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A NOTE ON THE HOMOGENIZATION OF FULLY NONLINEAR DEGENERATE ELLIPTIC EQUATIONS

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Abstract. In this note we describe some of results on the homogenization of fully nonlinear degenerate elliptic equations in the frame work of periodic homogenization, which have been obtained in a joint work with K. Shimano and P. E. Souganidis [4].

1. Periodic homogenization

We study the periodic homogenization for degenerate elliptic equations. Let $\Omega\subset\mathbf{R}^{N}$ be a bounded open set. Here $N=n+m$, with $n, m\in\mathbf{N}$, $\mathbf{R}^{N}=\mathbf{R}^{n}\times\mathbf{R}^{m}$, and a generic point $z\in\mathbf{R}^{N}$ will be denoted as $z=(x,y)$, with $x\in\mathbf{R}^{n}$ and $y\in\mathbf{R}^{m}$. We consider the Dirichlet problem

$$(1)\qquad \begin{cases} F\Big(D_{x}^{2}u^{\varepsilon}, D_{y}u^{\varepsilon}, x, y, \dfrac{x}{\varepsilon}, \dfrac{y}{\varepsilon}\Big)=0 & \text{in } \Omega,\\ u^{\varepsilon}=0 & \text{on } \partial\Omega, \end{cases}$$

where F is a real-valued continuous function on $\mathbf{S}^{n}\times\mathbf{R}^{m}\times\Omega\times\mathbf{R}^{n}\times\mathbf{R}^{m}$, $\mathbf{S}^{n}$ denotes the space of $n\times n$ real symmetric matrices, $u^{\varepsilon}=u^{\varepsilon}(x,y)$ represents the unknown function, and $\varepsilon>0$ is a parameter to be sent to zero.

Throughout this note we assume:

(A0) $F(X,q,z,\zeta)=F_{0}(X,z)+F_{1}(q,z,\zeta)$, where $F_{0}\in C(\mathbf{S}^{n}\times\overline{\Omega})$ and $F_{1}\in C(\mathbf{R}^{m}\times\overline{\Omega}\times\mathbf{R}^{N})$.

(A1) The functions $(\zeta)\mapsto F_{1}(q,z,\zeta)$ are periodic with period $\mathbf{Z}^{N}$, i.e., for all $(q,z)\in\mathbf{R}^{m}\times\overline{\Omega}$, $\zeta\in\mathbf{R}^{N}$, and $\zeta'\in\mathbf{Z}^{N}$,

$$F_{1}(q,z,\zeta+\zeta')=F_{1}(q,z,\zeta).$$

(A2) The function F_{0} is uniformly elliptic. That is, there are constants $0<\lambda\leq\Lambda<\infty$ such that for all $X, P\in\mathbf{S}^{n}$ and $z\in\overline{\Omega}$, if $P\geq 0$, then

$$-\Lambda\operatorname{tr}P\leq F_{0}(X+P,z)-F_{0}(X,z)\leq-\lambda\operatorname{tr}P.$$

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(A3) There are constants $C_0 > 0$ and $\kappa > 0$ such that for all $q \in \mathbf{R}^m$, $z \in \overline{\Omega}$, and $\zeta \in \mathbf{R}^N$,

$$C_0^{-1}|q|^\kappa - C_0 \le F_1(q, z, \zeta) \le C_0(|q|^\kappa + 1).$$

(A4) For each $R > 0$ there is a continuous non-decreasing function $\rho_R : [0, \infty) \to [0, \infty)$, with $\rho_R(0) = 0$, such that for all X, X', $Y \in \mathbf{S}^n$, $z, z' \in \overline{\Omega}$, and $\alpha > 1$, if $\|Y\| \le R$ and

$$-3\alpha \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & X' \end{pmatrix} \le 3\alpha \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix},$$

then

$$F_0(Y + X, z) - F_0(Y - X', z') \ge -\rho_R(\alpha|z - z'|^2 + |z - z'|).$$

Here and henceforth I_n denotes the unit matrix of order n.

(A5) There is a continuous non-decreasing function $\rho_1 : [0, \infty) \to [0, \infty)$, with $\rho_1(0) = 0$, such that for all $q \in \mathbf{R}^m$, $z, z' \in \overline{\Omega}$, and (ξ, η), $(\xi', \eta') \in \mathbf{R}^N$,

$$|F_1(q, z, \xi, \eta) - F_1(q, z', \xi', \eta')| \le \rho_1((|q| + 1)(|z - z'| + |\eta - \eta'|) + |\xi - \xi'|).$$

(A6) $F(0, 0, z, \zeta) \le 0$ for all $(z, \zeta) \in \overline{\Omega} \times \mathbf{R}^N$.

(A7) For compact subsets K of $\mathbf{R}^m$, the functions $(q, \eta) \mapsto F_1(q, z, \xi, \eta)$ are Lipschitz continuous on $K \times \mathbf{R}^m$. More precisely, for each compact $K \subset \mathbf{R}^m$ there is a constant $C_K > 0$ such that for all $q, q' \in K$, $z \in \overline{\Omega}$, $\xi \in \mathbf{R}^n$, and $\eta, \eta' \in \mathbf{R}^m$,

$$|F_1(q, z, \xi, \eta) - F_1(q', z, \xi, \eta')| \le C_K(|q - q'| + |\eta - \eta'|).$$

Our last assumption concerns the domain Ω, which is stated under the assumptions of (A0), (A1), and (A3). Set $M_0 = \max_{z \in \overline{\Omega}} \max\{-F_0(0, z), 0\}$. For any $r > 0$ we introduce a constant $A_r > 0$ which depends only on $r > 0$, M_0, the constants C_0, κ from (A3), the constants λ, Λ from (A2), and $\operatorname{diam}(\Omega)$. One possible choice of A_r is described as follows. We define $\alpha > 1$, $M_1 > 0$, $B > 1$, $L > 0$, and A_r in this order by

$$\alpha = 1 + \frac{\Lambda + 1}{\lambda r^2}, \qquad M_1 = [C_0(C_0 + M_0)]^{\frac{1}{\kappa}} \operatorname{diam}(\Omega),$$

$$B = 1 + \frac{e^\alpha M_0}{2} + \frac{M_1}{1 - e^{-1}}, \qquad L = [C_0(C_0 + M_0 + 2\alpha B \Lambda)]^{\frac{1}{\kappa}},$$

$$A_r = 1 + \frac{e^\alpha L}{r}.$$

For any $(\xi, \eta) \in \mathbf{R}^N$, $r > 0$, and $A > 0$ we set

$$E(\xi, \eta; r, A) = \{(x, y) \in \mathbf{R}^N \mid |x - \xi|^2 + A^2|y - \eta|^2 < r^2\},$$
$$I = \{(\xi, \eta) \in \mathbf{R}^N \mid E(\xi, \eta; r, A) \cap \Omega = \emptyset\}.$$

(A8) There are constants $r > 0$ and $A \geq A_r$ such that

$$\overline{\Omega} = \mathbf{R}^N \setminus \bigcup_{(\xi,\eta)\in I} E(\xi,\eta; r, A).$$

This condition may be considered as an "ellipse" version of the uniform exterior sphere condition.

Henceforth $\mathbf{T}^k$ denotes the k-dimensional torus $\mathbf{R}^k/\mathbf{Z}^k$. We identify any function f on $\mathbf{T}^k$ with the periodic function g with period $\mathbf{Z}^k$ given by $g(x) = f(\pi(x))$ for $x \in \mathbf{R}^k$, where π denotes the projection: $\mathbf{R}^k \ni x \mapsto x + \mathbf{Z}^k \in \mathbf{T}^k$.

Example. A typical example of equations which satisfies (A0)–(A3) is given by

$$-a(x,y)\Delta_x u^\varepsilon + b(x,y,\frac{y}{\varepsilon})|D_y u^\varepsilon| = f(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon}),$$

where $a \in C(\overline{\Omega})$, $b \in C(\overline{\Omega} \times \mathbf{T}^m)$, $f \in C(\overline{\Omega} \times \mathbf{T}^N)$, and

$$\min_{\overline{\Omega}} a > 0, \qquad \min_{\overline{\Omega}\times\mathbf{R}^m} b > 0.$$

If the functions a and b are Lipschitz continuous, then (A4) and (A5) are satisfied. If $f \geq 0$, then (A6) is satisfied. If $b(z,\eta)$ and $f(z,\xi,\eta)$ are Lipschitz continuous in η uniformly for $(x,y,\xi) \in \mathbf{R}^N \times \mathbf{R}^n$, then (A7) is satisfied.

Assumptions (A4) and (A5) are made so that the uniqueness of solutions for (1) and the Dirichlet problem for the effective equation (see (4) below) is valid. Assumption (A6) guarantees that the function $u(x,y) := 0$ is a subsolution of (1). Assumption (A8) is made in order to guarantee, together with (A2), (A3), and (A0), the existence of a supersolution for (1).

2. Effective equations

The following pair of cell problems describes the effective equation, i.e., the PDE which characterizes the limit function of the solutions u^ε of (1).

Cell problem I: given $(X,q,z,\eta) \in \mathbf{S}^n \times \mathbf{R}^m \times \overline{\Omega} \times \mathbf{R}^m$, find a constant $G(X,q,z,\eta)$ and a viscosity solution $w \in C(\mathbf{T}^n)$ of

(2) $$F(X + D^2 w(\xi), q, z, \xi, \eta) = G(X,q,z,\eta) \quad \text{in } \mathbf{R}^n.$$

Cell problem II: given $(X,q,z) \in \mathbf{S}^n \times \mathbf{R}^m \times \overline{\Omega}$ find a constant $H(X,q,z)$ and a viscosity solution $v \in C(\mathbf{T}^m)$ of

(3) $$G(X, q + Dv(\eta), z, \eta) = H(X,q,z) \quad \text{in } \mathbf{R}^m.$$

The limit function of solutions u^ε of (1) will turn out to be the unique solution of the Dirichlet problem for the effective equation:

$$\text{(4)} \qquad \begin{cases} H(D_x^2 u, D_y u, x, y) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Some properties of the effective functions G and H are given in the following propositions.

Proposition 1. For each $(X, q, z, \eta) \in \mathbf{S}^n \times \mathbf{R}^m \times \overline{\Omega} \times \mathbf{R}^m$ there is a unique constant $G(X, q, z, \eta) \in \mathbf{R}$ such that (2) has a viscosity solution $w \in C(\mathbf{T}^n)$.

Proposition 2. The function $G : \mathbf{S}^n \times \mathbf{R}^m \times \overline{\Omega} \times \mathbf{R}^m \to \mathbf{R}$ is continuous. Moreover G is uniformly elliptic, that is, for all $X, P \in \mathbf{S}^n$, $(q, z, \eta) \in \mathbf{R}^m \times \overline{\Omega} \times \mathbf{R}^m$, if $P \geq 0$, then

$$-\Lambda \operatorname{tr} P \leq G(X + P, q, z, \eta) - G(X, q, z, \eta) \leq -\lambda \operatorname{tr} P,$$

where the constants Λ and λ are those from (A2).

Proposition 3. For each $R > 0$ there is a continuous non-decreasing function $\bar\rho_R : [0, \infty) \to [0, \infty)$, with $\bar\rho_R(0) = 0$, such that for all $X, X', Y \in \mathbf{S}^n$, $q \in \mathbf{R}^m$, $z, z' \in \overline{\Omega}$, $\eta, \eta' \in \mathbf{R}^m$, and $\alpha > 1$, if $\|Y\| \leq R$ and

$$-3\alpha \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & X' \end{pmatrix} \leq 3\alpha \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix},$$

then

$$\begin{aligned} G(Y + X, q, z, \eta) - G(Y - X', q, z', \eta') \\ \geq -\bar\rho_R(\alpha |z - z'|^2 + (1 + |q|)(|z - z'| + |\eta - \eta'|)). \end{aligned}$$

Proposition 4. For all $(X, q, z, \eta) \in \mathbf{S}^n \times \mathbf{R}^m \times \overline{\Omega} \times \mathbf{R}^m$, we have

$$C_0^{-1} |q|^\kappa - C_0 \leq G(0, q, z, \eta) - F_0(0, z) \leq C_0(|q|^\kappa + 1),$$

where the constants C_0 and κ are those from (A3).

Proposition 5. For each $(X, q, z) \in \mathbf{S}^n \times \mathbf{R}^m \times \overline{\Omega}$ there is a unique constant $H(X, q, z) \in \mathbf{R}$ such that (3) has a viscosity solution $v \in C(\mathbf{T}^m)$.

Proposition 6. The function $H : \mathbf{S}^n \times \mathbf{R}^m \times \overline{\Omega} \to \mathbf{R}$ is continuous and uniformly elliptic, that is, for all $X, P \in \mathbf{S}^n$ and $(q, z) \in \mathbf{R}^m \times \mathbf{R}^N$, if $P \geq 0$, then

$$-\Lambda \operatorname{tr} P \leq H(X + P, q, z) - H(X, q, z) \leq -\lambda \operatorname{tr} P,$$

where Λ and λ are the constants from (A2).

Proposition 7. For each $R > 0$ there is a continuous non-decreasing function $\hat{\rho}_R : [0, \infty) \to [0, \infty)$, with $\hat{\rho}_R(0) = 0$, such that for all $X, X', Y \in \mathbf{S}^n$, $q \in \mathbf{R}^m$, $z, z' \in \overline{\Omega}$, and $\alpha > 1$, if $\|Y\| \le R$ and

$$-3\alpha \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & X' \end{pmatrix} \le 3\alpha \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix},$$

then

$$H(Y + X, q, z) - H(Y - X', q, z') \ge -\hat{\rho}_R(\alpha|z - z'|^2 + (1 + |q|)|z - z'|).$$

Proposition 8. For all $(X, q, z) \in \mathbf{S}^n \times \mathbf{R}^m \times \overline{\Omega}$, we have

$$\min_{\eta \in \mathbf{R}^m} G(X, q, z, \eta) \le H(X, q, z) \le \max_{\eta \in \mathbf{R}^m} G(X, q, z, \eta),$$

and, in particular,

$$C_0^{-1}|q|^\kappa - C_0 \le H(0, q, z) - F_0(0, z) \le C_0(|q|^\kappa + 1),$$

where C_0 and κ are the constants from (A3).

3. Homogenization

We begin with an existence theorem for (1) and (4).

Theorem 1. For each $\varepsilon \in (0,\ 1)$ there is a unique viscosity solution $u^\varepsilon \in C(\overline{\Omega})$ of $(1)_\varepsilon$ and a unique viscosity solution $u \in C(\overline{\Omega})$ of (4).

One can use the Perron method for the proof of the theorem above and then a crucial observation is that there is a non-negative function $\psi \in C(\overline{\Omega})$ vanishing on $\partial\Omega$ which is both a viscosity supersolution of (1) and of (4).

The main result in this note is the following:

Theorem 2. For each $\varepsilon \in (0, 1)$ let $u^\varepsilon \in C(\overline{\Omega})$ be the unique viscosity solution of (1) and u the unique viscosity solution of (4). Then, as $\varepsilon \to 0$,

$$u^\varepsilon(z) \to u(z) \quad \text{uniformly on } \overline{\Omega}.$$

Brief outline of proof. Part of the following arguments is heuristic, which simplifies the arguments.

First we define $\overline{u} \in \mathrm{USC}(\overline{\Omega})$ by

$$\overline{u}(x, y) = \limsup_{\varepsilon \searrow 0}{}^* u^\varepsilon(x, y).$$

By a barrier argument, we can show that

$$\overline{u}|_{\partial\Omega} \le 0.$$

In order to show that $\overline{u}$ is a viscosity subsolution of

$$(5) \qquad H(D_x^2 u, D_y u, x, y) = 0 \quad \text{in } \Omega,$$

let $\varphi \in C^2(\overline{\Omega})$ and assume that $\overline{u} - \varphi$ attains a strict maximum at $\bar{z} = (\bar{x}, \bar{y}) \in \Omega$. We need to show that

$$H(\bar{X}, \bar{q}, \bar{z}) \le 0,$$

where $\bar{X} = D_x^2\varphi(\bar{z})$ and $\bar{q} = D_y\varphi(\bar{z})$.

Let $v \in C(\mathbf{T}^m)$ be a viscosity solution of

$$G(\bar{X}, \bar{q} + Dv(\eta), \bar{z}, \eta) = H(\bar{X}, \bar{q}, \bar{z}) \quad \text{in } \mathbf{R}^m.$$

Let $w \in C(\mathbf{T}^n \times \mathbf{R}^m \times \mathbf{T}^m)$ be a function such that for each $(q, \eta) \in \mathbf{R}^{m+m}$ the function $u(\xi) := w(\xi, q, \eta)$ of

ξ is a viscosity solution of

$$F(\bar{X} + D^2u(\xi), \bar{q} + q, \bar{z}, \xi, \eta) = G(\bar{X}, \bar{q} + q, \bar{z}, \xi, \eta) \quad \text{in } \mathbf{R}^n.$$

Now, we make a strong assumption for simplicity of the arguments that

$$v \in C^2(\mathbf{T}^m), \qquad w \in C^2(\mathbf{T}^n \times \mathbf{R}^m \times \mathbf{T}^m).$$

For $0 < \varepsilon < 1$ we consider the function

$$u^\varepsilon(x, y) - \varphi(x, y) - \varepsilon v\Big(\frac{y}{\varepsilon}\Big) - \varepsilon^2 w\Big(\frac{x}{\varepsilon}, Dv\Big(\frac{y}{\varepsilon}\Big), \frac{y}{\varepsilon}\Big)$$

on $\overline{\Omega} \times \overline{\Omega}$ and let $z_\varepsilon \equiv (x_\varepsilon, y_\varepsilon)$ be one of its maximum points. In view of the definition of $\overline{u}$, we see that there is a sequence $\{\varepsilon_j\} \subset (0, 1)$ such that

$$\lim_{j\to\infty} \varepsilon_j = 0, \qquad \lim_{j\to\infty} z_{\varepsilon_j} = \bar{z}.$$

We will take the limit as $\varepsilon = \varepsilon_j$ and $j \to \infty$ in the following arguments. Hence we may assume that $z_\varepsilon \in \Omega$ for all $\varepsilon \in (0, 1)$ under considerations.

Now, in view of the definition of viscosity subsolutions, we have

$$F(X_\varepsilon, q_\varepsilon, z_\varepsilon, \zeta_\varepsilon) \le 0,$$

where $\zeta_\varepsilon \equiv (\xi_\varepsilon, \eta_\varepsilon) := z_\varepsilon/\varepsilon$ and

$$
\begin{aligned}
X_\varepsilon &:= D_x^2\varphi(z_\varepsilon) + D_\xi^2 w(\xi_\varepsilon, Dv(\eta_\varepsilon), \eta_\varepsilon),\\
q_\varepsilon &:= D_y\varphi(z_\varepsilon) + Dv(\eta_\varepsilon) + \varepsilon D^2 v(\eta_\varepsilon) Dw(\xi_\varepsilon, Dv(\eta_\varepsilon), \eta_\varepsilon) + \varepsilon D_\eta w(\xi_\varepsilon, Dv(\eta_\varepsilon), \eta_\varepsilon).
\end{aligned}
$$

Sending $j \to \infty$ along a subsequence, we find a point $\bar\zeta \equiv (\bar\xi, \bar\eta) \in \mathbf{T}^N$ such that

$$
F(\bar X + D^2 w(\bar\xi, Dv(\bar\eta), \bar\eta), \bar q + Dv(\bar\eta), \bar z, \bar\zeta) \le 0. \tag{6}
$$

On the other hand, by our choice of v and w, we get

$$
\begin{aligned}
&F(\bar X + D^2 w(\bar\xi, Dv(\bar\eta), \bar\eta), \bar q + Dv(\bar\eta), \bar z, \bar\zeta) = G(\bar X, \bar q + Dv(\bar\eta), \bar z, \bar\zeta),\\
&G(\bar X, \bar q + Dv(\bar\eta), \bar z, \bar\eta) = H(\bar X, \bar q, \bar z),
\end{aligned}
$$

which together yield

$$
F(\bar X + D^2 w(\bar\xi, Dv(\bar\eta), \bar\eta), \bar q + Dv(\bar\eta), \bar z, \bar\zeta) = H(\bar X, \bar q, \bar z).
$$

This combined with (6) guarantees that $H(\bar X, \bar q, \bar z) \le 0$, which was to be shown.

Similarly, we define $\underline{u} \in \mathrm{LSC}(\overline{\Omega})$ by

$$
\underline{u}(x, y) = \liminf_{\varepsilon \searrow 0}{}_* u^\varepsilon(x, y),
$$

and proceed as before to observe that $\overline{u}|_{\partial\Omega} \ge 0$ and $\underline{u}$ is a viscosity supersolution of (5).

By comparison, we find that $\overline{u} \le u \le \underline{u}$ in $\overline{\Omega}$, which shows that as $\varepsilon \to 0$,

$$
u^\varepsilon(x, y) \to u(x, y) \quad \text{uniformly on } \overline{\Omega}. \qquad \square
$$

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