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EIGENVALUES, SINGULAR VALUES, AND LITTLEWOOD-RICHARDSON COEFFICIENTS

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We briefly describe some recent results on inequalities relating the eigenvalues of the sum of Hermitian or real matrices, and how to use these them inequalities relating the eigenvalues and singular values of a matrix and its submatrices. These results are joint work with Poon, Fomin, and Fulton [4, 14, 15]. Some open problems and remarks are also mentioned.

1 Sum of Hermitian (Real Symmetric) Matrices

Let $\mathbf{H}_n$ be the set of $n \times n$ Hermitian matrices. Denote the vector of eigenvalues of $X \in \mathbf{H}_n$ by

$$\lambda(X) = (\lambda_1(X), \ldots, \lambda_n(X))$$

with $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$. There has been a great deal of interest in studying the following.

Problem Let $A, E \in \mathbf{H}_n$ and $\tilde{A} = A + E$. Determine inequalities relating the eigenvalues of $\tilde{A}$ and those of A and E.

One can regard E as a small perturbation of the matrix A. So, we are interested in the relations between the eigenvalues of the original matrix A and the perturbed matrix $\tilde{A}$.

One may see [6] for an excellent survey on this problem and its solution. To motivate our discussion, we collect several well known results by Weyl, Liskii, Mirsky, Wielandt, Thompson and Freede; see [1, 17].

- For $i = 1, \ldots, n$, $\lambda_n(E) \leq \lambda_i(\tilde{A}) - \lambda_i(A) \leq \lambda_1(E)$.
- Suppose $1 \leq i_1 < \cdots < i_m \leq n$ and $1 \leq j_1 < \cdots < j_m \leq n$. Then

$$\sum_{s=1}^{m} \lambda_{n-s+1}(E) \leq \sum_{s=1}^{m} (\lambda_{j_s}(\tilde{A}) - \lambda_{j_s}(A)) \leq \sum_{s=1}^{m} \lambda_s(E).$$

  Consequently, for any unitarily invariant norm $\|\cdot\|$ we have

$$-\|E\| \leq \|\tilde{A}\| - \|A\| \leq \|E\|.$$

- Suppose $1 \leq i_1 < \cdots < i_m \leq n$ and $1 \leq j_1 < \cdots < j_m \leq n$. If $i_m + j_m - m \leq n$, then

$$\sum_{s=1}^{m} \lambda_{i_s+j_s-s}(\tilde{A}) \leq \sum_{s=1}^{m} \lambda_{i_s}(A) + \sum_{s=1}^{m} \lambda_{j_s}(E).$$

  One may apply the result to $-\tilde{A} = -A - E$ to get a dual set of inequalities.

All the above and many more other early results suggest that there are inequalities of the form

$$\sum_{j\in J_0} \lambda_j(\tilde{A}) \leq \sum_{j\in J_1} \lambda_j(A) + \sum_{j\in J_2} \lambda_j(E)$$

for some suitable subsets J_0, J_1, J_2 of $\{1, \ldots, n\}$. It turns out that a complete set of inequalities can be described in this way; see [9] and also [6].

Theorem 1.1 *There exist $A, B, C \in \mathbf{H}_n$ satisfying $C = A + B$ with $\lambda(A) = (a_1, \ldots, a_n)$, $\lambda(B) = (b_1, \ldots, b_n)$, $\lambda(C) = (c_1, \ldots, c_n)$ if and only if we have the trace equality*

$$\sum_{s=1}^{n} c_s = \sum_{s=1}^{n} (a_s + b_s),$$

and for any $(J_0, J_1, J_2) \in LR_m^n$ with $m < n$

$$\sum_{j\in J_0} c_j \leq \sum_{j\in J_1} a_j + \sum_{j\in J_2} b_j.$$

In the theorem, we use the concepts of Littlewood-Richardson sequences LR_m^n. A good reference for this concept is [5]. Here we describe the formal definition and give a simple example.

Let $[n] = (1, \ldots, n)$ and $J = (j_1, \ldots, j_m)$ be an increasing subsequences of $[n]$, i.e., $1 \leq j_1 < \cdots < j_m \leq n$. Define

$$\mu(J) = (j_m - m, \ldots, j_1 - 1).$$

Suppose J_0, J_1, J_2 are increasing subsequences of $[n]$. Then $(J_0, J_1, J_2) \in LR_m^n$ if $\mu(J_0)$ can be **generated** from $\mu(J_1)$ and $\mu(J_2)$ according to the Littlewood-Richardson rules:

Display $\mu(J_0) = (r_1, \ldots, r_m)$, $\mu(J_1) = (s_1, \ldots, s_m)$, and $\mu(J_2) = (t_1, \ldots, t_m)$ as Young diagrams. Add $t_1 + \cdots + t_m$ entries from $\{1, \ldots, m\}$ to the rows of the Young diagram of $\mu(J_1)$ to generate the Young diagram of $\mu(J_0)$ so that:

- The entries i occurs exactly t_i so many times.
- The entries in each row is weakly increasing from left to right.
- The entries in each column is strictly increasing from top to bottom.
- For any p with $1 \leq p \leq \sum_{j=1}^{m} t_j$, define $p(i)$ to be the number of i in the first p assigned values counting from right to left and top to bottom, we have $p(i) \geq p(i+1)$.

In such a case, the Littlewood-Richardson coefficient $c_{\mu(J_1)\mu(J_2)}^{\mu(J_0)}$ of the three partitions $\mu(J_0), \mu(J_1)$, and $\mu(J_2)$ is positive.

Example 1.2 Suppose $\mu(J_0) = (5,4,3,1)$, $\mu(J_1) = (3,2,1,0)$, $\mu(J_2) = (3,2,2,0)$. Here are three examples of **good** constructions:

$$\begin{matrix} * & * & * & 1 & 1 \\ * & * & 1 & 2 & \\ * & 2 & 3 & & \\ 3 & & & & \end{matrix} \qquad \begin{matrix} * & * & * & 1 & 1 \\ * & * & 2 & 2 & \\ * & 1 & 3 & & \\ 3 & & & & \end{matrix} \qquad \begin{matrix} * & * & * & 1 & 1 \\ * & * & 2 & 2 & \\ * & 3 & 3 & & \\ 1 & & & & \end{matrix}$$

Here are three examples of **bad** constructions:

$$\begin{matrix} * & * & * & 1 & 1 \\ * & * & 2 & 1 & \\ * & 2 & 3 & & \\ 3 & & & & \end{matrix} \qquad \begin{matrix} * & * & * & 1 & 1 \\ * & * & 2 & 3 & \\ * & 2 & 1 & & \\ 3 & & & & \end{matrix} \qquad \begin{matrix} * & * & * & 1 & 1 \\ * & * & 1 & 2 & \\ * & 3 & 3 & & \\ 2 & & & & \end{matrix}$$

One can use the LR rules to explain the Weyl inequalities, and the standard inequalities of Thompson.

[Weyl's inequalities] $((j_0),(j_1),(j_2)) \in LR_1^n$ if and only if $j_0 = j_1 + j_2 - 1$.

[Thompson's standard inequalities] If $J_1 = (i_1, \ldots, i_m)$ and $J_2 = (j_1, \ldots, j_m)$ satisfy $i_m + j_m - m \leq n$, then $J_0 = (i_1 + j_1 - 1, \ldots, i_m + j_m - m)$ is admissible.

Note that one can do a good construction by adding $j_r - r$ to the $(m - r + 1)$st row of $\mu(J_1)$ to get $\mu(J_2)$.

In general, it is not easy to solve the following.

Problem How to generate all the (J_0, J_1, J_2) sequences, and do it efficiently?

By the result in [12], one can focus on (J_0, J_1, J_2) sequences with LR coefficient equal to one, i.e., there is a unique construction of $\mu(J_0)$ from $\mu(J_1)$ and $\mu(J_2)$.

However, it is hard to determine when the LR coefficient is positive or equals one. In particular, it is difficult to write a computer program to generate all LR sequences. in some situations, one may prefer to generate a class of sequences systematically even though the class may contain many redundant sequences. Taking this approach, one can use the Horn's consistent sequences (R, S, T), which is defined recursively as follows.

Let $R = (r_1, \ldots, r_m), S = (s_1, \ldots, s_m), T = (t_1, \ldots, t_m) \in [n]$.

- For $m \geq 1$, $\sum_{\ell=1}^m (r_\ell - \ell) = \sum_{\ell=1}^m (s_\ell + t_\ell - 2\ell)$.
- If $m > 1$, then for any consistent triple (U, V, W):

$$U = (u_1, \ldots, u_{m'}),\ V = (v_1, \ldots, v_{m'}),\ W = (w_1, \ldots, w_{m'})$$

with $m' \in \{1, \ldots, m-1\}$, we have

$$\sum_{\ell=1}^{m'} (r_{u_\ell} - \ell) \geq \sum_{\ell=1}^{m'} (s_{v_\ell} + t_{w_\ell} - 2\ell).$$

One can extend Theorem 1.1 to the sum of r Hermitian matrices over real, complex, or real quaternions; see [6].

Theorem 1.3 *There are* $A_1, \ldots, A_r \in \mathbf{H}_n$ *with* $\lambda(A_s) = (a_1^{(s)}, \ldots, a_n^{(s)})$ *for* $s = 1, \ldots, r$, *and* $\lambda(\sum A_j) = (a_1^{(0)}, \ldots, a_n^{(0)})$ *if and only if*

$$\sum_j a_j^{(0)} = \sum_j a_j^{(1)} + \cdots + \sum_j a_j^{(r)}$$

and for any $(J_0, J_1, \ldots, J_r) \in LR_m^n(r)$ *with* $m < n$

$$\sum_{j \in J_0} a_j^{(0)} \leq \sum_{j \in J_1} a_j^{(1)} + \cdots + \sum_{j \in J_r} a_j^{(r)}.$$

It is interesting that the same set of inequalities govern the eigenvalues of the sum of Hermitian matrices over real, complex, or real quaternions. To elaborate this comment, note that for every $A = [a_{ij}] \in \mathbf{H}_n$ there are $A_1, \ldots, A_n \in \mathbf{H}_n$ with the same eigenvalues as A such that

$$\operatorname{diag}(a_{11}, \ldots, a_{nn}) = \frac{1}{n}(A_1 + \cdots + A_n).$$

In fact, if $w = e^{2\pi i/n}$ and $D = \operatorname{diag}(1, w, \ldots, w^{n-1})$, then

$$\operatorname{diag}(a_{11}, \ldots, a_{nn}) = \frac{1}{n}\left(\sum_{j=1}^{n} D^j A (D^j)^*\right).$$

Now, by Theorem 1.3, the same result holds for real symmetric matrices. However, even for $n = 3$, it is hard to construct B_1, B_2, B_3! Let us consider the following.

Example 1.4 Let

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i2\pi/3} & 0 \\ 0 & 0 & e^{i4\pi/3} \end{pmatrix}.$$

Then $B_1 = D^*AD, B_2 = (D^2)^*AD^2, B_3 = A \in \mathbf{H}_3$ satisfy

(a) $\lambda(A) = \lambda(B_1) = \lambda(B_2) = \lambda(B_3)$, and

(b) $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{3}(B_1 + B_2 + B_3).$

Even for this specific example, it is not easy to construct $B_1, B_2, B_3 \in \mathbf{S}_3$ such that (a) and (b) hold.

2 Principal Submatrices of a Hermitian Matrix

Using the result on the sum of Hermitian matrices, we can obtain inequalities relating the eigenvalues of a Hermitian matrix and those of the principal submatrices. Here is the specific problem.
Problem Study the relations between the eigenvalues of $A \in \mathbf{H}_n$ and those of its principal submatrices.

Again, let us begin by describing some well known results; see [1, 3].

- There is $A \in \mathbf{H}_n$ with the vector of diagonal entries $(d_1, \ldots, d_n)$ if and only if it is majorized by $\lambda(A)$, i.e., $\sum_{j=1} d_j = \sum_{j=1}^{n} \lambda_j(A)$ and for $k = 1, \ldots, n-1$.

- There is $A \in \mathbf{H}_n$ with an $m \times m$ principal submatrix $B \in \mathbf{H}_m$ such that $\lambda(A) = (a_1, \ldots, a_n)$ and $\lambda(B) = (b_1, \ldots, b_m)$ if and only if

$$a_j \geq b_j \geq a_{n-m+j} \quad \text{for } j = 1, \ldots, m.$$

We have the following result; see [14].

Theorem 2.1 *There is* $A = \begin{pmatrix} A_1 & * \\ * & A_2 \end{pmatrix} \in \mathbf{H}_n$ *with* $\lambda(A) = (a_1, \ldots, a_n)$, $A_1 \in \mathbf{H}_k$ *and* $A_2 \in \mathbf{H}_{n-k}$ *such that*

$$\lambda(A_1) = (a_1^{(1)}, \ldots, a_k^{(1)}) \quad \text{and} \quad \lambda(A_2) = (a_1^{(2)}, \ldots, a_{n-k}^{(2)})$$

if and only if

$$\sum_j a_j = \sum_j a_j^{(1)} + \sum_j a_j^{(2)}$$

and for any $(J_0, J_1, J_2) \in LR_m^n$ *with* $m < n$

$$\sum_{j \in J_0} (a_j - a_n) \leq \sum_{j \in J_1} \left(a_j^{(1)} - a_n\right) + \sum_{j \in J_2} \left(a_j^{(2)} - a_n\right),$$

here $a_j^{(s)} = a_n$ *whenever* $j > n_s$.

More generally, we have the following.

Theorem 2.2 *Suppose* $n_1 + \cdots + n_r = n$. *There exists* $A = (A_{ij})_{1 \leq i,j \leq r} \in \mathbf{H}_n$ *such that* $\lambda(A) = (a_1, \ldots, a_n)$, *and* $A_{jj} \in \mathbf{H}_{n_j}$ *with* $\lambda(A_{jj}) = (a_1^{(j)}, \ldots, a_{n_j}^{(j)})$ *for* $j = 1, \ldots, r$ *if and only if*

$$\sum_j a_j = \sum_s \sum_j a_j^{(s)}$$

and for any $(J_0, J_1, \ldots, J_r) \in LR^n_m(r)$ *with* $m < n$

$$\sum_{j\in J_0} (a_j - a_n) \le \sum_{s=1}^{r} \sum_{j\in J_s} \left(a_j^{(s)} - a_n\right),$$

here $a_j^{(s)} = a_n$ *whenever* $j > n_s$.

Similar to the results on the sum of matrices, one would like to reduce the list of inequalities. For each $s = 1, \ldots, r$, only consider $J_s = (j_1^{(s)}, \ldots, j_m^{(s)})$ such that either $j_m^{(s)} \le n_s$ or the last p terms have the form: $n_s + 1, n_s + 2, \ldots, n_s + p$. Also, we did the case when $n_j \le 2$. To describe the result, we need some more notation.

Suppose $A_{ii} \in \mathbf{H}_2$ has eigenvalues $a_1^{(i)} \ge a_2^{(i)}$ for $1 \le i \le m$, and $A_{ii} = [a_1^{(i)}] \in \mathbf{H}_1$ for $m+1 \le i \le n-m$ Let $(i_1, \cdots, i_m)$ be a permutation of $(1, \cdots, m)$ such that $a_2^{(i_1)} \ge \cdots \ge a_2^{(i_m)}$. For any subset $R \subseteq \{1, \cdots, m\}$ with $|R| = r$, let $b_1^R \ge \cdots \ge b_{n-m-2r}^R$ be the eigenvalues of $\oplus_{i\notin R} A_i$.

Theorem 2.3 *There exists* $A = (A_{ij}) \in \mathbf{H}_n$ *with eigenvalues* $c_1 \ge \cdots \ge c_n$, *such that* $A_{ii} \in \mathbf{H}_2$ *has eigenvalues* $a_1^{(i)} \ge a_2^{(i)}$ *for* $1 \le i \le m$, *and* $A_{ii} = [a_1^{(i)}] \in \mathbf{H}_1$ *for* $m+1 \le i \le n-m$ *if and only if*

$$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n-m} a_1^{(i)} + \sum_{i=1}^{m} a_2^{(i)}$$

and for any $(s,t) \in \{0, \cdots, m\} \times \{0, \ldots, n-2s\}$ *with* $0 < s+t < n$ *and any* s *element subset* $S \subseteq \{i_1, \cdots, i_\ell\}$ *with* $\ell = \min\{m, s+t\}$, *we have*

$$\sum_{i=1}^{t} c_i + \sum_{i=t+2}^{s+t+1} c_i \ge \sum_{j\in S} a_2^{(j)} + \sum_{i=1}^{t} b_i^S.$$

3 Off-diagonal blocks

In this section, we study the following.

Problem Determine when a matrix $X \in M_{k,n-k}$ can be the off-diagonal block of a matrix $C \in \mathbf{H}_n$ with prescribed eigenvalues.

Observation There is $C = \begin{pmatrix} * & X \\ X^* & * \end{pmatrix} \in \mathbf{H}_n$ with $\lambda(C) = (c_1, \ldots, c_n)$ if and only if there are (for any) unitary matrices $U \in M_k$ and $V \in M_{n-k}$ the matrix

$$\tilde{C} = \begin{pmatrix} * & UXV \\ (UXV)^* & * \end{pmatrix} \in \mathbf{H}_n \text{ with } \lambda(\tilde{C}) = (c_1, \ldots, c_n).$$

Denote by $s(X) = (s_1(X), \ldots, s_k(X))$ the vector of singular values of $X \in M_{k,n-k}$ with entries arranged in descending order.

We have the following result; [15].

Theorem 3.1 *Let $c_1 \geq \cdots \geq c_n$ and $s_1 \geq \cdots \geq s_k \geq 0$ be given, where $k \leq n/2$. The following are equivalent.*

(a) *There is $C = \begin{pmatrix} * & X \\ X^* & * \end{pmatrix} \in \mathbf{H}_n$ such that $X \in M_{k,n-k}$, $\lambda(C) = (c_1, \ldots, c_n)$ and $s(X) = (s_1, \ldots, s_k)$.*

(b) *There exist $C_1, C_2, S \in \mathbf{H}_k$ such that $C_1 - C_2 \geq 2S$, where $\lambda(C_1) = (c_1, \ldots, c_k)$, $\lambda(C_2) = (c_{n-k+1}, \ldots, c_n)$, and $\lambda(S) = (s_1, \ldots, s_k)$.*

(c) *For each $(J_0, J_1, J_2) \in LR_m^k$ with $m \leq k$*

$$2 \sum_{j \in J_0} s_j \leq \sum_{j \in J_1} c_j - \sum_{j \in J_2} c_{n-j+1}.$$

There are some interesting consequences of this theorem. Let $\mathcal{S}_k(c)$ be the set of $k \times (n-k)$ matrices for the existence of $C = \begin{pmatrix} * & X \\ X^* & * \end{pmatrix} \in \mathbf{H}_n$ with $\lambda(C) = c = (c_1, \ldots, c_n)$.

- Suppose $X \in \mathcal{S}_k(c)$. Then for any contractions $R \in M_k$ and $T \in M_{n-k}$, we have $RXT \in \mathcal{S}_k(c)$. In particular, if $X \in \mathcal{S}_k(c)$ then $tX \in \mathcal{S}_k(c)$ for any $t \in [0,1]$. So, $\mathcal{S}_k(c)$ is star-shaped with the zero matrix as the star-center.
- The set $\mathcal{S}_k(c)$ is convex if and only if $(c_1, \ldots, c_k)$ and $(c_{n-k+1}, \ldots, c_n)$ are arithmetic progressions with the same common difference.

4 Complex Symmetric Matrices

In this section, we consider the following.

Problem Study the singular values of the off-diagonal blocks of complex symmetric and general matrices.

It turns out that there are not much difference between complex symmetric or general matrices with real symmetric matrices! We have the following result; see [4].

Theorem 4.1 *Let $\gamma_1 \geq \cdots \geq \gamma_n$ and $s_1 \geq \cdots \geq s_k \geq 0$ be given, where $k \leq n/2$. The following are equivalent.*

(a) *There is a symmetric matrix $C = \begin{pmatrix} * & X \\ X^t & * \end{pmatrix} \in M_n$ with $X \in M_{k,n-k}$, $s(C) = (\gamma_1, \ldots, \gamma_n)$ and $s(X) = (s_1, \ldots, s_k)$.*

(b) *There is* $C = \begin{pmatrix} * & Y \\ Z & * \end{pmatrix} \in M_n$ *with* $s(C) = (\gamma_1, \ldots, \gamma_n)$ *such that* $Y, Z \in M_{k,n-k}$ *have a combined list of singular values:* $s_1, s_1, s_2, s_2, \ldots, s_k, s_k$.

(c) *There exists a real symmetric matrix* $C = \begin{pmatrix} * & X \\ X^t & * \end{pmatrix}$ *such that* $X \in M_{k,n-k}$, $s(X) = (s_1, \ldots, s_k)$, *and*

$$\lambda(C) = (\gamma_1, -\gamma_2, \gamma_3 \ldots, (-1)^n \gamma_n).$$

(d) *There exist* $C_1, C_2, S \in \mathbf{H}_k$ *such that* $C_1 + C_2 \geq 2S$ *with* $\lambda(S) = (s_1, \ldots, s_k)$,

$$\lambda(C_1) = (\gamma_1, \gamma_3 \ldots, c_{2k-1}), \quad \textit{and} \quad \lambda(C_2) = (\gamma_2, \gamma_4, \ldots, \gamma_{2k}).$$

(e) *For each* $(J_0, J_1, J_2) \in LR_m^k$ *with* $m \leq k$

$$2 \sum_{j \in J_0} s_j \leq \sum_{j \in J_1} \gamma_{2j-1} + \sum_{j \in J_2} \gamma_{2j}.$$

Again, there are some interesting consequences of this result.

- If $X \in M_{k,n-k}$ is the $(1,2)$ block of $C \in \mathbf{H}_n$ with eigenvalues values $c_1, \ldots, c_n$, such that $|c_1| \geq \cdots \geq |c_n|$, then X is the $(1,2)$ block of $\tilde{C} \in \mathbf{H}_n$ with eigenvalues

$$|c_1|, -|c_2|, |c_3|, -|c_4|, \ldots.$$

- If $A, B, C \in \mathbf{H}_n$ satisfies $C = A + B$ and the combined list of eigenvalues of A and B is $\gamma_1 \geq \cdots \geq \gamma_{2n}$, then $C = \tilde{A} + \tilde{B}$ such that

$$\lambda(\tilde{A}) = (\gamma_1, \gamma_3, \ldots, \gamma_{2n-1}) \quad \text{and} \quad \lambda(\tilde{B}) = (\gamma_2, \gamma_4, \ldots, \gamma_{2n}).$$

- Same result work for $A_0 = A_1 + \cdots + A_r$, we can rearrange the eigenvalues:

  $\tilde{A}_1$ has eigenvalues $\gamma_1, \gamma_{r+1}, \gamma_{2r+1}, \ldots$

  $\tilde{A}_2$ has eigenvalues $\gamma_2, \gamma_{r+2}, \gamma_{2r+2}, \ldots$

  $\tilde{A}_3$ has eigenvalues $\gamma_3, \gamma_{r+3}, \gamma_{2r+3}, \ldots$

5 Future research

There are many interesting problems deserve further study. We mention a few of them in the following.

- Determine numerical algorithms to construct the matrices with desired properties.
- Study the relations between the singular values of complementary blocks; see [4, 16].
- Study the relations between the singular values of C and those of S, or those of R and T, for

$$C = \begin{pmatrix} R & 0 \\ S & T \end{pmatrix};$$

  see [13].
- Study the implications of the results in the real world!

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