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THERMODYNAMICS OF SYSTEMS WITH NONZERO EXPONENTS: MORE QUESTIONS THAN ANSWERS

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1. Thermodynamic formalism in dynamical systems

We discuss the thermodynamic formalism, i.e., an adaptation of the formalism of equilibrium statistical physics to dynamical systems. It was developed in the classical works of Sinai, Ruelle, and Bowen and it deals with a continuous map f of a compact metric space X and a continuous function φ on X. Its main constituent components are: 1) the *topological pressure* $P(\varphi)$; 2) the *variational principle*, $P(\varphi)=\sup_{\mu} E(\varphi,\mu)$ where $E(\varphi,\mu)=h_{\mu}(f)+\int_{X}\varphi\,d\mu$ is (up to a normalizing factor) the *free energy* and the supremum is taken over all f-invariant Borel probability measures on X; 3) the *equilibrium measures* μ_{φ} for φ for which the above supremum is attained.

Existence and uniqueness of equilibrium measures depend upon the dynamical properties of the map f and the regularity of the potential φ. The classical result claims that if f is a topologically transitive Anosov or (an axiom A diffeomorphism) and φ is a Hölder continuous function, there exists a unique equilibrium measure for φ (see for example, [22]). Many important invariant measures are equilibrium measures. Among them are:

(1) measures of maximal entropy (corresponding to $\varphi=0$);
(2) absolutely continuous invariant measures or more general Sinai-Ruelle-Bowen (SRB)-measres (corresponding to $\varphi(x)=-\log \operatorname{Jac}(df|E^{u}(x))$ where $E^{u}(x)$ is the unstable subspace at the point x);
(3) measures of maximal dimension (corresponding to $\varphi_{t}(x)=-t\log \operatorname{Jac}(df|E^{u}(x))$ where t is the root of the *Bowen equation* $P(\varphi_{t}(x))=0$).

Developing thermodynamics of systems which are nonuniformly hyperbolic faces several obstacles. One of them is the discontinuity of the potential. For example, the natural potential $\varphi(x)=-\log \operatorname{Jac}(df|E^{u}(x))$ is a measurable but not necessarily continuous function (since the unstable distribution $E^{u}(x)$ depends measurably on the base point x; it may not even be bounded as in the case of one-dimensional maps with critical points). For these potentials a new concept of the topological pressure is required. Furthermore, one may have to reduce the class of invariant measures under consideration to the one for which the potential is integrable thus making a new setup for the variational principle. Finally, new methods are needed to establish the existence and uniqueness of equilibrium measures.

It has been one of the major achievements in the recent theory of dynamical systems to construct absolutely continuous invariant measures for some one-dimensional maps (unimodal and some multimodal) and SRB-measures for some two or higher dimensional nonuniformly

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hyperbolic attractors (Hénon-like attractors, partially hyperbolic attractors with negative central exponents, etc.). As far as other equilibrium measures are concerned, Bruin and Keller [10] have shown that for a unimodal map satisfying the Collet-Eckmann condition and any t sufficiently closed to 1 there exists a unique equilibrium measure for the potential $\varphi_t(x) = -t \log \operatorname{Jac}(df|E^u(x))$. On another but related direction, Hofbauer, [19, 20] (see also [23]) established existence of measures of maximal entropy for piecewise monotonic transformations of an interval with positive topological entropy, obtained an upper bound for their number and in particular, proved their uniqueness in the case of unimodal maps; Buzzi, [13, 14] extended these results to multidimensional piecewise expanding maps and also showed that under some very general assumptions acim can be viewed as equilibrium measures.

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2. Thermodynamics for nonuniformly hyperbolic maps

We describe an approach to the thermodynamic formalism for nonuniformly hyperbolic maps which can be represented as a tower of a special type. This class of maps was introduced by Young in [37]. In the one-dimensional case this class includes unimodal and multimodal maps. Let us begin with this case.

Consider a one-parameter family of unimodal maps f_a of a compact interval I. It is well-known (see [36] and [31]) that for a typical transverse family there is a set A of parameters of positive Lebesgue measure such that for every $a \in A$ the map $f = f_a$ admits an "inducing scheme" (S, τ), where S is a countable collection of disjoint closed intervals and $\tau : S \to \mathbb{N}$ a positive integer-valued function such the the following properties hold:

(H1) $f^{\tau(J)}(J) \supseteq \mathcal{W}$ where $\mathcal{W} = \bigcup_{J\in S} J$ is the *inducing domain*;

(H2) there exists $\lambda > 1$ with $|dF(x)| > \lambda$ where $F(x) = f^{\tau(J)}|J(x)$, $x \in J$ is the *induced map*;

(H3) (*Markov property*) if $J, J' \in S$, $0 \leq i \leq \tau(J) - 1$, and $f^i(J) \cap J' \neq \emptyset$, then $f^i(J) \cap J' = f^i(J)$;

(H4) for all $n \geq 0$,

$$\sum_{J\in S:\,\tau(J)\geq n} |J| \leq c_1 \lambda_1^{-n};$$

(H5) (Bounded Distortion) for each $n \geq 0$, each interval $I_{[b_0,\dots,b_n]}$ and each $x, y \in I_{[b_0,\dots,b_n]}$,

$$\left| \frac{dF^n(x)}{dF^n(y)} - 1 \right| \leq c_5 |F^n(x) - F^n(y)|.$$

In fact, for $a \in A$ the map $f = f_a$ satisfies the Collet-Eckmann condition, i.e., there exist positive constants c and $\theta > 1$ such that

$$|Df^n(f(0))| > c\, \theta^n$$

for every $n \geq 0$. Note that in [36] and [31] the authors require that f has negative Schwarzian derivative, however, this requirement can be dropped in view of [18].

Set

$$W = \bigcap_{n\geq 0} F^{-n}(\mathcal{W}) = \lim_{n\to\infty} F^{-n}(\mathcal{W}), \quad X = \bigcup_{J\in S} \bigcup_{k=0}^{\tau(J)-1} f^k(W\cap J).$$

The set W is F-invariant and for $a \in A$ it has positive Lebesgue measure. The inducing scheme represents (X, f) as a tower with the base (W, F).

Let $\mathcal{R}$ be the cover of $\mathcal{W}$ by the intervals $J \in S$. It is "almost" a cover: any two intervals are either disjoint or intersect by their endpoints. This cover induces a cover of W which we denote by the same letter. In view of (H2) the induced map F is equivalent up to a countable set to the *full shift of countable type* $(S^{\mathbb{N}}, \sigma)$ where $S^{\mathbb{N}}$ is the space of one-sided infinite sequences with elements in S and σ the shift operator. One can apply some recent results of Mauldin and Urbanski, [25], of Sarig, [29], [30], of Buzzi and Sarig [12] and of Yuri [38, 39, 17] on the existence and uniqueness of Gibbs measures for the Bernoulli shift (or more general subshifts of countable type). More precisely, let $\Phi : S^{\mathbb{N}} \to \mathbb{R}$ be a function. Assume that Φ has *summable variations*, i.e.,

$$\sum_{n\geq 1} V_n(\Phi) < \infty, \tag{1}$$

where $V_n(\Phi) = \sup_{[b_0,\dots,b_{n-1}]} \sup_{a,a'\in[b_0,\dots,b_{n-1}]}\{|\Phi(a)-\Phi(a')|\}$ and $[b_0,\dots,b_{n-1}]$ is the cylinder. A measure $\nu = \nu_\Phi$ on $S^{\mathbb{N}}$ is said to be a *Gibbs measure* for Φ if there exist constants $C_1 > 0$ and $C_2 > 0$ such that for any cylinder set $[b_0,\dots,b_{n-1}]$ and any $a \in [b_0,\dots,b_{n-1}]$ we have

$$C_1 \leq \frac{\nu([b_0,\dots,b_{n-1}])}{\exp\left(-nP_G(\Phi) + \Phi_n(a)\right)} \leq C_2. \tag{2}$$

Here

$$P_G(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n(a)=a} \exp \Phi_n(a)$$

equals to the *Gurevich pressure* of Φ and $\Phi_n(a) = \sum_{k=0}^{n-1} \Phi(\sigma^k(a))$ (under Assumption (1) the limit exists and does not depend on a). For the following result see [1], [2], [25], [30], [12]) and [38, 39, 17].

Theorem 2.1. *Assume that the function Φ satisfies: 1) $\sup\Phi < \infty$; 2) $P_G(\Phi) < \infty$; 3) Φ has summable variations (with respect to F). Then the variational principle for Φ holds:*

$$P_G(\Phi) = \sup_{\nu\in\mathcal{M}_\Phi} \{h_\nu(\sigma) + \int \Phi d\nu \},$$

where $\mathcal{M}_\Phi$ is the space of all shift invariant Borel probability measures ν on $S^{\mathbb{N}}$ for which $\int \Phi d\nu > -\infty$. Moreover, there exists an ergodic shift invariant Gibbs measure ν_Φ for Φ. If, in addition, the entropy $h_{\nu_\Phi}(\sigma) < \infty$ then $\nu_\Phi \in \mathcal{M}_\Phi$ and is a unique Gibbs and equilibrium measure.

As a corollary one has equilibrium (Gibbs) measures for the induced map F. One can lift them from the inducing domain to the tower. The latter procedure is quite subtle and requires integrability of the inducing time with respect to the Gibbs measures and their lifts. It also uses a correspondence between measures on W invariant under the induced map F and measures on X invariant under f and, in particular, a relation between their entropies

– the so-called Abramov formula. For general towers this has been recently obtained by Zweimüller [41] (see also [23] and [9]).

As a result, given a unimodal map f, admitting an inducing scheme (S, τ), one can single out a class of potentials for which the thermodynamic formalism can be affected. In fact, it is shown (see [28]) that *any* one-dimensional map admitting an inducing scheme as described above possesses a class of potentials for which there exists a unique equilibrium measure supported on X. More precisely, given a potential $\varphi : X \to \mathbb{R}$ define the *induced potential* $\phi : W \to \mathbb{R}$ by $\phi(x) = \sum_{k=0}^{\tau(J)-1} \varphi(f^k(x))$, where $x \in J$.

Theorem 2.2 ([28]). *Assume that the potential φ is such that the induced potential ϕ satisfies Conditions 1-3 of Theorem 2.1. Then there exists a unique invariant Borel ergodic measure μ_φ for which*

$$h_{\mu_\varphi}(f) + \int_X \varphi\, d\mu_\varphi = \sup\{h_\mu(f) + \int_X \varphi\, d\mu\}, \tag{3}$$

where the supremum is taken over all invariant Borel probability measures on X for which the inducing time is integrable, $\tau \in L^1(X, \mu)$.

As shown in [28] Theorem 2.2 applies to the family of potential functions $\varphi_t(x) = -t \log |df(x)|$, $x \in I$ for $t \geq 0$ (to prove this one may have to reduce the set of parameters under consideration to a set $A' \subset A$ which still has positive measure). One can further apply Theorem 2.2 to multimodal maps using recent results of Bruin, Luzzatto and van Strien [11] who constructed inducing schemes and the corresponding Markov extensions for multimodal maps with a finite critical set and no stable or neutral periodic point.

Extending the above approach from one-dimensional to multi-dimensional systems, consider the class of dynamical systems introduced by Young in [37]. Namely, let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M. Assume that there exist

(Y1) a set Λ with a hyperbolic product structure (the structure of a "horseshoe") generated by transverse families of stable and unstable disks, γ^s and γ^u; it is assumed that $\mathrm{Leb}_{\gamma^u}\{\gamma^u \cap \Lambda\} > 0$ for every $\gamma^u \in \Gamma^u$, that f is absolutely continuous along γ^s and that it has the distortion property along γ^u;

(Y2) a return map f^R ($R = R(x)$ is a measurable integer-valued function and is not necessarily the first return time) from Λ to itself such that f is a Markov extension over $F = f^R$; the function R is assumed to be integrable, i.e., for some $\gamma^u \in \Gamma^u$,

$$\int_{\gamma^u \cap \Lambda} R\, d\mu_{\gamma^u} < \infty.$$

Unlike the one-dimensional case the induced map F is modeled by the full shift acting on the space of *two-sided* infinite sequences. Still, Proposition 2.1 can be used to establish existence and uniqueness of equilibrium measures.

3. Open problems

Here are some challenging open problems in the thermodynamics of systems with nonzero Lyapunov exponents.

Problem 3.1. *Given a diffeomorphism f satisfying Conditions (Y1) and (Y2) above, describe a class $\mathcal{P}$ of potential functions φ for which there exists a unique equilibrium measure μ_φ.*

This class should include the family of potentials $\varphi_t(x) = -t \log \operatorname{Jac}(df|E^u(x))$ for t in some interval (t_0, t_1) around 1 and hence, it should include SRB-measures. It is of principle interest to include also the value $t = 0$ and thus to establish existence and uniqueness of measures of maximal entropy.

Problem 3.2. *Under what conditions the equilibrium measures are ergodic, Bernoulli, have exponential decay of correlations and satisfy the Central Limit Theorem.*

One should expect an exponential rate of decay of correlations and the Central Limit Theorem provided R has an exponential tail with respect to the function $\varphi \in \mathcal{P}$, i.e., there exist $K > 0$ and $0 < \theta < 1$ such that for all $n > 0$,

$$\mu_\varphi(\{x \in \Lambda : R(x) \geq n\}) \leq K\theta^n.$$

Problem 3.3. *Extend the notion of the topological pressure $P(\varphi)$ to functions in $\mathcal{P}$ and establish the variational principle.*

Since these functions are not in general continuous, the classical notion of the topological pressure does not apply and one may use an approach based on representing the pressure as a Carathéodory dimension. This approach was developed in [27] and allows one to extend the classical concept of the topological pressure to systems acting on non-compact spaces, to some sequences of potential functions (the so-called *non-additive* pressure) which may be used to approximate a given potential, etc.

A progress in solving these problems may, in particular, help study continuity, differentiability and other properties of the *pressure function* $P(t) = P(-t \log \operatorname{Jac}(df|E^u(x)))$ which, in turn, determine the multifractal analysis of non-uniformly hyperbolic systems described above.

4. Examples

We describe two examples of dynamical systems admitting Markov extensions, for which the above problems are of particular interest.

4.1. **The Hénon family.** This is the family of maps given by $H_{a,b}(x, y) = (1 - ax^2 + by, x)$. Observe that for $b = 0$ the family $H_{a,b}$ reduces to the logistic family Q_a. By continuity, given $a \in (0, 2)$, there is a rectangle in the plane which is mapped by $H_{a,b}$ into itself. and hence, $H_{a,b}$ has an attractor provided b is sufficiently small. Benedicks and Carleson [5], treating $H_{a,b}$ as small perturbations of Q_a, developed highly sophisticated techniques to describe the dynamics near the attractor. Building on this analysis, Benedicks and Young [7, 8] established existence of SRB-measures for the Hénon attractors and described their ergodic properties. In [33], Wang and Young introduced a 2-parameter family of maps of the plane to which the above results extend. Later they have further generalized their construction to higher dimensions, [34]. Mora and Viana [26] modified Benedicks and Carleson's approach in a way which allowed them to treat Hénon-like maps using some techniques from the general bifurcation theory such as homoclinic tangencies. Later Viana [32] extended results from [26] to higher dimensions.

Based on these results one should be able to construct an inducing scheme in the above mentioned sense and hence, to represent the Hénon map $H_{a,b}$ as a Markov extension for values of parameters (a, b) of positive Lebesgue measure.

4.2. **The Katok's map.** Unlike the Hénon map, this example is an area-preserving diffeomorphism f of the 2-torus with non-zero Lyapunov exponents constructed by Katok in [21] (see also [4]). Starting with a hyperbolic toral automorphism A, one obtains f by slowing down A near the origin, so that the time the trajectories spend near 0 increases. By the Poincaré recurrence theorem for almost every trajectory the average increase in time is not significant and cannot substantially change the values of the Lyapunov exponent along the trajectory. However, for some trajectories (which form a set of zero area) the average increase in time may be abnormally high to allow zero Lyapunov exponents.

Observe that Katok's map lies in the core of the construction of a volume preserving Bernoulli C^∞ diffeomorphism with nonzero Lyapunov exponents on any manifold (see [16]). Solving the above problem for Katok's map will allow to construct diffeomorphisms with Markov extensions on any manifold and thus, affect the thermodynamic formalism for these diffeomorphisms.

References

[1] J. Aaronson, *An introduction to infinite ergodic theory*, Mathematical Surveys and Monographs, AMS **50**, 1997.

[2] J. Aaronson, M. Denker and M. Urbański, *Ergodic theory for Markov fibred systems and parabolic rational maps*, Trans. AMS **337**, (1993), 495–548.

[3] L. Barreira, *A non-additive theromodynamic formalism and applications to dimension theory of hyperbolic dynamical systems*, Ph.D. thesis, The Pennsylvania State Univ. (1999).

[4] L. Barreira and Ya. Pesin, *Lyapunov exponents and smooth ergodic theory*, University Lecture Series, **23**, AMS, 2002.

[5] M. Benedicks and L. Carleson, *The dynamics of the Hénon map*, Ann. of Math., **133**, (1991), 73–169.

[6] M. Benedicks and M. Viana, *Solution of the basin problem for Hénon-like attractors*, Invent. Math., **143**, (2001), 375–434.

[7] M. Benedicks and L.-S. Young, *Sinai-Bowen-Ruelle measures for certain Hénon maps*, Invent. Math., **112**, (1993), 541–576.

[8] M. Benedicks and L.-S. Young, *Markov extensions and decay of correlations for certain Hénon maps*, Astérisque, 2000.

[9] H. Bruin, *Induced maps, Markov extensions and invariant measures in one-dimensional dynamics*, Comm. Math. Phys., **168**, (1995), 571–580.

[10] H. Bruin and G. Keller, *Equilibrium states for S-unimodal maps*, Ergodic Theory Dynam. Systems, **18**, (1998), 765–789.

[11] H. Bruin, S. Luzzatto and S. van Strien, *Rates of decay of correlation for one-dimensional dynamics*, Ann. Sci. École Norm. Sup. (4) **36** (2003), 621–646.

[12] J. Buzzi and O. Sarig, *Uniqueness of equilibrium states for countable Markov shifts and multidimensional piesewise expanding maps*, Ergodic Theory Dynam. Systems, **23**, (2003), 1383–1400.

[13] J. Buzzi, *Number of equilibrium states of piecewise monotonic maps of the interval*, Proc. Amer. Math. Soc., **123** (1995), 2901–2907.

[14] J. Buzzi, *Thermodynamical formalism for piecewise invertible maps: absolutely continuous invariant measures as equilibrium states*, in the book: *Smooth ergodic theory and its applications*, Amer. Math. Soc., (2001), 749–783.

[15] P. Collet, J.-P. Eckmann, *Instability and fronts in extended systems*, Princeton Univ. Press, NY 1990.

[16] D. Dolgopyat and Ya. Pesin, *Every compact manifold carries a completely hyperbolic diffeomorphism*, Ergodic Theory Dynam. Systems **22** (2002), 409–435.

[17] D. Fiebig, U.-R. Fiebig, M. Yuri, *Pressure and equilibrium states for countable state Markov shifts*, Israel J. Math. **131**, (2002), 221–257.

[18] J. Graczyk, D. Sands and G. Świpokatek, *La drive schwarzienne en dynamique unimodale*, C. R. Acad. Sci. Paris Sr. I Math. **332** (2001), 329–332.

[19] F. Hofbauer, *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy*, Israel J. Math., **34** (1979), 213–237.

[20] F. Hofbauer, *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy, II*, Israel J. Math., **38** (1981), 107–115.

[21] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Inst. Hautes Études Sci. Publ. Math. **51** (1980), 137–173.

[22] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press, 1995.

[23] G. Keller,, *Lifting measures to Markov extensions*, Monatsh. Math. **108** (1989), 183–200.

[24] G. Knieper, *The uniqueness of the measure of maximal entropy for the geodesic flows on rank* 1 *manifolds*, Ann. of Math. (2) **148** (1998), 291–314.

[25] D. Mauldin and M. Urbański, *Gibbs states on the symbolic space over an infinite alphabet*, Israel J. Math. **125**, (2001), 93–130.

[26] L. Mora and M. Viana, *Abundance of strange attractors*, Acta Math. **171** (1993), 1–71.

[27] Ya. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*, Chicago Lectures in Mathematics, Chicago University Press, 1998.

[28] Ya. Pesin and S. Senti, *Equilibrium Measures for Some One Dimensional Maps*, in preparation.

[29] O. Sarig, *Characterization of Existence of Gibbs Measures for Countable Markov Shifts*, Proc. AMS **131:6**, (2003), 1751–1758.

[30] O. Sarig, *Thermodynamic formalism for countable Markov shifts*, Ergodic Theory Dynam. Systems **19**, (1999), 1565–1593.

[31] S. Senti, *Dimension of Weakly Expanding Points for Quadratic Maps*, Bull. Soc. Math. France, **131**, (2003), 399–420.

[32] M. Viana, *Strange attractors in higher dimensions*, Bol. Soc. Brasil. Mat. (N.S.) **24** (1993), 13–62.

[33] Q. Wang and L.-S.Young, *Strange attractors with one direction of instability*, Comm. Math. Phys. **218** (2001), 1–97.

[34] Q. Wang and L.-S.Young, *From invariant curves to strange attractors*, Comm. Math. Phys. **225** (2002), 275–304.

[35] M. Yuri, *Thermodynamic formalism for certain nonhyperbolic maps*, Ergodic Theory Dynam. Systems **19**, (1999), 1365–1378.

[36] J.-C., Yoccoz, *Jakobson's Theorem*, Manuscript of Course at Collège de France, 1997.

[37] L. S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Ann. of Math., **147**, (1998), 585–650.

[38] M. Yuri, *Thermodynamic formalism for certain nonhyperbolic maps*, Ergodic Theory Dynam. Systems, **19**, (1999), 1365–1378.

[39] M. Yuri, *Thermodynamic formalism for countable to one Markov systems*, Trans. Amer. Math. Soc., **355**, (2003), 2949–2971

[40] A. Zorich, *Rauzy induction, Veech zippered rectangles, Mazur polygons, and Lyapunov exponents of Teichmüer geodesic flow*, in the "Handbook of Dynamical Systems", **1B**, Edited by B. Hasselblatt and A. Katok, Elsevier, Amsterdam, 2005 (to appear).

[41] R. Zweimüller, *Invariant measures for general(ized) induced tranformations*, Preprint, 2003.

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