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RECENT ADVANCES IN THE THEORY OF HOMOGENIZATION FOR FULLY NONLINEAR FIRST- AND SECOND-ORDER PDE IN STATIONARY ERGODIC MEDIA

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In this note I review recent results about the behavior, as $\varepsilon \to 0$, of the (viscosity) solution $u^\varepsilon \in BUC(\mathbb{R}^N)$ of Hamilton-Jacobi equations

$$H(Du^\varepsilon, u^\varepsilon, x, \varepsilon^{-1}x, \omega) = 0 \ \text{ in } \ \mathbb{R}^N , \tag{1}$$

"viscous" Hamilton-Jacobi equations

$$-\varepsilon \operatorname{tr} A(x, \varepsilon^{-1}x, \omega) D^2 u^\varepsilon + H(Du^\varepsilon, u^\varepsilon, x, \varepsilon^{-1}x, \omega) = 0 \ \text{ in } \ \mathbb{R}^N , \tag{2}$$

and fully nonlinear, uniformly elliptic equations

$$F(D^2u^\varepsilon, Du^\varepsilon, u^\varepsilon, x, \varepsilon^{-1}x, \omega) = 0 \ \text{ in } \ \mathbb{R}^N . \tag{3}$$

The theory also applies to the Cauchy as well as Dirichlet boundary and initial boundary value problems associated with the above equations.

Here $(\Omega, \mathcal{F}, P)$ is a general probability space and $\omega \in \Omega$, $BUC(\mathbb{R}^N)$ is the space of bounded uniformly continuous functions defined on $\mathbb{R}^N$, and, for each $p, x \in \mathbb{R}^N$, $X \in S^N$, the space of symmetric $N \times N$-matrices, and $r \in \mathbb{R}$, if y denotes the fast variable $\varepsilon^{-1}x$, then

$$\begin{cases} H(p, r, x, \cdot, \cdot), A(x, \cdot, \cdot) \ \text{ and } \ F(X, p, r, x, \cdot, \cdot) \\ \text{are stationary ergodic with respect to } \ (y, \omega) \ . \end{cases} \tag{4}$$

A stochastic process $f : \mathbb{R}^N \times \Omega \to \mathbb{R}$ is stationary if its distribution function is independent of its location in space, i.e., for each $\alpha \in \mathbb{R}$, $P(\{\omega \in \Omega : f(y, \omega) > \alpha\})$ is independent of $y \in \mathbb{R}^N$. This is usually quantified by assuming that, for $y \in \mathbb{R}^N$, there exists a measure preserving transformation $\tau_y : \Omega \to \Omega$. Then f is stationary if, for all $y, y' \in \mathbb{R}^N$ and $\omega \in \Omega$,

$$f(y, y', \omega) = f(y, \tau_{y'}\omega) .$$

In this note we say that a stationary process is ergodic, if the underlying measure preserving transformation τ is ergodic, i.e., if the only τ-invariant sets $A \in \mathcal{F}$ have probability either 0 or 1.

Typical examples of stationary ergodic media are random spherical inclusions and regular and irregular random chessboards – see, for example, [DM], [CSW], etc.. Of course, periodic,

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quasi-periodic and almost periodic functions can be imbedded into a stationary ergodic settings.

The aim of the theory is to identify effective (averaged) nonlinearities $\bar{H}$ and $\bar{F}$ such that solutions of (1), (2) and (3) converge, as $\varepsilon \to 0$ and a.s. in ω, to solutions of the effective equations

(5) $$\bar{H}(D\bar{u}, \bar{u}, x) = 0 \text{ in } \mathbb{R}^N ,$$

and

(6) $$\bar{F}(D^2\bar{u}, D\bar{u}, \bar{u}, x) = 0 \text{ in } \mathbb{R}^N .$$

In addition to (4), the other key assumptions are that, for each r, x, y and ω,

(7) $$H \text{ is coercive with respect to } p \text{ uniformly in } r, x, y \text{ and } \omega ,$$

i.e., as $|p| \to \infty$ and uniformly in all the other arguments, $H(p, r, x, y, \omega) \to \infty$,

(8) $$p \mapsto H(p, r, x, y, \omega) \text{ is convex,}$$

(9) $$A \in S^N \text{is degenerate elliptic,}$$

i.e., there exists $\Lambda > 0$ such that, for all $x, y, \xi \in \mathbb{R}^N$ and $\omega \in \Omega$, $0 \leqq (A(x,y)\xi, \xi) \leqq \Lambda|\xi|^2$,

(10) $$F \text{ is uniformly elliptic,}$$

i.e., there exist constants $\Lambda, \lambda > 0$ such that, for all $X, Y \in S^N$ with $Y \geqq 0$, $p, x, y \in \mathbb{R}^N$, $r \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lambda\|Y\| \leqq F(X, p, r, x, y, \omega) - F(X + Y, p, r, x, y, \omega) \leqq \Lambda\|Y\|,$$

where $\|Y\|$ denotes the the usual L^2-type norm of S^N,

(11) $$\text{there exists } \Sigma(\cdot, \cdot, \omega) \in C^{0,1}(\mathbb{R}^N \times \mathbb{R}^N; \mathcal{M}^{N\times M}) \text{ such that } A = \Sigma\Sigma^T,$$

where $\mathcal{M}^{N\times M}$ is the space of $N \times M$ matrices, and

(12) $$\begin{cases} \text{there exists } \theta \in (0,1) \text{ such that, for all } p, x, y, r \text{ and } \omega, \\ \underline{\lim}_{|p|\to\infty} |p|^{-2}(\theta(1-\theta)H^2 + \theta\|\Sigma\|^2 D_y H \cdot p) > \|D_y\Sigma\|^2\|\Sigma\|^2 . \end{cases}$$

Finally it is necessary to assume a number of technical hypotheses guaranteeing the well-posedness of (1), (2) and (3). Such conditions can be found in standard references about viscosity solutions like [B], [BC] and [CIL]. Instead of listing in detail these conditions, we assume that

(13) $$\begin{cases} H, A \text{ and } F \text{ satisfy all the assumptions guaranteeing, for each } \varepsilon > 0 \text{ and } \omega \in \Omega, \\ \text{the existence and uniqueness of viscosity solutions of (1), (2) and (3).} \end{cases}$$

The main results are:

Theorem 1. (i) *Assume that A and H satisfy* (4), (7), (8), (9), (12) *and* (13). *There exists $\bar{H} \in C(\mathbb{R}^N \times \mathbb{R} \times U)$ satisfying* (7), (8) *and* (13) *such that, if $u^\varepsilon(\cdot, \omega)$ and $\bar{u}$ are respectively, for each ω, the viscosity solutions of* (2) *and* (5), *then, as $\varepsilon \to 0$ and a.s. in ω, $u^\varepsilon(\cdot, \omega) \to \bar{u}$ in $C(\mathbb{R}^N)$.*

(ii) *Assume that A, H and A_n, H_n satisfy* (4), (7), (8), (9), (12) *and* (13) *and that, as $n \to \infty$ and a.s. in ω, $A_n \to A$ and $H_n \to H$ in $C(\mathbb{R}^N \times \mathbb{R}^N)$ and $C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)$ respectively. Let $\bar{H}$ and $\bar{H}_n$ be the respective effective nonlinearities. Then, as $n \to \infty$, $\bar{H}_n \to \bar{H}$ in $C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$.*

Since, in view of (9), it is possible to have $A \equiv 0$, the above theorem also provides the homogenization result for (1).

Theorem 2. (i) *Assume that F satisfies (4), (10) and (13). There exists $\bar{F} \in C(S^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ satisfying (10) and (13) such that, if $u^\varepsilon(\cdot,\omega)$ and $\bar{u}$ are respectively, for each ω, the solutions of (3) and (6), then, as $\varepsilon \to 0$ and a.s. in ω, $u^\varepsilon(\cdot,\omega) \to \bar{u}$ in $C(\mathbb{R}^N)$.*

(ii) *Assume that F and F_n satisfy (4), (10) and (13) and that, as $n \to \infty$ and a.s. in ω, $F_n \to F$ in $C(S^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)$. Let $\bar{F}_n$ and $\bar{F}$ be the respective effective nonlinearities. Then, as $n \to \infty$, $\bar{F}_n \to \bar{F}$ is $C(S^n \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$.*

Associated with (1), (2) and (3) and for each X, p, r, x and ω are the macroscopic problems

$$H(p + Dv, r, x, y, \omega) = \bar{H}(p, r, x) \quad \text{in} \quad \mathbb{R}^N, \tag{14}$$

$$-\operatorname{tr} A(x, y, \omega) D^2 v + H(p + Dv, r, x, y, \omega) = \bar{H}(p, r, x) \quad \text{in} \quad \mathbb{R}^N, \tag{15}$$

and

$$F(X + D^2 v, p, r, x, y, \omega) = \bar{F}(X, p, r, x) \quad \text{in} \quad \mathbb{R}^N. \tag{16}$$

In order for the constants $\bar{H}(p,r,x)$ and $\bar{F}(X,p,r,x)$ to be unique, it is necessary to find solutions $v \in UC(\mathbb{R}^N)$, the space of uniformly continuous functions on $\mathbb{R}^N$, which have, a.s. in ω, strictly sub-linear (for (14) and (15)) and strictly sub-quadratic (for (16)) growth at infinity. This, in general, is not possible, as it was shown in [LS1] and [LS2].

There is an extensive literature about the homogenization, in the periodic setting, of (1), (2) and (3) and their generalizations. The results are based on the fact that in such settings it is possible to solve the associated macroscopic problems, which are now set in the periodic cell and, therefore, are usually called the cell problems. Replacing the cell problem by an equation in $\mathbb{R}^N$ (the macroscopic equation), which admits appropriate approximate solutions, it is also possible to study, following [I] and [LS3], the homogenization of (1), (2) and (3) and their generalizations in almost periodic settings, which have some strong compactness properties.

The situation is, however, quite different in the stationary ergodic setting, since, due to the luck of compactness, in general, it is not possible to solve, either exactly or approximately, the macroscopic problem. It is therefore necessary to follow a different strategy making use of the sub-additive ergodic theorem.

Papanicolaou and Varadhan [PV1], [PV2] and Kozlov [K] (see also [JKO]) were the first to consider the problem of homogenizing linear, uniformly elliptic/parabolic operators. Their results were later generalized to particular quasi-linear problems by Bensoussan and Blakenship [BB] and Castel [Cas] - see also [BP] for other more recent results in the linear setting. The first nonlinear result in the variational setting was obtained by Dal Maso and Modica [DM].

The homogenization of fully nonlinear, convex, first-order (Hamilton-Jacobi) equations, i.e, Theorem 1 with $A \equiv 0$, was studied in [Sol] (see also [RT]). In a subsequent work [LS1], it was shown that, in general, in this case, there are no solutions (correctors) of the associated macroscopic problem.

The homogenization result for (2), i.e., Theorem 1, was obtained in [LS2], where it was also shown that, in general, appropriate correctors do not exist. The homogenization of fully nonlinear, uniformly elliptic equations, i.e., Theorem 2, was studied by Caffarelli, Souganidis and Wang in [CSW]. Finally, in a forthcoming paper ([CLS]) it is shown that correctors exist for convex, fully nonlinear, uniformly elliptic equations in the stationary ergodic setting.

The proof of Theorem 1 relies very heavily on the facts that H is convex and and A is independent of p provide formulae for the solutions of (1) and (2), which are based on the control interpretation of the equations. The formula for (1) is sub-additive and so it is possible to apply directly the sub-additive ergodic theorem which yields the convex dual of $\bar{H}$. The stochastic control formula for (2) is not, however, (sub-)additive, and, hence, it is not possible to apply directly the (sub-additive) ergodic theorem at least in the degenerate case. When H grows faster than quadratically in p, there exists a convenient sub-additive representation, which, however, yields only a super-solution for (2). Nevertheless, in this case, it is possible to show that, as $\varepsilon \to 0$, the difference between this super-solution and the solution of (2) tends to 0. This in turn identifies the averaged limit of the u^ε's as the averaged limit of the particular super-solutions. When H does not have this growth, one argues by penalizing the equation and obtaining bounds independent of the penalization.

In the generality of Theorem 2 there are no suitable representations for the solutions of (3). The approach taken in [CSW] is, therefore, very different. Roughly speaking the effective nonlinearity is defined by identifying all the matrices which belong to its level sets. This in turn is achieved by studying the obstacle problem associated with the equation with quadratic obstacle. The sub-additive ergodic theorem allows to identify the "critical" quadratics. The uniform ellipticity of F plays a fundamental role here.

Asymptotic problems like (1), (2) and (3) arise in a variety of applications ranging from front propagation to turbulent combustion, large deviations for diffusion in random environments, etc..

An example in front propagation which can be analyzed using Theorem 1 is the level set pde

$$\begin{cases} u_t^\varepsilon + v(\varepsilon^{-1}x,\omega)|Du^\varepsilon| = 0 \ \text{ in } \ \mathbb{R}^N \times (0,T] \ , \\ u^\varepsilon = g \ \text{ on } \ \mathbb{R}^N \times \{0\} \ , \end{cases}$$

which describes the generalized evolution of the level sets of g with normal velocity

$$V = -v(\varepsilon^{-1}x,\omega).$$

A typical problem in theory of large deviations is the following: If $(\Omega,\mathcal{F},P)$ is a given probability space, let $V : \mathbb{R}^N \times \mathbb{R}^N \times \Omega \to [0,\infty)$ be a stationary ergodic random variable and $(X_t^\varepsilon)_{t\geq 0}$ be a diffusion process evolving according to the sde

$$\begin{cases} dX_t^\varepsilon = b(\varepsilon^{-1}X_t^\varepsilon,\omega) + \sqrt{2\varepsilon}\Sigma(\varepsilon^{-1}X_t^\varepsilon,\omega)\, dB_t \qquad (t>0) \ , \\ X_0^\varepsilon = x \ , \end{cases}$$

where B_t is a standard M-dimensional Brownian motion on a different probability space, b is a Lipshitz continuous stationary ergodic vector field and Σ is a Lipshitz continuous and stationary ergodic $N \times M$-symmetric matrix.

The diffusion in the potential V is governed by the weighted probability

$$Q_{t,\omega}^\varepsilon(d\omega_0) = S_{t,\omega}^{-1} \exp\left\{-\varepsilon^{-1}\int_0^t V(X_s^\varepsilon(\omega),\varepsilon^{-1}(X_s^\varepsilon(\omega)),\omega)\right\} P_0(d\omega_0) \ ,$$

where $S_{t,\omega}$ is a normalizing factor.

It turns out that the a.s. asymptotics, as $\varepsilon \to 0$, of events related to the above diffusion process are governed by the a.s. asymptotics, as $\varepsilon \to 0$, of the solutions of

$$\begin{cases} u_t^\varepsilon - \varepsilon \operatorname{tr}(A(\varepsilon^{-1}x,\omega)D^2u^\varepsilon) + (A(\varepsilon^{-1}x,\omega)Du^\varepsilon, Du^\varepsilon) \\ -b(\varepsilon^{-1}x,\omega)\cdot Du^\varepsilon - V(x,\varepsilon^{-1}x,\omega) = 0 \text{ in } \mathbb{R}^N \times (0,T]\ , \end{cases}$$

for appropriate initial conditions.

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