$ which is generated by the sets of the form

$$\bigcap_{k=0}^{n} T^{-(n-k)} A_k$$

for any $n \geq 0$ and $A_0, A_1, \dots, A_n \in \mathbf{B}$. Then we define the measure $\tilde{m}$ on $\tilde{\mathbf{X}}$ by

$$\tilde{m}(\{(x_0, x_1, x_2, \dots) \in \tilde{\mathbf{X}} : x_k \in A_k,\ 0 \leq k \leq n\}) = m\left(\bigcap_{k=0}^{n} T^{-(n-k)} A_k\right). \tag{1.5}$$

It is possible to show that $\tilde{T}$ on $(\tilde{\mathbf{X}}, \tilde{\mathbf{B}}, \tilde{m})$ is a representation of the natural extension of T on $(X, \mathbf{B}, m)$, see p90, [1].

2 Main theorem

We denote by $(\tilde{\mathbf{I}}_\alpha, \tilde{\mu}_\alpha, \tilde{T}_\alpha)$ and $(\tilde{\mathcal{J}}_\alpha, \tilde{\nu}_\alpha, \tilde{F}_\alpha)$ the natural extensions of T_α and F_α given by (1.3) and (1.4), respectively, that is,

$$\tilde{\mathbf{I}}_\alpha = \{(x_0, x_1, x_2, \dots) \in \prod_0^\infty \mathbf{I}_\alpha : x_i = T_\alpha(x_{i+1}),\ i \geq 0\},$$

$$\tilde{T}_\alpha(x_0, x_1, x_2, \dots) = (T_\alpha x_0, x_0, x_1, \dots) \quad \text{for } (x_0, x_1, x_2, \dots) \in \tilde{\mathbf{I}}_\alpha$$

and

$$\tilde{\mathcal{J}}_\alpha = \{(z_0, z_1, z_2, \dots) \in \prod_0^\infty \mathcal{J}_\alpha : z_i = F_\alpha(z_{i+1}),\ i \geq 0\},$$

$$\tilde{F}_\alpha(z_0, z_1, z_2, \dots) = (F_\alpha z_0, z_0, z_1, \dots) \quad \text{for } (z_0, z_1, z_2, \dots) \in \tilde{\mathcal{J}}_\alpha.$$

Moreover, $\tilde{\mu}_\alpha$ and $\tilde{\nu}_\alpha$ are the measures given by (1.5). We do not mention the σ-algebras specifically since they are induced by cylinder sets and the construction of measures $\tilde{\mu}_\alpha$ and $\tilde{\nu}_\alpha$ make it clear what these σ-algebras are.

Concerning the isomorphism problem for α-Gauss maps T_α, the following results hold: For any α and α', $\frac{\sqrt{5}-1}{2} \leq \alpha \neq \alpha' \leq 1$, T_α and $T_{\alpha'}$ ($\tilde{T}_\alpha$ and $\tilde{T}_{\alpha'}$) are not isomorphic, since their metrical entropy values are different each other. On the other hand, for $\frac{1}{2} \leq \alpha \leq \frac{\sqrt{5}-1}{2}$, C. Kraaikamp has proved that $\tilde{T}_\alpha$ are isomorphic each other, see [4]. Here we note the following result.

Theorem 1. *For any α and α', $\frac{1}{2} \leq \alpha \neq \alpha' \leq \frac{\sqrt{5}-1}{2}$, we have T_α and $T_{\alpha'}$ are not isomorphic.*

We can prove this theorem by showing that the set of Jacobian is different each other for different values of α. Concerning the isomorphism problem for α-Farey maps F_α, we have the following result, which is the main theorem.

Theorem 2 ([8]). *For any α and α', $\frac{1}{2} \leq \alpha \neq \alpha' \leq 1$, we have*
(i) F_α and $F_{\alpha'}$ are not isomorphic,
(ii) $\tilde{F}_\alpha$ and $\tilde{F}_{\alpha'}$ are isomorphic.

Now, we prove Theorem 2 in several steps. We note that we prove the assertion (ii) at first and then the assertion (i). As the first step, we show a relation between $\tilde{F}_\alpha$ and $\tilde{T}_\alpha$.

Put

$$\tilde{\mathcal{J}}_{\alpha,0} = \{\mathbf{z} = (z_0, z_1, z_2, \dots) \in \tilde{\mathcal{J}}_\alpha : z_1 \in \mathcal{J}_{\alpha,3} \cup \mathcal{J}_{\alpha,4}\}.$$

We denote by $(\tilde{F}_\alpha)_{\tilde{\mathcal{J}}_{\alpha,0}}$ the induced transformation of $\tilde{F}_\alpha$ to $\tilde{\mathcal{J}}_{\alpha,0}$, that is,

$$(\tilde{F}_\alpha)_{\tilde{\mathcal{J}}_{\alpha,0}}(\mathbf{z}) = \tilde{F}_\alpha^{\tau_\alpha(z_0)+1}(\mathbf{z}).$$

Proposition 2. *$(\tilde{F}_\alpha)_{\tilde{\mathcal{J}}_{\alpha,0}}$ and $\tilde{T}_\alpha$ are isomorphic.*

Next, we consider different representations of the natural extensions of T_α and F_α, respectively, for the proof of the main theorem.

We put

$$\hat{\mathbf{I}}_\alpha = \begin{cases} [\alpha - 1, \frac{1-2\alpha}{\alpha}) \times [-\infty, -\frac{\sqrt{5}+3}{2}] \cup [\frac{1-2\alpha}{\alpha}, \frac{2\alpha-1}{1-\alpha}) \times [-\infty, -2] \\ \cup [\frac{2\alpha-1}{1-\alpha}, \alpha] \times [-\infty, -\frac{\sqrt{5}+1}{2}] & \text{if } \frac{1}{2} \leq \alpha \leq \frac{\sqrt{5}-1}{2}, \\ [\alpha - 1, \frac{1-\alpha}{\alpha}) \times [-\infty, -2] \cup [\frac{1-\alpha}{\alpha}, \alpha] \times [-\infty, -1] & \text{if } \frac{\sqrt{5}-1}{2} \leq \alpha \leq 1 \end{cases}$$

and

$$\hat{\mathcal{J}}_\alpha = \begin{cases} [\alpha-1, \frac{1-2\alpha}{\alpha}) \times [-\infty, -\frac{\sqrt{5}+3}{2}] \cup [\frac{1-2\alpha}{\alpha}, 0) \times [-\infty, -2] \cup [0,\alpha) \times [-\infty, 0] \\ \cup [\alpha, \frac{1-\alpha}{\alpha}) \times [-\frac{\sqrt{5}+1}{2}, 0] \cup [\frac{1-\alpha}{\alpha}, 1] \times [-1, 0] \qquad \text{if } \frac{1}{2} \le \alpha \le \frac{\sqrt{5}-1}{2}, \\ [\alpha-1, 0) \times [-\infty, -2] \cup [0,\alpha) \times [-\infty, 0] \cup [\alpha, 1] \times [-1, 0] \\ \qquad \text{if } \frac{\sqrt{5}-1}{2} \le \alpha \le 1. \end{cases}$$

Define a measure $\hat{\nu}_\alpha$ on $\hat{\mathcal{J}}_\alpha$ by

$$d\hat{\nu}_\alpha = \hat{g}_\alpha(x,y)dxdy \quad \text{for} \quad (x,y) \in \hat{\mathcal{J}}_\alpha$$

with $\hat{g}_\alpha(x,y) = \frac{1}{(x-y)^2}$, and $\hat{\mu}_\alpha$ denotes the restriction of $\hat{\nu}_\alpha$ to $\hat{\mathbf{I}}_\alpha$, that is,

$$d\hat{\mu}_\alpha = \hat{g}_\alpha(x,y)dxdy \quad \text{for} \quad (x,y) \in \hat{\mathbf{I}}_\alpha.$$

We define maps $\hat{T}_\alpha$ of $\hat{\mathbf{I}}_\alpha$ and $\hat{F}_\alpha$ of $\hat{\mathcal{J}}_\alpha$ by

$$\hat{T}_\alpha(x,y) = \left(\frac{\varepsilon_{\alpha,1}(x)}{x} - c_{\alpha,1}(x),\ \frac{\varepsilon_{\alpha,1}(x)}{y} - c_{\alpha,1}(x)\right)$$

and

$$\hat{F}_\alpha(x,y) = \begin{cases} (-\frac{x}{1+x}, -\frac{y}{1+y}) & \text{if } x \in \mathcal{J}_{\alpha,1} \\ (\frac{x}{1-x}, \frac{y}{1-y}) & \text{if } x \in \mathcal{J}_{\alpha,2} \\ (\frac{1-2x}{x}, \frac{1-2y}{y}) & \text{if } x \in \mathcal{J}_{\alpha,3} \\ (\frac{1-x}{x}, \frac{1-y}{y}) & \text{if } x \in \mathcal{J}_{\alpha,4}, \end{cases}$$

respectively.

Proposition 3. *$\hat{F}_\alpha$ is a one-to-one onto map of $\hat{\mathcal{J}}_\alpha$ except for a set of Lebesgue measure 0 and $\hat{\nu}_\alpha$-preserving.*

We will construct an isomorphism from $(\tilde{F}_\alpha, \tilde{\nu}_\alpha)$ to $(\hat{F}_\alpha, \hat{\nu}_\alpha)$. We note that the x-marginal density of $\hat{\mu}_\alpha$ coincides with $h_\alpha(x)$, see (1.1) and (1.2). Moreover the marginal distribution of $\hat{\nu}_\alpha$ gives the absolutely continuous invariant measure ν_α for F_α, that is,

$$d\nu_\alpha(x) = g_\alpha(x)dx$$

with

$$g_\alpha(x) = \int_{\{y:(x,y)\in\hat{\mathcal{J}}_\alpha\}} \hat{g}_\alpha(x,y)dy. \tag{2.1}$$

Note that $\hat{\nu}_\alpha(\hat{\mathcal{J}}_\alpha) = \nu_\alpha(\mathcal{J}_\alpha) = \infty$. If we change y to $w = -\frac{1}{y}$ for $(x,y) \in \hat{\mathbf{I}}_\alpha$, then we get the natural extension that was discussed in [6].

Proposition 4. *There exists an isomorphism $\xi : \tilde{T}_\alpha \to \hat{T}_\alpha$ such that $\tilde{\mu}_\alpha \circ \xi^{-1} = \hat{\mu}_\alpha$.*

Proposition 5. *$\hat{T}_\alpha$ and $(\hat{F}_\alpha)_{\hat{\mathbf{I}}_\alpha}$ are isomorphic.*

Proposition 6. *$\hat{F}_\alpha$ and $\tilde{F}_\alpha$ are isomorphic.*

Proposition 7. *$\hat{F}_\alpha$ and $\hat{F}_1$ are isomorphic.*

We note that for any $\alpha, \frac{1}{2} \le \alpha \le 1$, $\hat{F}_\alpha$ and $\hat{F}_1$ are isomorphic with the isomorphism $\hat{\psi} : \hat{F}_\alpha \to \hat{F}_1$ which is given by the following :

$$\hat{\psi}(x, y) = \begin{cases} (x+1, y+1) & \text{if } \ x \in [\alpha - 1, 0) \\ (x, y) & \text{if } \ x \in [0, 1]. \end{cases}$$

The statement (ii) of Theorem 2 follows from Proposition 6 and 7. Now we show the statement (i). We start with the following lemma.

Lemma 1. *Suppose that $\tilde{\pi}$ is an isomorphism from $\tilde{F}_\alpha$ to $\tilde{F}_{\alpha'}$ such that $\tilde{\nu}_\alpha \circ \tilde{\pi}^{-1} = \tilde{\nu}_{\alpha'}$, then $\tilde{\nu}_\alpha \circ \tilde{\pi}'^{-1} = \tilde{\nu}_{\alpha'}$ for any isomorphism $\tilde{\pi}'$ from $\tilde{F}_\alpha$ to $\tilde{F}_{\alpha'}$.*

Proof. The induced transformation of F_1 to $[\frac{1}{2}, 1]$ is isomorphic to T_1 by the isomorphism $\frac{1}{x} - 1$. Thus, F_1 is pointwise dual ergodic since T_1 is continued fraction mixing. Hence, $\tilde{F}_1$ is rationally ergodic which implies $\tilde{F}_\alpha$ is also rationally ergodic for any α, $\frac{1}{2} \le \alpha \le 1$. The rational ergodicity of $\tilde{F}_\alpha$ implies that $\tilde{F}_\alpha$ has a law of large numbers in the sense of [1]. Then $\tilde{\nu}_\alpha \circ \tilde{\eta}^{-1} = \tilde{\nu}_\alpha$ for any isomorphism $\tilde{\eta} : \tilde{F}_\alpha \to \tilde{F}_\alpha$, see 3.3.1 and Definition and Remark 1, p96 in [1]. This shows the assertion of the lemma since $\tilde{\pi}'^{-1} \circ \tilde{\pi} : \tilde{F}_\alpha \to \tilde{F}_\alpha$ is an isomorphism. We refer to pp93 – 99 and pp118 – 128, [1] for the detail. □

Let

$$E_{\alpha,i} = \{x \in \mathcal{J}_\alpha : \sharp F_\alpha^{-1}(x) = i\}, \quad i = 1, 2.$$

We also need the following lemma :

Lemma 2. *We have*

$$(\nu_\alpha(E_{\alpha,1}),\ \nu_\alpha(E_{\alpha,2})) \ne (\nu_{\alpha'}(E_{\alpha',1}),\ \nu_{\alpha'}(E_{\alpha',2}))$$

for any α and α', $\frac{1}{2} \le \alpha \ne \alpha' \le 1$.

Proof of Theorem 2 (i).
Suppose that $\phi : F_\alpha \to F_{\alpha'}$ is an isomorphism. Then ϕ induces the isomorphism $\tilde{\phi}$ from $\tilde{F}_\alpha$ to $\tilde{F}_{\alpha'}$ by $(z_0, z_1, z_2, \dots)$ to $(\phi(z_0), \phi(z_1), \phi(z_2), \dots)$. From Proposition 6, Proposition 7 and their proofs, it is possible to construct an isomorphism $\tilde{\vartheta} : \tilde{F}_\alpha \to \tilde{F}_{\alpha'}$ such that $\tilde{\nu}_\alpha \circ \tilde{\vartheta}^{-1} = \tilde{\nu}_{\alpha'}$. From Lemma 1, we see $\tilde{\nu}_\alpha \circ \tilde{\phi}^{-1} = \tilde{\nu}_{\alpha'}$. This implies $\nu_\alpha \circ \phi^{-1} = \nu_{\alpha'}$. Since

$$\phi^{-1}(E_{\alpha',i}) = E_{\alpha,i}, \quad i = 1, 2,$$

we have

$$\nu_{\alpha'}(E_{\alpha',i}) = \nu_\alpha(\phi^{-1}(E_{\alpha',i})) = \nu_\alpha(E_{\alpha,i}).$$

This is impossible by Lemma 2. $\square$

Remark. If "$\frac{\nu_\alpha(E_{\alpha,1})}{\nu_\alpha(E_{\alpha,2})} \neq \frac{\nu_{\alpha'}(E_{\alpha',1})}{\nu_{\alpha'}(E_{\alpha',2})}$" holds for any $\alpha \neq \alpha'$, then we do not need Lemma 1. However, for some α, there exists $\alpha' \neq \alpha$ such that

$$\frac{\nu_\alpha(E_{\alpha,1})}{\nu_\alpha(E_{\alpha,2})} = \frac{\nu_{\alpha'}(E_{\alpha',1})}{\nu_{\alpha'}(E_{\alpha',2})}$$

holds.

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