数理解析研究所講究録 1408 巻 2004 年 123-142



# *Cavity QED and Quantum Computation in the Weak Coupling Regime*

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**概要**

In this paper we consider a model of quantum computation based on n atoms of laser–cooled and trapped linearly in a cavity and realize it as the n atoms Tavis–Cummings Hamiltonian interacting with n external (laser) fields.

We solve the Schrödinger equation of the model in the case of n=2 and construct the controlled NOT gate by making use of a resonance condition and rotating wave approximation associated to it.

We also present an idea of the construction of three controlled NOT gates in the case of n=3, which gives the controlled–controlled NOT gate.

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# 1 Introduction

Quantum Computation (or Computer) is a challenging task in this century for not only physicists but also mathematicians. Quantum Computation is in a usual understanding based on qubits which are based on two level systems (two energy levels or fundamental spins) of atoms, see [1] as for general theory of two level systems.

In a realistic image of Quantum Computer we need at least one hundred atoms. However, then we may meet a very severe problem called Decoherence which destroy a superposition of quantum states in the process of unitary evolution of our system. At the current time it is not easy to control the decoherence. See for example [2] as an introduction.

An optical system like Cavity QED may have some advantage on this problem, therefore we consider a quantum computation based on Cavity QED. As an approximate model we realize it as the n atoms Tavis–Cummings Hamiltonian interacting with n external (laser) fields. As to the Tavis–Cummings model see [3]. To perform the quantum computation we must first of all show that our system is universal [4]. To show it we must construct the controlled NOT operator (gate) explicitly in the case of $n = 2$, [4], [5].

For that we must embed a system of two–qubits in a space of wave functions of the model and solve the Schrödinger equation. In a reduced system we can construct the controlled NOT by use of some resonance condition and the rotating wave approximation associated to it. Then we need to assume that the coupling constants are small enough (the weak coupling regime in the title).

Next we want to construct the controlled–controlled NOT operator in the case of $n = 3$. For that purpose the construction of three controlled NOT gates is required [1] because three atoms are in our system trapped **linearly** in the cavity, and we present an idea

[1] In the study of Cavity QED Quantum Computation this point is missed as far as we know

toward explicit construction. If this difficulty will be overcome our system of quantum computation may become complete.

## 2 A Model Based on Cavity QED

We consider a quantum computation model based on n atoms of laser–cooled and trapped linearly in a cavity and realize it as the n atoms Tavis–Cummings Hamiltonian interacting with n external (laser) fields. This is of course an approximate theory. In a more realistic model we must add other dynamical variables such as positions of atoms and their momenta etc. However, such a model is almost impossible to solve, therefore we consider a simple one.

The Hamiltonian is given by

$$H = \omega 1_L \otimes a^\dagger a + \frac{\Delta}{2}\sum_{j=1}^{n} \sigma_j^{(3)} \otimes \mathbf{1} + g\sum_{j=1}^{n}\left(\sigma_j^{(+)} \otimes a + \sigma_j^{(-)} \otimes a^\dagger\right) + \sum_{j=1}^{n} h_j \left(\sigma_j^{(+)} e^{i(\Omega_j t+\phi_j)} + \sigma_j^{(-)} e^{-i(\Omega_j t+\phi_j)}\right) \otimes \mathbf{1} \tag{1}$$

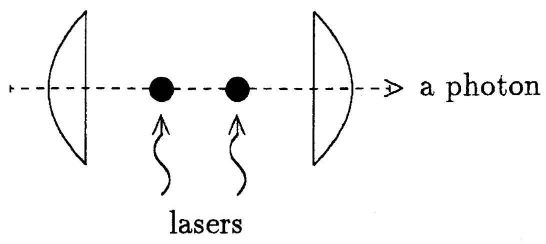
where $\omega$ is the frequency of radiation field, $\Delta$ the energy difference of two level atoms, $a$ and $a^\dagger$ are annihilation and creation operators of the field, and $g$ a coupling constant, $\Omega_j$ the frequencies of external fields which are treated as classical fields, $h_j$ coupling constants, and $L = 2^n$. Here $\sigma_j^{(+)}$, $\sigma_j^{(-)}$ and $\sigma_j^{(3)}$ are given as

$$\sigma_j^{(s)} = 1_2 \otimes \cdots \otimes 1_2 \otimes \sigma_s \otimes 1_2 \otimes \cdots \otimes 1_2 \ (j - \text{position}) \ \in \ M(L, \mathbf{C}) \tag{2}$$

where $s$ is $+$, $-$ and 3 respectively and

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

When $n = 2$ (which is the target through this paper) see the following picture.




Here let us rewrite the hamiltonian (1). If we set

$$S_+ = \sum_{j=1}^{n} \sigma_j^{(+)}, \quad S_- = \sum_{j=1}^{n} \sigma_j^{(-)}, \quad S_3 = \frac{1}{2}\sum_{j=1}^{n} \sigma_j^{(3)}, \tag{4}$$

then (1) can be written as

$$\begin{aligned} H &= \omega 1_L \otimes a^\dagger a + \Delta S_3 \otimes \mathbf{1} + g\left(S_+ \otimes a + S_- \otimes a^\dagger\right) + \\ &\sum_{j=1}^{n} h_j \left(\sigma_j^{(+)} \mathrm{e}^{i(\Omega_j t + \phi_j)} + \sigma_j^{(-)} \mathrm{e}^{-i(\Omega_j t + \phi_j)}\right) \otimes \mathbf{1} \equiv H_0 + V(t), \end{aligned} \tag{5}$$

which is relatively clear. $H_0$ is the Tavis–Cummings Hamiltonian and we treat it as an unperturved one. We note that $\{S_+, S_-, S_3\}$ satisfy the $su(2)$–relation

$$[S_3, S_+] = S_+, \quad [S_3, S_-] = -S_-, \quad [S_+, S_-] = 2S_3. \tag{6}$$

However, the representation $\rho$ defined by

$$\rho(\sigma_+) = S_+, \quad \rho(\sigma_-) = S_-, \quad \rho(\sigma_3/2) = S_3$$

is a full representation of $su(2)$, which is of course not irreducible.

We would like to solve the Schrödinger equation

$$i\frac{d}{dt}U = HU = (H_0 + V)\,U, \tag{7}$$

where $U$ is a unitary operator. We can solve this equation by using the **method of constant variation**. The equation $i\frac{d}{dt}U = H_0 U$ is solved to be

$$U(t) = \left(\mathrm{e}^{-it\omega S_3} \otimes \mathrm{e}^{-it\omega N}\right) \mathrm{e}^{-itg\left(S_+ \otimes a + S_- \otimes a^\dagger\right)} U_0$$

where $N = a^\dagger a$ is the number operator and $U_0$ a constant unitary. Here we have used the resonance condition

$$\omega = \Delta \tag{8}$$

, see for example [6]. By changing $U_0 \longmapsto U_0(t)$ and substituting into (7) we have the equation

$$i\frac{d}{dt}U_0 = \mathrm{e}^{itg\left(S_+\otimes a+S_-\otimes a^\dagger\right)}\left(\mathrm{e}^{it\omega S_3}\otimes \mathrm{e}^{it\omega N}\right)V(t)\left(\mathrm{e}^{-it\omega S_3}\otimes \mathrm{e}^{-it\omega N}\right)\mathrm{e}^{-itg\left(S_+\otimes a+S_-\otimes a^\dagger\right)}U_0 \tag{9}$$

after some algebras. We would like to calculate the right hand side of (9) explicitly, which is however a very hard task due to the term $\mathrm{e}^{-itg\left(S_+\otimes a+S_-\otimes a^\dagger\right)}$. It has been done only for $n = 1$ and 2 as far as we know, [6], [7]. The case $n = 1$ which is just the Jaynes–Cummings model is not interesting from the point of view of a quantum computation, so we restrict to the case $n = 2$ in the following.

First let us write down each term (9). From the result in [6] we have

$$\mathrm{e}^{-itg\left(S_+\otimes a+S_-\otimes a^\dagger\right)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \tag{10}$$

where

$$a_{11} = \frac{N+2+(N+1)\mathrm{cos}\left(tg\sqrt{2(2N+3)}\right)}{2N+3}, \quad a_{12} = a_{13} = -i\frac{\sin\left(tg\sqrt{2(2N+3)}\right)}{\sqrt{2(2N+3)}}a,$$

$$a_{14} = \frac{-1+\cos\left(tg\sqrt{2(2N+3)}\right)}{2N+3}a^2, \quad a_{21} = a_{31} = -i\frac{\sin\left(tg\sqrt{2(2N+1)}\right)}{\sqrt{2(2N+1)}}a^\dagger,$$

$$a_{22} = a_{33} = \frac{1+\cos\left(tg\sqrt{2(2N+1)}\right)}{2}, \quad a_{23} = a_{32} = \frac{-1+\cos\left(tg\sqrt{2(2N+1)}\right)}{2},$$

$$a_{24} = a_{34} = -i\frac{\sin\left(tg\sqrt{2(2N+1)}\right)}{\sqrt{2(2N+1)}}a, \quad a_{41} = \frac{-1+\cos\left(tg\sqrt{2(2N-1)}\right)}{2N-1}(a^\dagger)^2,$$

$$a_{42} = a_{43} = -i\frac{\sin\left(tg\sqrt{2(2N-1)}\right)}{\sqrt{2(2N-1)}}a^\dagger, \quad a_{44} = \frac{N-1+N\mathrm{cos}\left(tg\sqrt{2(2N-1)}\right)}{2N-1},$$

and

$$V(t)=\begin{pmatrix} 0 & h_2 e^{i(\Omega_2 t+\phi_2)} & h_1 e^{i(\Omega_1 t+\phi_1)} & 0 \\ h_2 e^{-i(\Omega_2 t+\phi_2)} & 0 & 0 & h_1 e^{i(\Omega_1 t+\phi_1)} \\ h_1 e^{-i(\Omega_1 t+\phi_1)} & 0 & 0 & h_2 e^{i(\Omega_2 t+\phi_2)} \\ 0 & h_1 e^{-i(\Omega_1 t+\phi_1)} & h_2 e^{-i(\Omega_2 t+\phi_2)} & 0 \end{pmatrix} \otimes \mathbf{1}, \tag{11}$$

$$e^{-it\omega S_3} \otimes e^{-it\omega N} = \begin{pmatrix} e^{-it\omega} & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{it\omega} \end{pmatrix} \otimes e^{-it\omega N}. \tag{12}$$

Some algebras from (11) and (12) lead us to

$$\left(e^{it\omega S_3} \otimes e^{it\omega N}\right) V(t) \left(e^{-it\omega S_3} \otimes e^{-it\omega N}\right)$$
$$=\begin{pmatrix} 0 & h_2 e^{i\{(\Omega_2+\omega)t+\phi_2\}} & h_1 e^{i\{(\Omega_1+\omega)t+\phi_1\}} & 0 \\ h_2 e^{-i\{(\Omega_2+\omega)t+\phi_2\}} & 0 & 0 & h_1 e^{i\{(\Omega_1+\omega)t+\phi_1\}} \\ h_1 e^{-i\{(\Omega_1+\omega)t+\phi_1\}} & 0 & 0 & h_2 e^{i\{(\Omega_2+\omega)t+\phi_2\}} \\ 0 & h_1 e^{-i\{(\Omega_1+\omega)t+\phi_1\}} & h_2 e^{-i\{(\Omega_2+\omega)t+\phi_2\}} & 0 \end{pmatrix} \otimes \mathbf{1}. \tag{13}$$

Therefore we can calculate the term

$$F(t) \equiv e^{itg\left(S_+\otimes a+S_-\otimes a^\dagger\right)} \left(e^{it\omega S_3} \otimes e^{it\omega N}\right) V(t) \left(e^{-it\omega S_3} \otimes e^{-it\omega N}\right) e^{-itg\left(S_+\otimes a+S_-\otimes a^\dagger\right)} \tag{14}$$

from (10) and (13). However, we omit the explicit form because of being too complicated.

Here let us introduce a brief notation on (10) for the latter convenience.

$$e^{-itg\left(S_+\otimes a+S_-\otimes a^\dagger\right)} =$$
$$\begin{pmatrix} \frac{2N+2}{2N+3}f(N+1)+1 & -ik(N+1)a & -ik(N+1)a & \frac{2}{2N+3}f(N+1)a^2 \\ -ik(N)a^\dagger & f(N)+1 & f(N) & -ik(N)a \\ -ik(N)a^\dagger & f(N) & f(N)+1 & -ik(N)a \\ \frac{2}{2N-1}f(N-1)a^{\dagger 2} & -ik(N-1)a^\dagger & -ik(N-1)a^\dagger & \frac{2N}{2N-1}f(N-1)+1 \end{pmatrix} \tag{15}$$

where

$$f(N)=\frac{-1+\cos\left(tg\sqrt{2(2N+1)}\right)}{2},\quad k(N)=\frac{\sin\left(tg\sqrt{2(2N+1)}\right)}{\sqrt{2(2N+1)}}.$$

Next let us go to a quantum computation based on two atoms of laser–cooled and trapped linearly in a cavity.

# 3 Quantum Computation and Controlled NOT Gate

Let us make a short review of two–qubits. Each element can be written as

$$\psi=a_{++}|+\rangle\otimes|+\rangle+a_{+-}|+\rangle\otimes|-\rangle+a_{-+}|-\rangle\otimes|+\rangle+a_{--}|-\rangle\otimes|-\rangle$$

with two bases $|+\rangle$ and $|-\rangle$ and $|a_{++}|^2+|a_{+-}|^2+|a_{-+}|^2+|a_{--}|^2=1$. Here if we identify

$$|+\rangle=\begin{pmatrix}1\\0\end{pmatrix},\quad |-\rangle=\begin{pmatrix}0\\1\end{pmatrix},$$

then $\psi$ above becomes

$$\psi=\begin{pmatrix}a_{++}\\a_{+-}\\a_{-+}\\a_{--}\end{pmatrix}. \tag{16}$$

How do we embed two–qubits in our quantized system ? It is not known at the present which will depend on experimentalists. Therefore let us consider the simplest one like

$$|\psi(t)\rangle=\begin{pmatrix}a_{++}(t)\\a_{+-}(t)\\a_{-+}(t)\\a_{--}(t)\end{pmatrix}\otimes|0\rangle, \tag{17}$$

where $|0\rangle$ is the ground state of the radiation field ($a|0\rangle = 0$). We note that in full theory we must consider the following superpositions

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} \begin{pmatrix} a_{++,n}(t) \\ a_{+-,n}(t) \\ a_{-+,n}(t) \\ a_{--,n}(t) \end{pmatrix} \otimes |n\rangle$$

as a wave function, which is however too complicated to solve.

To determine a dynamics that the coefficients $a_{++}, a_{+-}, a_{-+}, a_{--}$ will satisfy we substitute (17) into the equation

$$i\frac{d}{dt}|\psi(t)\rangle = F(t)|\psi(t)\rangle. \tag{18}$$

The left hand side of (18) is

$$\begin{pmatrix} i\frac{d}{dt}a_{++}(t) \\ i\frac{d}{dt}a_{+-}(t) \\ i\frac{d}{dt}a_{-+}(t) \\ i\frac{d}{dt}a_{--}(t) \end{pmatrix} \otimes |0\rangle \tag{19}$$

, while each of the right hand side becomes

$$\begin{aligned}
&\text{1-component} = h_1 e^{i\{(\Omega_1+\omega)t+\phi_1\}} \times \\
&\Big[-ia_{++}\Big\{k(1) + \frac{4}{5}k(1)f(2) - \frac{4}{3}f(1)k(2)\Big\}|1\rangle + a_{+-}\Big\{f(0) + \frac{2}{3}f(0)f(1) + k(0)k(1)\Big\}|0\rangle \\
&+a_{-+}\Big\{1 + f(0) + \frac{2}{3}f(1) + \frac{2}{3}f(0)f(1) + k(0)k(1)\Big\}|0\rangle\Big] + \\
&h_2 e^{i\{(\Omega_2+\omega)t+\phi_2\}} \times \\
&\Big[-ia_{++}\Big\{k(1) + \frac{4}{5}k(1)f(2) - \frac{4}{3}f(1)k(2)\Big\}|1\rangle \\
&+a_{+-}\Big\{1 + f(0) + \frac{2}{3}f(1) + \frac{2}{3}f(0)f(1) + k(0)k(1)\Big\}|0\rangle \\
&+a_{-+}\Big\{f(0) + \frac{2}{3}f(0)f(1) + k(0)k(1)\Big\}|0\rangle\Big],
\end{aligned} \tag{20}$$

$$
\begin{aligned}
&\text{2-component} = h_1 e^{i\{(\Omega_1+\omega)t+\phi_1\}} \times \\
&\left[\sqrt{2}a_{++}\left\{\frac{2}{3}f(1)+k(1)k(2)+\frac{2}{3}f(1)f(2)\right\}|2\rangle - ia_{+-}\{k(0)-f(0)k(1)+k(0)f(1)\}|1\rangle\right. \\
&-ia_{-+}\{k(0)-k(1)-f(0)k(1)+k(0)f(1)\}|1\rangle + a_{--}\{1+f(0)\}|0\rangle] + \\
&h_2 e^{i\{(\Omega_2+\omega)t+\phi_2\}} \times \\
&\left[\sqrt{2}a_{++}\left\{k(1)k(2)+\frac{2}{3}f(1)f(2)\right\}|2\rangle + ia_{+-}\{k(1)+f(0)k(1)-k(0)f(1)\}|1\rangle\right. \\
&+ia_{-+}\{f(0)k(1)-k(0)f(1)\}|1\rangle + a_{--}f(0)|0\rangle] + \\
&h_1 e^{-i\{(\Omega_1+\omega)t+\phi_1\}} a_{++}\left\{f(0)+\frac{2}{3}f(0)f(1)+k(0)k(1)\right\}|0\rangle + \\
&h_2 e^{-i\{(\Omega_2+\omega)t+\phi_2\}} a_{++}\left\{1+f(0)+\frac{2}{3}f(1)+\frac{2}{3}f(0)f(1)+k(0)k(1)\right\}|0\rangle, \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
&\text{3-component} = h_1 e^{i\{(\Omega_1+\omega)t+\phi_1\}} \times \\
&\left[\sqrt{2}a_{++}\left\{k(1)k(2)+\frac{2}{3}f(1)f(2)\right\}|2\rangle + ia_{+-}\{f(0)k(1)-k(0)f(1)\}|1\rangle\right. \\
&+ia_{-+}\{k(1)+f(0)k(1)-k(0)f(1)\}|1\rangle + a_{--}f(0)|0\rangle] + \\
&h_2 e^{i\{(\Omega_2+\omega)t+\phi_2\}} \times \\
&\left[\sqrt{2}a_{++}\left\{\frac{2}{3}f(1)+k(1)k(2)+\frac{2}{3}f(1)f(2)\right\}|2\rangle - ia_{+-}\{k(0)-k(1)-f(0)k(1)+k(0)f(1)\}|1\rangle\right. \\
&-ia_{-+}\{k(0)-f(0)k(1)+k(0)f(1)\}|1\rangle + a_{--}\{1+f(0)\}|0\rangle] + \\
&h_1 e^{-i\{(\Omega_1+\omega)t+\phi_1\}} a_{++}\left\{1+f(0)+\frac{2}{3}f(1)+\frac{2}{3}f(0)f(1)+k(0)k(1)\right\}|0\rangle + \\
&h_2 e^{-i\{(\Omega_2+\omega)t+\phi_2\}} a_{++}\left\{f(0)+\frac{2}{3}f(0)f(1)+k(0)k(1)\right\}|0\rangle, \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
&\text{4-component} = h_1 e^{i\{(\Omega_1+\omega)t+\phi_1\}} \times \\
&\left[-2\sqrt{6}ia_{++}\left\{\frac{1}{5}k(1)f(2)-\frac{1}{3}f(1)k(2)\right\}|3\rangle + \sqrt{2}a_{+-}\left\{\frac{2}{3}f(0)f(1)+k(0)k(1)\right\}|2\rangle\right. \\
&\left.+\sqrt{2}a_{-+}\left\{\frac{2}{3}f(1)+\frac{2}{3}f(0)f(1)+k(0)k(1)\right\}|2\rangle + ia_{--}k(0)|1\rangle\right] + \\
&h_2 e^{i\{(\Omega_2+\omega)t+\phi_2\}} \times \\
&\left[-2\sqrt{6}ia_{++}\left\{\frac{1}{5}k(1)f(2)-\frac{1}{3}f(1)k(2)\right\}|3\rangle + \sqrt{2}a_{+-}\left\{\frac{2}{3}f(1)+\frac{2}{3}f(0)f(1)+k(0)k(1)\right\}|2\rangle\right. \\
&\left.+\sqrt{2}a_{-+}\left\{\frac{2}{3}f(0)f(1)+k(0)k(1)\right\}|2\rangle + ia_{--}k(0)|1\rangle\right] +
\end{aligned}
$$

$$h_1 e^{-i\{(\Omega_1+\omega)t+\phi_1\}} \times$$
$$\left[ia_{++}\left\{k(0)-k(1)+\frac{2}{3}k(0)f(1)-2f(0)k(1)\right\}|1\rangle + a_{+-}\left\{1+f(0)\right\}|0\rangle + a_{-+}f(0)|0\rangle\right] +$$
$$h_2 e^{-i\{(\Omega_2+\omega)t+\phi_2\}} \times$$
$$\left[ia_{++}\left\{k(0)-k(1)+\frac{2}{3}k(0)f(1)-2f(0)k(1)\right\}|1\rangle + a_{+-}f(0)|0\rangle + a_{-+}\left\{1+f(0)\right\}|0\rangle\right] \quad (23)$$

after a long calculation by making use of (14).

The equation is not satisfied under the restrictive ansatz (17). However, excited states $|1\rangle$, $|2\rangle$, $|3\rangle$ which have no corresponding kinetic terms contain the coupling constants $h_1$ and $h_2$, so the equation is approximately satisfied if they are small enough (namely, in the weak coupling regime).

Therefore the (full) equation is reduced to the equations of $\{a_{++}, a_{+-}, a_{-+}, a_{--}\}$ at the ground state.

$$\begin{aligned}
&i\frac{d}{dt}a_{++} = \\
&\left[h_1 e^{i\{(\Omega_1+\omega)t+\phi_1\}}\left\{f(0)+\frac{2}{3}f(0)f(1)+k(0)k(1)\right\}+\right. \\
&\quad\left. h_2 e^{i\{(\Omega_2+\omega)t+\phi_2\}}\left\{1+f(0)+\frac{2}{3}f(1)+\frac{2}{3}f(0)f(1)+k(0)k(1)\right\}\right]a_{+-}+ \\
&\left[h_1 e^{i\{(\Omega_1+\omega)t+\phi_1\}}\left\{1+f(0)+\frac{2}{3}f(1)+\frac{2}{3}f(0)f(1)+k(0)k(1)\right\}+\right. \\
&\quad\left. h_2 e^{i\{(\Omega_2+\omega)t+\phi_2\}}\left\{f(0)+\frac{2}{3}f(0)f(1)+k(0)k(1)\right\}\right]a_{-+},
\end{aligned} \quad (24)$$

$$\begin{aligned}
&i\frac{d}{dt}a_{+-} = \\
&\left[h_1 e^{-i\{(\Omega_1+\omega)t+\phi_1\}}\left\{f(0)+\frac{2}{3}f(0)f(1)+k(0)k(1)\right\}+\right. \\
&\quad\left. h_2 e^{-i\{(\Omega_2+\omega)t+\phi_2\}}\left\{1+f(0)+\frac{2}{3}f(1)+\frac{2}{3}f(0)f(1)+k(0)k(1)\right\}\right]a_{++}+ \\
&\left[h_1 e^{i\{(\Omega_1+\omega)t+\phi_1\}}\left\{1+f(0)\right\}+h_2 e^{i\{(\Omega_2+\omega)t+\phi_2\}}f(0)\right]a_{--},
\end{aligned} \quad (25)$$

$$\begin{aligned}
&i\frac{d}{dt}a_{-+} = \\
&\left[h_1 e^{-i\{(\Omega_1+\omega)t+\phi_1\}}\left\{1+f(0)+\frac{2}{3}f(1)+\frac{2}{3}f(0)f(1)+k(0)k(1)\right\}+\right.
\end{aligned}$$

$$h_2 e^{-i\{(\Omega_2+\omega)t+\phi_2\}}\left\{f(0)+\frac{2}{3}f(0)f(1)+k(0)k(1)\right\}\Big]\, a_{++} +$$

$$\Big[h_1 e^{i\{(\Omega_1+\omega)t+\phi_1\}} f(0) + h_2 e^{i\{(\Omega_2+\omega)t+\phi_2\}}\{1+f(0)\}\Big]\, a_{--}, \tag{26}$$

$$i\frac{d}{dt}a_{--} =$$

$$\Big[h_1 e^{-i\{(\Omega_1+\omega)t+\phi_1\}}\{1+f(0)\} + h_2 e^{-i\{(\Omega_2+\omega)t+\phi_2\}} f(0)\Big]\, a_{+-} +$$

$$\Big[h_1 e^{-i\{(\Omega_1+\omega)t+\phi_1\}} f(0) + h_2 e^{-i\{(\Omega_2+\omega)t+\phi_2\}}\{1+f(0)\}\Big]\, a_{-+} \tag{27}$$

or in a matrix form

$$i\frac{d}{dt}\begin{pmatrix} a_{++}(t) \\ a_{+-}(t) \\ a_{-+}(t) \\ a_{--}(t) \end{pmatrix} = \begin{pmatrix} & \sharp & \sharp & \\ \sharp & & & \sharp \\ \sharp & & & \sharp \\ & \sharp & \sharp & \end{pmatrix} \begin{pmatrix} a_{++}(t) \\ a_{+-}(t) \\ a_{-+}(t) \\ a_{--}(t) \end{pmatrix} \tag{28}$$

where $\sharp$ is the corresponding matrix element from the above equations.

We obtained the system of complete equations, which is still complicated. How do we solve it ? We use some resonance condition and the rotating wave approximation associated to it. Since

$$f(0) = \left\{-1+\cos\left(tg\sqrt{2}\right)\right\}/2, \quad f(1) = \left\{-1+\cos\left(tg\sqrt{6}\right)\right\}/2,$$
$$k(0) = \sin\left(tg\sqrt{2}\right)/\sqrt{2}, \quad k(1) = \sin\left(tg\sqrt{6}\right)/\sqrt{6},$$

the products $f(0)f(1)$ and $k(0)k(1)$ contain the term $e^{-itg(\sqrt{2}+\sqrt{6})}$ by the Euler formulas $\cos(\theta) = (e^{i\theta}+e^{-i\theta})/2$, $\sin(\theta) = (e^{i\theta}-e^{-i\theta})/2i$. Noting

$$e^{i\{(\Omega_1+\omega)t+\phi_1\}} e^{-itg(\sqrt{2}+\sqrt{6})} = e^{i\{(\Omega_1+\omega-(\sqrt{2}+\sqrt{6})g)t+\phi_1\}},$$

we set a new resonance condition

$$\Omega_1+\omega-(\sqrt{2}+\sqrt{6})g = 0. \tag{29}$$

All terms in (28) except for the constant one $e^{i\{(\Omega_1+\omega-(\sqrt{2}+\sqrt{6})g)t+\phi_1\}} = e^{i\phi_1}$ contain ones like $e^{i(t\theta+\alpha)}$ $(\theta \neq 0)$, so we neglect all such oscillating terms (a rotating wave approximation). Then (28) reduces to a very simple matrix equation

$$i\frac{d}{dt}\begin{pmatrix} a_{++}(t) \\ a_{+-}(t) \\ a_{-+}(t) \\ a_{--}(t) \end{pmatrix} = \frac{-(\sqrt{3}-1)h_1}{24}\begin{pmatrix} 0 & e^{i\phi_1} & e^{i\phi_1} & 0 \\ e^{-i\phi_1} & 0 & & \\ e^{-i\phi_1} & & 0 & \\ 0 & & & 0 \end{pmatrix}\begin{pmatrix} a_{++}(t) \\ a_{+-}(t) \\ a_{-+}(t) \\ a_{--}(t) \end{pmatrix}. \tag{30}$$

The solution is easily obtained to be

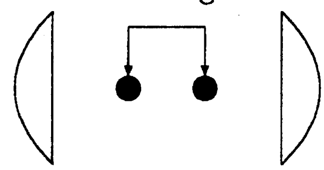
$$\begin{aligned}\begin{pmatrix} a_{++}(t) \\ a_{+-}(t) \\ a_{-+}(t) \\ a_{--}(t) \end{pmatrix} &= \exp\left\{\frac{i(\sqrt{3}-1)h_1 t}{24}\begin{pmatrix} 0 & e^{i\phi_1} & e^{i\phi_1} & 0 \\ e^{-i\phi_1} & 0 & & \\ e^{-i\phi_1} & & 0 & \\ 0 & & & 0 \end{pmatrix}\right\}\begin{pmatrix} a_{++}(0) \\ a_{+-}(0) \\ a_{-+}(0) \\ a_{--}(0) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha t) & \frac{ie^{i\phi_1}}{\sqrt{2}}\sin(\alpha t) & \frac{ie^{i\phi_1}}{\sqrt{2}}\sin(\alpha t) & 0 \\ \frac{ie^{-i\phi_1}}{\sqrt{2}}\sin(\alpha t) & \frac{1+\cos(\alpha t)}{2} & \frac{-1+\cos(\alpha t)}{2} & 0 \\ \frac{ie^{-i\phi_1}}{\sqrt{2}}\sin(\alpha t) & \frac{-1+\cos(\alpha t)}{2} & \frac{1+\cos(\alpha t)}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{++}(0) \\ a_{+-}(0) \\ a_{-+}(0) \\ a_{--}(0) \end{pmatrix} \\ &\equiv U(t)\begin{pmatrix} a_{++}(0) \\ a_{+-}(0) \\ a_{-+}(0) \\ a_{--}(0) \end{pmatrix}\end{aligned} \tag{31}$$

where we have set $\alpha = \frac{\sqrt{6}-\sqrt{2}}{24}h_1$. That is, we obtained the unitary operator $U(t)$. In particular, if we choose $t_0$ satisfying $\cos(\alpha t_0) = -1$ $(\sin(\alpha t_0) = 0)$, then

$$U(t_0) = \begin{pmatrix} -1 & & & \\ & 0 & -1 & \\ & -1 & 0 & \\ & & & 1 \end{pmatrix} = -\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & -1 \end{pmatrix}. \tag{32}$$

At this stage we use a very skillful method [2]. That is, we exchange two atoms (see [9]) in the cavity

exchange



which introduces the exchange (swap) operator

$$P = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}. \tag{33}$$

Multiplying $U(t_0)$ by $P$ gives

$$PU(t_0) = -\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}. \tag{34}$$

This is just the controlled $\sigma_z$ operator except for the overall constant $-1$ (an overall constant can be always neglected). From this it is easy to construct the controlled NOT

[2] $U(t_0)$ is imprimitive in the sense of [8], so the main theorem in it says that our system is universal (namely, we can construct any element in $U(4)$) . However, how to construct a unitary element explicitly is not given in [8]

operator, namely

$$C_{NOT} = (\mathbf{1}_2 \otimes W) C_{\sigma_z} (\mathbf{1}_2 \otimes W) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

where $W$ is the Walsh–Hadamard operator given by

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = W^{-1}. \tag{35}$$

See for example [5]. As to a construction of $W$ by making use of Rabi oscillations see [10]. Therefore our system is universal [4], [8].

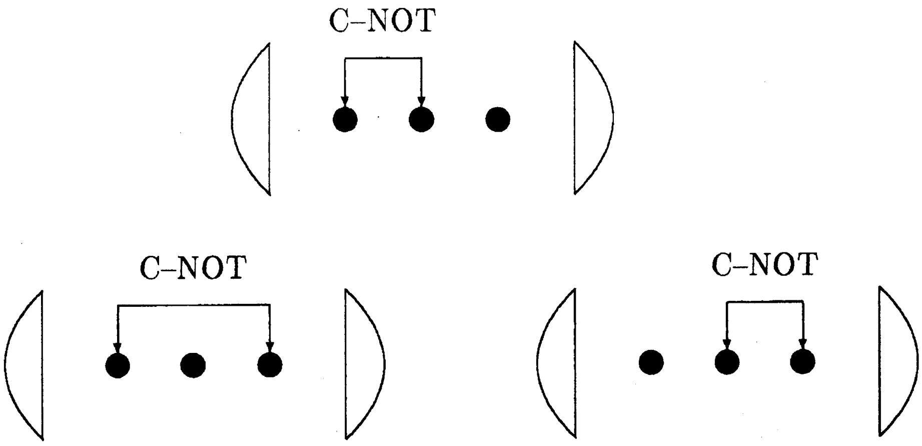
A comment is in order. In the equation (28) we can set another resonance condition in place of (29) and obtain a unitary operator like $U(t)$ in (31). We leave it to the readers.

## 4 Controlled-Controlled NOT Gate $\cdots$ A Problem

Our quantum computation model is based on n atoms of laser–cooled and trapped **linearly** in a cavity, so we have another problem on the controlled NOT operators (of three types) when $n = 3$.

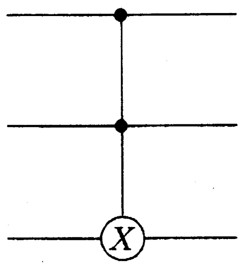
**Problem :** Let us consider the case of three atoms in a cavity. How can we construct C-NOT (or C-Unitary) operators for any two atoms among them ?

See the following pictures :




These constructions are very crucial in realizing quantum logic gates, for example, the controlled–controlled NOT gate shown as a picture

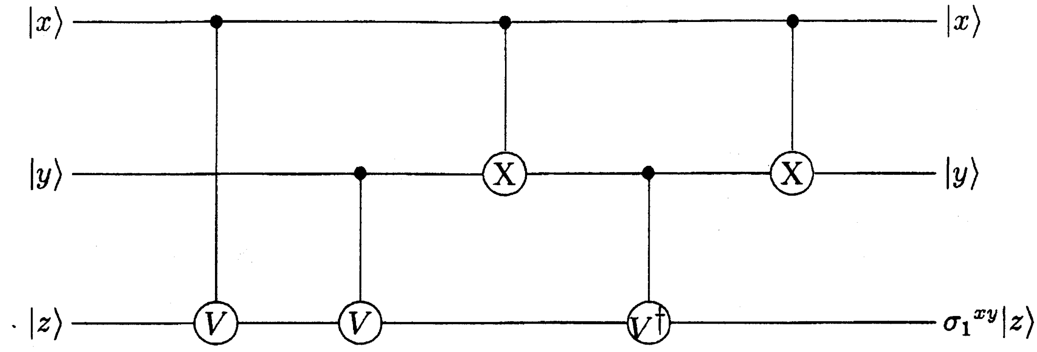
$CC_{NOT}$ =



or in a matrix form

$$CC_{NOT} = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{pmatrix}.$$

The (usual) construction by making use of controlled NOT or controlled U gates is shown as a picture ([5], [4])

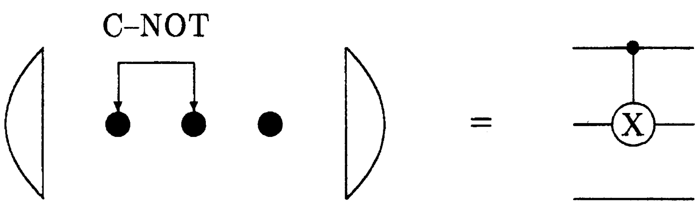



where $V$ is a unitary matrix given by

$$V = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \implies V^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 .$$

However, we have not seen "realistic" constructions in any references, so we must give the explicit construction.

To solve this let us state our idea. First we consider the construction of controlled NOT operator between the first and second atoms, namely




$$= C_{NOT} \otimes \mathbf{1}$$

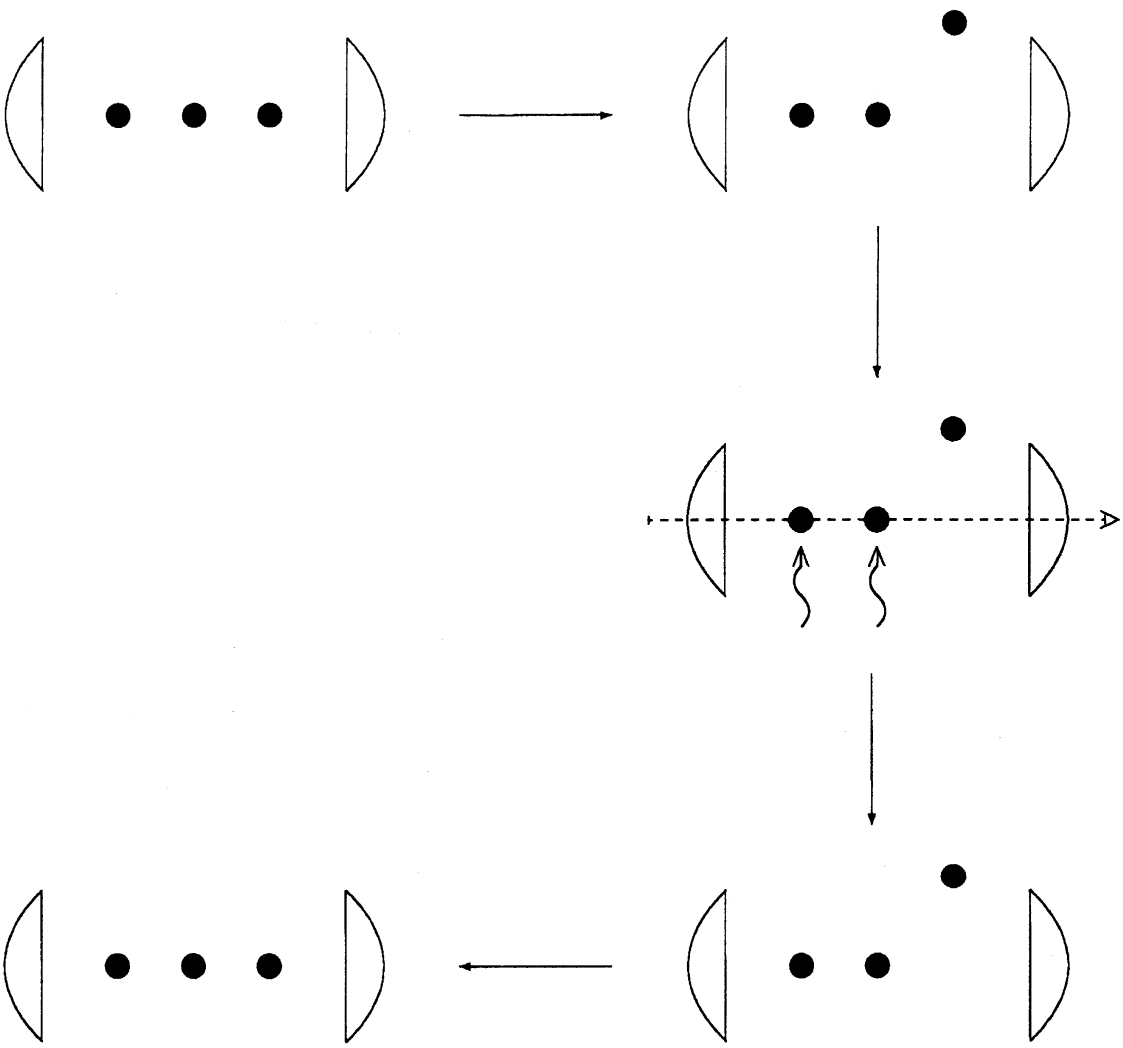
Our strategy is as follows.

(i) We move the third atom from the cavity.

(ii) We subject a photon in the cavity as two atoms interact with it, which gives the controlled NOT operator as shown in the preceding section.

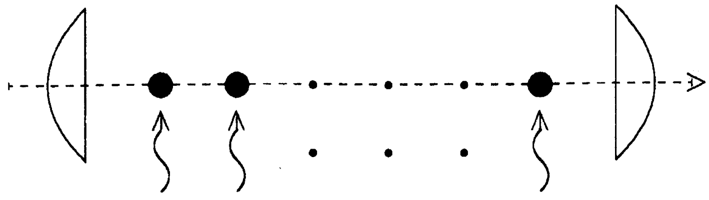
(iii) We return the third atom (outside the cavity) to the former position.

See the following sequence of pictures :



If an influence of the "getting the third atom in and out" on the states space is small enough (namely, the unitary operator induced is near to the identity $\mathbf{1}_4$), then we certainly obtain the controlled NOT gate (namely, $C_{NOT} \otimes \mathbf{1}_2$) that we are looking for. Similarly we can obtain the remaining two ones.

It is easy to generalize our idea to the $n$–atoms case. To perform a quantum computation we need to construct (many) controlled–controlled NOT gates or controlled–controlled unitary ones for three atoms among $n$–atoms, see §7 in [4]. The method is almost same, so we leave it to the readers.




Then we must estimate an influence of the "getting atoms (which are not our target) in and out" on the whole states space, which is however difficult in our model. We must add in (1) further terms necessary to calculate it.

By the way, we have given the exact form of evolution operator for the three atoms case [6], therefore we can in principle track the same line shown in this paper and it may be possible to get many unitary gates directly (without combining many elementary gates like the above construction of controlled–controlled NOT gate). However, the calculation becomes very difficult because we must treat $8 \times 8$ matrices at each step. We will attempt it in the forthcoming paper.

# 5 Discussion

In this paper we constructed the controlled NOT operator in the quantum computation based on Cavity QED which showed that our system is universal. We also constructed the controlled–controlled NOT operator (under some assumption). Therefore we can in principle perform a quantum computation. **We expect strongly that some experimentalists will check whether our method works good or not.**

We conclude this paper by making a comment (which is important at least to us). The Tavis–Cummings model is based on only two energy levels of atoms. However, an atom has in general infinitely many energy levels, so it is natural to use this possibility. We are also studying a quantum computation based on multi–level systems of atoms (a qudit theory) [11]. Therefore we would like to extend the Tavis–Cummings model based on two–levels to a model based on multi–levels. This is a very challenging task.

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