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On a connection problem of simple pole type operators of second order in exact WKB analysis

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§0. Introduction

In our exact WKB theoretic study of simple pole type operators ([AKKT2]), we announced that the connection problem for WKB solutions of any simple pole type operators can be reduced to that of a second order equation of the form

$$\left(-\frac{d^2}{dx^2}+\eta^2 V(x,\eta)\right)\psi=0, \tag{0.1}$$

where η denotes a large parameter,

$$V(x,\eta)=\frac{V_0(x)}{x}+\eta^{-1}\frac{V_1(x)}{x}+\eta^{-2}\frac{V_2(x)}{x^2}+\sum_{j\geq 3}\eta^{-j}\frac{V_j(x)}{x}, \tag{0.2}$$

$\{V_j\}$ are holomorphic functions near the origin, and $V_0(0)\neq 0$. Actually, this equation is obtained if we eliminate the first order term of simple pole type operator of the second order by changing the unknown functions.

In [K1] and [K2] the case where $V_1=0$ and $V_j=0$ for $j\geq 3$ was considered. There it was shown that although the origin is a regular singular point of (0.1), it plays the same role as a turning point. For example the logarithmic derivative $S(x,\eta)=\eta S_{-1}+S_0+\eta^{-1}S_1+\cdots$ of WKB solutions behaves like $S_j=O(x^{-j/2-1})$ near the origin. That is, the order of singularity of S_j becomes worse and worse as j increases. This is a typical feature of the logarithmic derivative of WKB solutions near turning points. This situation is also observed for (0.1) and we expect that the origin also plays the same role as a turning point. This expectation has been validated in [K1] and [K2], by analyzing the singularity structure of the Borel transform of WKB solutions of (0.1) when $V_1=0$ and $V_j=0$ for $j\geq 3$. In this paper we extend the analysis of [K1] and [K2] to a more general potential as given in (0.2).

§1. Connection formulas for simple pole type operators of second order

We consider

$$\left(-\frac{d^2}{dx^2}+\eta^2 Q(x,\eta)\right)\psi=0, \tag{1.1}$$

where η denotes a large parameter, and

$$Q(x,\eta)=\frac{Q_0(x)}{x}+\eta^{-1}\frac{Q_1(x)}{x}+\sum_{j\geq 2}\eta^{-j}\frac{Q_j(x)}{x^2}. \tag{1.2}$$

with $\{Q_j\}$ satisfying the conditions (A.1), (A.2) and (A.3) below:

(A.1) Each Q_j are holomorphic in a neighborhood $U\subset\mathbb{C}$ of the origin, and $Q_0(0)\neq 0$.

(A.2) $\{Q_j\}$ is pre-Borel summable in U, i.e., for any compact set K in U, there exist constants $A_K, C_K>0$ for which

$$\sup_{x\in K}|Q_j(x)|\leq A_K {C_K}^j j! \tag{1.3}$$

holds for any $j\geq 0$.

(A.3) $Q_j(0)=0$ for $j\geq 3$.

We study the analytic structure of Borel transform of WKB solutions of (1.1). We note that the equation (1.1) has a slightly different form from (0.1); instead we assume (A.3). This is for a notational simplicity.

For this equation we can construct WKB solutions of the form

$$\psi_\pm=\frac{1}{\sqrt{S_{\mathrm{odd}}(x,\eta)}}\exp\left(\pm\int_0^x S_{\mathrm{odd}}(x,\eta)dx\right), \tag{1.4}$$

where

$$\begin{aligned} S_{\mathrm{odd}}(x,\eta) &= (S^+(x,\eta)-S^-(x,\eta))/2 \\ &= \eta S_{\mathrm{odd},-1}(x)+S_{\mathrm{odd},0}(x)+\eta^{-1}S_{\mathrm{odd},1}(x)+\cdots \end{aligned} \tag{1.5}$$

with $S_{\mathrm{odd},-1}(x) = \sqrt{Q_0(x)/x}$, and

$$S^{\pm}(x,\eta) = \eta S^{\pm}_{-1}(x) + S^{\pm}_0(x) + \eta^{-1}S^{\pm}_1(x) + \cdots \tag{1.6}$$

with $S^{\pm}_{-1}(x) = \pm\sqrt{Q_0(x)/x}$ are two formal solutions of the Riccati equation

$$S^2 + \frac{dS}{dx} = \eta^2 Q(x,\eta) \tag{1.7}$$

associated to (1.1). We call WKB solutions of (1.1) normalized as (1.4) as WKB solutions normalized at the origin.

WKB solutions (1.4) have the following expansion:

$$\psi_{\pm} = e^{\pm\eta s(x)} \sum_{j=0}^{\infty} \psi_{\pm,j}(x)\eta^{-j-1/2}, \tag{1.8}$$

where

$$s(x) = \int_0^x S_{\mathrm{odd},-1}(x)dx = \int_0^x \sqrt{\frac{Q_0(x)}{x}}dx. \tag{1.9}$$

Then the Borel transform of WKB solutions (1.4) are defined to be

$$\psi_{\pm,B}(x,y) = \sum_{j=0}^{\infty} \frac{\psi_{\pm,j}(x)}{\Gamma(j+1/2)}(y \pm s(x))^{j-1/2}. \tag{1.10}$$

Concerning the analyticity of the Borel transform $\psi_{\pm,B}$ of the WKB solutions we obtain

Theorem 1. *We assume conditions* (A.1), (A.2) *and* (A.3). *Then for the Borel transform* (1.10) *of WKB solutions of* (1.1), *we can find a positive constant r_0 for which the following hold:*

(i) $(y+s(x))^{1/2}\psi_{+,B}$ *and* $(y-s(x))^{1/2}\psi_{-,B}$ *converge and define holomorphic functions in* $W_+(r_0)$ *and* $W_-(r_0)$ *respectively, where*

$$W_{\pm}(r_0) = \{(x,y) \in \mathbb{C}; 0 < |x| < r_0,\ |y \pm s(x)| < 2|s(x)|\}. \tag{1.11}$$

(ii) $\psi_{+,B}$ *and* $\psi_{-,B}$ *can be analytically continued and define multi-valued analytic functions in* $W_-(r_0) \setminus \{y = s(x)\}$ *and* $W_+(r_0) \setminus \{y = -s(x)\}$, *respectively.*

(iii) *The discontinuity of* $\psi_{+,B}(x,y)$ *(resp.* $\psi_{-,B}$*) along the cut*

$$\{(x,y) \in \mathbb{C}^2; \operatorname{Im} y = \operatorname{Im} s(x),\ \operatorname{Re} y \geq \operatorname{Re} s(x)\} \tag{1.12}$$

$$(\textit{resp.}\quad \{(x,y) \in \mathbb{C}^2; \operatorname{Im} y = \operatorname{Im}(-s(x)),\ \operatorname{Re} y \geq \operatorname{Re}(-s(x))\}) \tag{1.13}$$

coincides with

$$2i\cos(\pi\sqrt{1+4Q_2(0)})\psi_{-,B}(x,y) \tag{1.14}$$

$$(\textit{resp.}\quad 2i\cos(\pi\sqrt{1+4Q_2(0)})\psi_{-,B}(x,y)). \tag{1.15}$$

Here we note that $\sqrt{1+4Q_2(0)}$ is a difference value of two characteristic exponents of (1.1) at the regular singular point $x = 0$.

If we assume *endless continuability* and some appropriate growth conditions on $\psi_{\pm,B}(x,y)$ (See [DP]. See also [V] and [DDP].), we obtain the following connection formulas of Borel sum of WKB solutions from Theorem thm:main: first, in view of singularity structure of $\psi_{\pm,B}$, we define the Stokes curve γ

$$\operatorname{Im}(s(x)) = \operatorname{Im}(-s(x)) \quad \Leftrightarrow \quad \operatorname{Im} \int_0^x \sqrt{\frac{Q_0(x)}{x}} dx = 0. \tag{1.16}$$

Then when we cross γ in a counterclockwise manner with a center the origin, we obtain

$$\psi_+ \mapsto \psi_+ + 2i\cos(\pi\sqrt{1+4Q_2(0)})\psi_-, \tag{1.17}$$

$$\psi_- \mapsto \psi_-, \tag{1.18}$$

if $\operatorname{Re} \int_0^x \sqrt{Q_0(x)/x}\,dx > 0$ holds along γ, or

$$\psi_+ \mapsto \psi_+, \tag{1.19}$$

$$\psi_- \mapsto \psi_- + 2i\cos(\pi\sqrt{1+4Q_2(0)})\psi_+, \tag{1.20}$$

if $\operatorname{Re} \int_0^x \sqrt{Q_0(x)/x}\,dx < 0$ holds along γ.

Remark. *The Stokes multiplier* $2i\cos(\pi\sqrt{1+4Q_2(0)})$ *depends on the subleading terms* $Q_2(x)$ *of the potential* $Q(x,\eta)$*. This fact was essential in the argument discussed in* [AKKT2].

§2. Sketch of the proof of the connection formulas

To prove Theorem 1, we first transform (1.1) to a canonical equation

$$\left(-\frac{d^2}{dx^2}+\eta^2\left(\frac{1}{x}+\eta^{-2}\frac{\lambda}{x^2}\right)\right)\psi=0, \tag{2.1}$$

where $\lambda=\lambda_0+\eta^{-1}\lambda_1+\cdots$ with $\lambda_j\in\mathbb{C}$. Here, to distinguish the independent variable and the unknown functions of (1.1) from (2.1), we use $\tilde{x}$ and $\tilde{\psi}$ as the independent variable and the unknown functions of (1.1) respectively. In fact we can prove the following:

Proposition 2. *Assume (A.1) and (A.2). Then we can find a neighborhood V of $\tilde{x}=0$ and*

$$x=x(\tilde{x},\eta)=x_0(\tilde{x})+\eta^{-1}x_1(\tilde{x})+\cdots \tag{2.2}$$

where $\{x_j(\tilde{x})\}_{j\geq 0}$ are holomorphic functions in some open neighborhood $V\subset U$, and satisfy the following:

(i) $x_0(0)=0$, $(dx_0/d\tilde{x})(0)\neq 0$.

(ii) $x_j(0)=0$ *for* $j\geq 1$.

(iii) *there exist positive constants A, C for which the following holds in V:*

$$|x_j(\tilde{x})|\leq AC^{j-1}j!. \tag{2.3}$$

(iv) *The following relation holds degree by degree with respect to η:*

(2.4)
$$Q(\tilde{x},\eta)=\left(\frac{\partial x}{\partial\tilde{x}}\right)^2\left(\frac{1}{x(\tilde{x},\eta)}+\eta^{-2}\frac{\lambda}{x(\tilde{x},\eta)^2}\right)-\frac{1}{2}\eta^{-2}\left\{x(\tilde{x},\eta);\tilde{x}\right\},$$

where $\lambda=\lambda_0+\eta^{-1}\lambda_1+\cdots$ is given by $\lambda_j=Q_{j+2}(0)$. (Hence $\lambda_j=0$ for $j\geq 1$ if (A.3) *holds.) Here $\{x,\tilde{x}\}$ denotes the Schwarzian derivative, i.e.,*

$$\{x;\tilde{x}\}=\frac{x'''}{x'}-\frac{3}{2}\left(\frac{x''}{x'}\right)^2, \tag{2.5}$$

where $'$ denotes the differentiation with respect to $\tilde{x}$.

(v) *The following relation holds among the WKB solutions $\tilde{\psi}_\pm$ of (1.1) normalized at the origin, and $\psi_\pm$ of (2.1):*

$$\tilde{\psi}_\pm(\tilde{x},\eta) = \left(\frac{\partial x}{\partial \tilde{x}}\right)^{-1/2} \psi_\pm(x(\tilde{x},\eta),\eta). \tag{2.6}$$

Proof of this Proposition 2 will be given in the subsequent sections.

Once this proposition is proved, then we can prove Theorem 1 in the same manner as in [K2]. In fact, by considering the Borel transform of (2.6), we obtain

$$\tilde{\psi}_{\pm,B}(\tilde{x},y) = (P_{\mp 2\sqrt{x}}\psi_{\pm,B}(x,y))_{x=x_0(\tilde{x})}, \tag{2.7}$$

where

(2.8)

$$\begin{aligned} &P_{y_0}(x;\frac{\partial}{\partial x},\frac{\partial}{\partial y}) \\ &\quad = J(x)\sum_{N=0}^{\infty}\sum_{\substack{\mu+\nu+m+n=N\\ \mu,\nu\geq 0\\ m,n\geq 0}}\sum_{\substack{\mu_1+\cdots+\mu_n=\mu\\ \nu_1+\cdots+\nu_n=\nu\\ \mu_j,\nu_j\geq 0}}(-1)^n\frac{\Gamma(n+1/2)}{\Gamma(1/2)m!n!} \\ &\qquad\qquad \times\tilde{x}_{\mu_1+1}(x)\cdots\tilde{x}_{\mu_m+1}(x)\tilde{x}'_{\nu_1+1}(x)\cdots\tilde{x}'_{\nu_n+1}(x)\left(\frac{\partial}{\partial x}\right)^m\left(\frac{\partial}{\partial y}\right)^{-N}_{y_0}, \end{aligned}$$

and

$$(x_0'(\tilde{x}))^{-1/2} = J(x_0(\tilde{x})),\quad x_j(\tilde{x}) = \tilde{x}_j(x_0(\tilde{x})). \tag{2.9}$$

(See [K2] for the definition of $(\partial/\partial y)^{-N}_{y_0}$.) In [K2] we study the analyticity of the right-hand side of (2.7) by using the explicit description of $\psi_{\pm,B}(x,y)$, which are expressed by Gauss hypergeometric functions (Here we note that $\lambda=\lambda_0$ since we assume (A.3) in Theorem 1.):

$$\psi_{+,B}(x,y) = \frac{1}{\sqrt{4\pi}}s^{-1/2}\,F\left(\alpha-\frac{1}{2},\beta-\frac{1}{2},\frac{1}{2};s\right)\Bigg|_{s=\frac{y}{4\sqrt{x}}+\frac{1}{2}},$$

$$\psi_{-,B}(x,y) = \frac{1}{\sqrt{-4\pi}}(1-s)^{-1/2}\,F\left(\alpha-\frac{1}{2},\beta-\frac{1}{2},\frac{1}{2};1-s\right)\Bigg|_{s=\frac{y}{4\sqrt{x}}+\frac{1}{2}},$$

where α and β are constants satisfying $\alpha+\beta=2$ and $\alpha\beta=-4\lambda_0$. There we have only used the properties in Proposition 2. Hence once Proposition 2 is proven, then we can use the same argument to obtain Theorem 1.

§3. Formal coordinate transformation to a canonical equation

In this section we construct the transformation function $\{x_j(\tilde{x})\}$ so that it satisfies (2.4), and then this $\{x_j\}$ satisfies the properties (i), (ii), (iv) and (v) in Proposition 2. The proof of Proposition 2 (iii) will be given in §4.

By comparing (2.4) degree by degree with respect to η, we obtain

$$
(3.1) \quad \begin{cases} \left(\dfrac{dx_0}{d\tilde{x}}\right)^2 \dfrac{1}{x_0} = \dfrac{Q_0(\tilde{x})}{\tilde{x}}, & (3.1.0) \\ \left(2\dfrac{x_0'}{x_0}\dfrac{d}{d\tilde{x}} - \left(\dfrac{x_0'}{x_0}\right)^2\right) x_n(\tilde{x}) = F_n(\tilde{x}) - \left(\dfrac{x_0'}{x_0}\right)^2 \lambda_{n-2} & (3.1.n) \end{cases}
$$

for $n \geq 1$ (we set $\lambda_{-1} = 0$ for the convenience). Here

$$
(3.2) \quad F_1(\tilde{x}) = \frac{Q_1(\tilde{x})}{\tilde{x}},
$$

$$
(3.3) \quad F_2(\tilde{x}) = \frac{Q_2(\tilde{x})}{\tilde{x}^2} - \frac{{x_1'}^2}{x_0} - \left(\frac{x_0' x_1}{x_0}\right)^2 + \frac{2x_0' x_1' x_1}{x_0^2} + \frac{1}{2}\{x_0; \tilde{x}\},
$$

and

(3.4)

$$
\begin{aligned}
F_n(\tilde{x}) = {} & \frac{Q_n(\tilde{x})}{\tilde{x}^2} + \sum_{\mu+\nu+l=n} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu \\ \nu_1+\nu_2=\nu \\ 0\leq\mu_j\leq n-2 \\ 0\leq\nu_j\leq n-1}} (-1)^{l+1} x'_{\nu_1} x'_{\nu_2} \frac{x_{\mu_1+1}\cdots x_{\mu_l+1}}{{x_0}^{l+1}} \\
& + \sum_{\substack{\mu+\nu+l+k=n-2 \\ k\neq n-2}} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu \\ \nu_1+\nu_2=\nu}} (-1)^{l+1}(l+1)\lambda_k x'_{\nu_1} x'_{\nu_2} \frac{x_{\mu_1+1}\cdots x_{\mu_l+1}}{{x_0}^{l+2}} \\
& + \frac{1}{2} \sum_{\mu+l+k=n-2} \sum_{\mu_1+\cdots+\mu_l=\mu} (-1)^l x_k'''(x) \frac{x'_{\mu_1+1}\cdots x'_{\mu_l+1}}{{x_0'}^{l+1}} \\
& + \frac{3}{4} \sum_{\mu+l+k=n-2} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu \\ \nu_1+\nu_2=\nu}} (-1)^{l+1}(l+1) x''_{\nu_1} x''_{\nu+2} \frac{x'_{\mu_1+1}\cdots x'_{\mu_l+1}}{{x_0'}^{l+1}}.
\end{aligned}
$$

We will now solve (3.1.n) for $n \geq 0$ step by step. First we obtain

$$
(3.5) \quad x_0(\tilde{x}) = \left(\frac{1}{2}\int_0^{\tilde{x}} \sqrt{\frac{Q_0(\tilde{x})}{\tilde{x}}}\, d\tilde{x}\right)^{1/2}
$$

from (3.1.0). Then we easily confirm that (i) is satisfied since $x_0(\tilde{x})$ can be expanded as

$$(3.6) \qquad x_0(\tilde{x}) = \sqrt{Q_0(0)}\tilde{x} + O(\tilde{x}^2)$$

near the origin, and $Q_0(0) \neq 0$ by our assumption (A.1). We take the neighborhood $U_0 \subset U$ of the origin so that x_0 is holomorphic in U_0 and x_0' does not vanish in U_0. The holomorphic solution (3.1.1) is obtained by

$$(3.7) \qquad x_1(\tilde{x}) = \sqrt{x_0(\tilde{x})} \int_0^{\tilde{x}} \frac{\sqrt{x_0(\tilde{x})}}{2x_0'(\tilde{x})} F_1(\tilde{x}) d\tilde{x}$$

since $F_1(\tilde{x})$ has a simple pole at the origin (See (3.2).). This x_1 is holomorphic in U_0 and vanish at the origin. Now we solve (3.1.2). We first observe that

$$(3.8) \qquad x_2(\tilde{x}) = \sqrt{x_0(\tilde{x})} \int_0^{\tilde{x}} \frac{\sqrt{x_0(\tilde{x})}}{2x_0'(\tilde{x})} \left(F_2(\tilde{x}) - \left(\frac{x_0'}{x_0}\right)^2 \lambda_0 \right) d\tilde{x}$$

give a holomorphic solution of (3.1.2). Then we choose

$$(3.9) \qquad \lambda_0 = \left(\frac{x_0}{x_0'}\right)^2 F_2(\tilde{x}) \bigg|_{\tilde{x}=0}$$

to ensure that this solution vanish at the origin.

Remark. *At the level $n = 2$, we can obtain a holomorphic solution of (3.1.n) even if we do not assume the condition (3.9). However, the resulting solution $x_2(\tilde{x})$ does not vanish at the origin. Hence $F_n(\tilde{x})$ would have higher order (≥ 2) poles at the origin by its definition, in general. Thus we can not expect a holomorphic solution of (3.1.n) near the origin. We can obtain holomorphic solution of (3.1.2) without the condition (3.9).*

We now inductively determine $x_n(\tilde{x})$ for $n \geq 3$. We first note that $F_n(\tilde{x})$ have a double pole at the origin by our induction hypothesis. Then we choose λ_{n-2} as

$$(3.10) \qquad \lambda_{n-2} = \left(\frac{x_0}{x_0'}\right)^2 F_n(\tilde{x}) \bigg|_{\tilde{x}=0} .$$

By this choice of the constant λ_{n-2}, we obtain a holomorphic solution

$$x_n(\tilde{x}) = \sqrt{x_0(\tilde{x})} \int_0^{\tilde{x}} \frac{\sqrt{x_0(\tilde{x})}}{2x_0'(\tilde{x})} \left(F_n(\tilde{x}) - \left(\frac{x_0'}{x_0}\right)^2 \lambda_0 \right) d\tilde{x} \tag{3.11}$$

of (3.1.n), which vanishes at the origin. This $x_n(\tilde{x})$ is holomorphic in U_0 Hence our induction runs and the construction of $\{x_j(\tilde{x})\}$ has been completed.

We then prove $\lambda_j = Q_{j+2}(0)$ for $j \geq 0$. Multiplying (2.4) by $\tilde{x}^2$, and taking the limit $\tilde{x}$ tends to zero, we obtain

$$\lim_{\tilde{x}\to 0} \tilde{x}^2 Q(\tilde{x}, \eta) = \eta^{-2}\lambda \lim_{\tilde{x}\to 0} \left(\frac{\partial x}{\partial \tilde{x}}\right)^2 \frac{\tilde{x}^2}{x(\tilde{x},\eta)^2}. \tag{3.12}$$

The right-hand side of (3.12) becomes

$$\eta^{-2}(Q_2(0) + \eta^{-1}Q_3(0) + \cdots) \tag{3.13}$$

while the lefthand side of (3.12) becomes $\eta^{-2}\lambda$ because $x_j(0) = 0$ for all j. Hence we obtain

$$\lambda = Q_2(0) + \eta^{-1}Q_3(0) + \cdots . \tag{3.14}$$

Since (v) is a direct consequence of (2.4), the remaining part of the proof of Proposition 2 is (iii), the pre-Borel summability of the transformation function $x(\tilde{x}, \eta)$. We prove this pre-Borel summability in the next section.

§4. Pre-Borel summability of the transformation function.

In this section we prove Proposition 2 (iii). We can assume that there exist positive constants B, D and R so that following hold:

(a) $x_0(\tilde{x})$ is holomorphic in $\{x; |x| \leq R\}$.

(b) $x_0(\tilde{x})$ is holomorphic in $\{x; |x| \leq R\}$.

(c) For every n,

$$\sup_{|x|\leq R} \left| \left(\frac{x_0}{x_0'}\right)^2 \frac{Q_n(\tilde{x})}{\tilde{x}^2} \right| \leq n! B D^n. \tag{4.1}$$

Then we can find a positive constant C_1 so that

$$\sup_{|x|\le R} |x_0(x)| \le C_1 \quad \text{and} \quad \frac{1}{C_1} \le \sup_{|x|\le R} |x_0'(x)| \le C_1. \tag{4.2}$$

Furthermore, as we have shown in the previous section, $x_j(\tilde{x})$ for $j \ge 1$ is holomorphic in $\{x; |x| \le R\}$.

The pre-Borel summability of $\{x_j(\tilde{x})\}$ near the origin is a consequence of the following:

Lemma 3. *We can find positive constants A and C so that the following inequalities hold for any sufficiently small $\epsilon > 0$ and $n \ge 1$:*

$$\begin{cases} \sup\limits_{|\tilde{x}|\le R-\epsilon} |x_n(\tilde{x})| \le n! A C^{n-1} \epsilon^{-n}, \\ \sup\limits_{|\tilde{x}|\le R-\epsilon} |x_n'(\tilde{x})| \le n! A C^{n-1} \epsilon^{-n}, \\ \sup\limits_{|\tilde{x}|\le R-\epsilon} \left|\dfrac{x_n(\tilde{x})}{x_0(\tilde{x})}\right| \le n! A C^{n-1} \epsilon^{-n}. \end{cases} \tag{4.3}$$

To prove this lemma we prepare the following:

Lemma 4. *([K1, Lemma 2.3]) Let R' be a positive number, $v(t)$ a holomorphic function on $\{t \in \mathbb{C}; |t| < R'\}$ satisfying $v(0) = 0$. Then the differential equation*

$$\left(t\frac{d}{dt} - \frac{1}{2}\right) u(t) = v(t) \tag{4.4}$$

has a unique holomorphic solution on $\{t \in \mathbb{C}; |t| < R'\}$, which satisfies the following inequalities for any positive $R'' < R'$:

$$\sup_{|t|\le R''} |u(t)| \le 2 \sup_{|t|\le R''} |v(t)|, \tag{4.5}$$

$$\sup_{|t|\le R''} |u'(t)| \le \frac{2}{R''} \sup_{|t|\le R''} |v(t)|, \tag{4.6}$$

$$\sup_{|t|\le R''} \left|\frac{u(t)}{t}\right| \le \frac{2}{R''} \sup_{|t|\le R'} |v(t)|. \tag{4.7}$$

See [K1, pp.43–44] for the proof of Lemma 4.

By changing a local coordinate through $t = x_0(\tilde{x})$ in (3.1), we obtain

$$\left(t\frac{d}{dt} - \frac{1}{2}\right) x_n = \frac{1}{2}\left\{\left(\frac{x_0}{x_0'}\right)^2 F_n - \lambda_{n-2}\right\}. \tag{4.8}$$

Since we choose λ_n as (3.10), the right-hand side of (4.8) have a zero at the origin. Hence by Lemma 4, we find a positive constant C_2 such that for any sufficiently small positive ϵ,

$$\sup_{|\tilde{x}|\le R-\epsilon} |x_n(\tilde{x})|, \ \sup_{|\tilde{x}|\le R-\epsilon} |x_n'(\tilde{x})| \quad \text{and} \quad \sup_{|\tilde{x}|\le R-\epsilon} \left|\frac{x_n(\tilde{x})}{x_0(\tilde{x})}\right| \tag{4.9}$$

are dominated by

$$C_2 \sup_{|\tilde{x}|\le R-\epsilon} \left|\left(\frac{x_0}{x_0'}\right)^2 F_n - \lambda_{n-2}\right|. \tag{4.10}$$

To give the estimation of (4.10) we decompose F_n as

$$F_n(\tilde{x}) = \frac{Q_n(\tilde{x})}{\tilde{x}^2} + F_{n,\mathrm{I}} + F_{n,\mathrm{II}} + F_{n,\mathrm{III}} + F_{n,\mathrm{IV}}, \tag{4.11}$$

where

$$\begin{aligned}
F_{n,\mathrm{I}} &= \sum_{\mu+\nu+l=n} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu \\ \nu_1+\nu_2=\nu \\ 0\le\mu_j\le n-2 \\ 0\le\nu_j\le n-1}} (-1)^{l+1} x_{\nu_1}' x_{\nu_2}' \frac{x_{\mu_1+1}\cdots x_{\mu_l+1}}{{x_0}^{l+1}}, \\
F_{n,\mathrm{II}} &= \sum_{\substack{\mu+\nu+l+k=n-2 \\ k\neq n-2}} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu \\ \nu_1+\nu_2=\nu}} (-1)^{l+1}(l+1)\lambda_k x_{\nu_1}' x_{\nu_2}' \frac{x_{\mu_1+1}\cdots x_{\mu_l+1}}{{x_0}^{l+2}}, \\
F_{n,\mathrm{III}} &= \frac{1}{2} \sum_{\mu+l+k=n-2} \sum_{\mu_1+\cdots+\mu_l=\mu} (-1)^l x_k'''(x) \frac{x_{\mu_1+1}'\cdots x_{\mu_l+1}'}{{x_0'}^{l+1}}, \\
F_{n,\mathrm{IV}} &= \frac{3}{4} \sum_{\mu+\nu+l=n-2} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu \\ \nu_1+\nu_2=\nu}} (-1)^{l+1}(l+1) x_{\nu_1}'' x_{\nu_2}'' \frac{x_{\mu_1+1}'\cdots x_{\mu_l+1}'}{{x_0'}^{l+1}}.
\end{aligned}$$

In the following we give the estimation of $F_{n,\mathrm{I}}$, $F_{n,\mathrm{II}}$, $F_{n,\mathrm{III}}$ and $F_{n,\mathrm{IV}}$ respectively. Without loss of generality, we can assume that C is so large that

$$D < C \quad \text{and} \quad \epsilon < 1 \tag{4.12}$$

holds.
1°) To give the estimation of $F_{n,\mathrm{I}}$, we write $F_{n,\mathrm{I}} = F_{n,\mathrm{I}}^{(1)} + 2F_{n,\mathrm{I}}^{(2)} + F_{n,\mathrm{I}}^{(3)}$ with

$$F_{n,\mathrm{I}}^{(1)} = \sum_{\mu+l=n} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu \\ 0\le\mu_j\le n-2}} (-1)^{l+1} x'_0 x'_0 \frac{x_{\mu_1+1}\cdots x_{\mu_l+1}}{x_0{}^{l+1}}, \tag{4.13}$$

$$F_{n,\mathrm{I}}^{(2)} = \sum_{\substack{\mu+k+l=n \\ 1\le k\le n-1}} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu \\ 0\le\mu_j\le n-2}} (-1)^{l+1} x'_0 x'_k \frac{x_{\mu_1+1}\cdots x_{\mu_l+1}}{x_0{}^{l+1}}, \tag{4.14}$$

$$F_{n,\mathrm{I}}^{(3)} = \sum_{\mu+\nu+l=n} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu \\ \nu_1+\nu_2=\nu \\ 0\le\mu_j\le n-2 \\ 1\le\nu_j\le n-1}} (-1)^{l+1} x'_{\nu_1} x'_{\nu_2} \frac{x_{\mu_1+1}\cdots x_{\mu_l+1}}{x_0{}^{l+1}}. \tag{4.15}$$

We fist estimate $F_{n,\mathrm{I}}^{(1)}$. We obtain

$$\begin{aligned} \left(\frac{x_0}{x'_0}\right)^2 F_{n,\mathrm{I}}^{(1)} &\le \frac{|x_0|^2}{|x'_0|^2} \sum_{\mu+l=n} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu \\ 0\le\mu_j\le n-2}} \frac{|x'_0||x'_0|}{|x_0|} \left|\frac{x_{\mu+1}}{x_0}\right| \cdots \left|\frac{x_{\mu+1}}{x_0}\right| \\ &= |x_0| \sum_{\mu+l=n} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu \\ 0\le\mu_j\le n-2}} \left|\frac{x_{\mu+1}}{x_0}\right| \cdots \left|\frac{x_{\mu+1}}{x_0}\right|. \end{aligned} \tag{4.16}$$

By using (4.2) and (4.3) we find that $|(x_0/x'_0)^2 F_{n,\mathrm{I}}^{(1)}|$ is dominated by

$$\begin{aligned} & C_1 \sum_{\mu+l=n} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu \\ 0\le\mu_j\le n-2}} A^l C^\mu \epsilon^{-\mu-l} (\mu_1+1)!\cdots(\mu_l+1)! \\ &= C_1 C^n \epsilon^{-n} \sum_{l=2}^{n} \left(\frac{A}{C}\right)^l \sum_{\substack{\mu_1+\cdots+\mu_l=n-l \\ 0\le\mu_j\le n-2}} (\mu_1+1)!\cdots(\mu_l+1)! \end{aligned} \tag{4.17}$$

Then we use the formula

$$\sum_{\substack{n_1+n_2+\cdots+n_l=n \\ n_j\ge1}} n_1!\cdots n_l! \le n! \tag{4.18}$$

to obtain

$$\begin{aligned} \left(\frac{x_0}{x'_0}\right)^2 F_{n,\mathrm{I}}^{(1)} &\le n! C_1 C^n \epsilon^{-n} \sum_{l=2}^{n} \left(\frac{A}{C}\right)^l \\ &\le n! A C^{n-1} \epsilon^{-n} \cdot \frac{AC_1}{C(1-A/C)}. \end{aligned} \tag{4.19}$$

In a similar manner we can give the estimation of $F^{(2)}_{n,\mathrm{I}}$ and $F^{(3)}_{n,\mathrm{I}}$ as follows:

(4.20)

$$\begin{aligned}
\left(\frac{x_0}{x_0'}\right)^2 F^{(2)}_{n,\mathrm{I}} &\le C_1^2 \sum_{\substack{\mu+k+l=n\\ 1\le k\le n-1}} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu\\ 0\le\mu_j\le n-2}} A^{l+1}C^{\mu+\nu-1}\epsilon^{-\mu-\nu-l} \\
&\qquad\times \nu!(\mu_1+1)!\cdots(\mu_l+1)! \\
&= AC_1^2C^{n-1}\epsilon^{-n}\sum_{l=1}^{n}\left(\frac{A}{C}\right)^l \sum_{\substack{\mu_1+\cdots+\mu_l+k=n-l\\ 0\le\mu_j\le n-2, 1\le k\le n-1}} k!(\mu_1+1)!\cdots(\mu_l+1)! \\
&\le n!AC_1^2C^{n-1}\epsilon^{-n}\sum_{l=1}^{n}\left(\frac{A}{C}\right)^l \\
&\le n!AC^{n-1}\epsilon^{-n}\frac{C_1^2}{C(1-A/C)},
\end{aligned}$$

(4.21)

$$\begin{aligned}
\left(\frac{x_0}{x_0'}\right)^2 F^{(3)}_{n,\mathrm{I}} &\le C_1^3 \sum_{\mu+\nu+l=n} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu\\ \nu_1+\nu_2=\nu\\ 0\le\mu_j\le n-2\\ 1\le\nu_j\le n-1}} A^{l+2}C^{\mu+\nu}\epsilon^{-\mu-\nu-l} \\
&\qquad\times \nu_1!\nu_2!(\mu_1+1)!\cdots(\mu_l+1)! \\
&\le A^2C_1^3\sum_{l=0}^{n}\left(\frac{A}{C}\right)^l \sum_{\substack{\mu_1+\cdots+\mu_l+\nu_1+\nu_2=n-l\\ \nu_1,\nu_2\ge 1}} \nu_1!\nu_2!(\mu_1+1)!\cdots(\mu_l+1)! \\
&\le n!AC^{n-1}\epsilon^{-n}\frac{AC_1^3}{C(1-A/C)}.
\end{aligned}$$

2°) We give the estimation of $F_{n,\mathrm{II}}$. We first write $F_{n,\mathrm{II}} = F^{(1)}_{n,\mathrm{II}} + F^{(2)}_{n,\mathrm{II}} + F^{(3)}_{n,\mathrm{II}}$ with

(4.22)
$$\begin{aligned}
F^{(1)}_{n,\mathrm{II}} &= \sum_{\substack{\mu+l+k=n-2\\ k\neq n-2}} \sum_{\mu_1+\cdots+\mu_l=\mu} (-1)^{l+1}(l+1)\lambda_k x'_0 x'_0 \frac{x_{\mu_1+1}\cdots x_{\mu_l+1}}{{x_0}^{l+2}},\\
F^{(2)}_{n,\mathrm{II}} &= \sum_{\substack{\mu+\nu+l+k=n-2\\ k\neq n-2,\nu\neq 0}} \sum_{\mu_1+\cdots+\mu_l=\mu} (-1)^{l+1}(l+1)\lambda_k x'_0 x'_\nu \frac{x_{\mu_1+1}\cdots x_{\mu_l+1}}{{x_0}^{l+2}},\\
F^{(3)}_{n,\mathrm{II}} &= \sum_{\substack{\mu+\nu+l+k=n-2\\ k\neq n-2}} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu\\ \nu_1+\nu_2=\nu\\ \nu_1,\nu_2\neq 0}} (-1)^{l+1}(l+1)\lambda_k x'_{\nu_1} x'_{\nu_2} \frac{x_{\mu_1+1}\cdots x_{\mu_l+1}}{{x_0}^{l+2}}.
\end{aligned}$$

We first note that we obtain

$$|\lambda_k| \leq k!B'D^k \quad (k \geq 0) \tag{4.23}$$

for some positive constant B' since $\lambda_k = Q_k(0)$ and (4.1). By using the similar argument as 1°), we find

$$\left|\left(\frac{x_0}{x'_0}\right) F^{(1)}_{n,\mathrm{II}}\right| \leq B'C^{n-2}\epsilon^{-n+2}\sum_{l=0}^{n-2}\left(\frac{A}{C}\right)^l \sum_{k=0}^{n-l-2}\left(\frac{D\epsilon}{C}\right)^k k!(n-2-k)!. \tag{4.24}$$

Since we assume (4.12), we obtain

$$\sum_{k=0}^{n-l-2}\left(\frac{D\epsilon}{C}\right)^k k!(n-2-k)! \leq \sum_{k=0}^{n-l-2} k!(n-2-k)! \leq 3(n-2)!. \tag{4.25}$$

Hence we conclude that

$$\left|\left(\frac{x_0}{x'_0}\right)^2 F^{(1)}_{n,\mathrm{II}}\right| \leq (n-2)!AC^{n-1}\epsilon^{-n}\frac{3B'}{AC(1-A/C)}. \tag{4.26}$$

In a similar manner we obtain

$$\left|\left(\frac{x_0}{x'_0}\right) F^{(2)}_{n,\mathrm{II}}\right| \leq (n-2)!AC^{n-1}\epsilon^{-n}\frac{3B'C_1\epsilon^2}{C^2(1-A/C)}, \tag{4.27}$$

$$\left|\left(\frac{x_0}{x'_0}\right) F^{(3)}_{n,\mathrm{II}}\right| \leq (n-2)!AC^{n-1}\epsilon^{-n}\frac{3AB'C_1^2\epsilon^2}{C^3(1-A/C)}. \tag{4.28}$$

3°) We give the estimation of $F_{n,\mathrm{III}}$. As in the previous estimation we write $F_{n,\mathrm{III}} = F^{(1)}_{n,\mathrm{III}} + F^{(2)}_{n,\mathrm{III}}$ with

$$
\begin{aligned}
(4.29)\qquad F^{(1)}_{n,\mathrm{III}} &= \frac{1}{2}\sum_{\mu+l=n-2}\ \sum_{\mu_1+\cdots+\mu_l=\mu}(-1)^l x_0'''(x)\frac{x'_{\mu_1+1}\cdots x'_{\mu_l+1}}{x_0'^{\,l+1}},\\
(4.30)\qquad F^{(2)}_{n,\mathrm{III}} &= \frac{1}{2}\sum_{\substack{\mu+l+k=n-2\\ k\neq 0}}\ \sum_{\mu_1+\cdots+\mu_l=\mu}(-1)^l x_k'''(x)\frac{x'_{\mu_1+1}\cdots x'_{\mu_l+1}}{x_0'^{\,l+1}}.
\end{aligned}
$$

By a straightforward computation we obtain

$$
\begin{aligned}
(4.31)\qquad \left|\left(\frac{x_0}{x_0'}\right)^2 F_{n,\mathrm{III}\text{-}1}\right| &\le (n-2)!AC^{n-1}\epsilon^{-n}\frac{C_1^6}{2AC(1-AC_1/C)},\\
(4.32)\qquad \left|\left(\frac{x_0}{x_0'}\right)^2 F_{n,\mathrm{III}\text{-}2}\right| &\le n!AC^{n-1}\epsilon^{-n}\frac{e^3AC_1^5}{2C^2(1-AC_1/C)}.
\end{aligned}
$$

Here we have used the inequality

$$(4.33)\qquad \sup_{|\tilde{x}|\le R-\epsilon}\left|\frac{d^k x_0}{d\tilde{x}^k}\right| \le C_1\epsilon^{n-1}$$

and

$$(4.34)\qquad \sup_{|\tilde{x}|\le R-\epsilon}\left|\frac{d^k x_n}{d\tilde{x}^k}\right| \le (n+k)!AC^{n-1}\epsilon^{-n-k}e^k \quad (n\ge 1).$$

In fact, (4.33) follows from

$$(4.35)\qquad \frac{d^k x_0}{d\tilde{x}^k} = \frac{1}{2\pi i}\oint_{|\zeta-x|=\epsilon}\frac{x_0'(\zeta)}{(\zeta-\tilde{x})^n}d\zeta,$$

and (4.34) can be obtained inductively by using

$$(4.36)\qquad \frac{d^k x_n}{d\tilde{x}^k} = \frac{1}{2\pi i}\oint_{|\zeta-x|=\frac{\epsilon}{n+1}}\frac{x_n^{(k-1)}(\zeta)}{(\zeta-\tilde{x})^2}d\zeta.$$

4°) We give the estimation of $F_{n,\mathrm{IV}}$. We write $F_{n,\mathrm{IV}} = F^{(1)}_{n,\mathrm{IV}} + 2F^{(2)}_{n,\mathrm{IV}} + F^{(3)}_{n,\mathrm{IV}}$ with

(4.37)
$$F^{(1)}_{n,\mathrm{IV}} = \frac{3}{4}\sum_{\mu+l=n-2}\sum_{\mu_1+\cdots+\mu_l=\mu}(-1)^{l+1}(l+1)x_0''x_0''\frac{x'_{\mu_1+1}\cdots x'_{\mu_l+1}}{x_0'^{\,l+1}},$$

(4.38)
$$F^{(2)}_{n,\mathrm{IV}} = \frac{3}{4}\sum_{\substack{\mu+l+\nu=n-2,\\ \nu\neq 1}}\sum_{\mu_1+\cdots+\mu_l=\mu}(-1)^{l+1}(l+1)x_0''x_\nu''\frac{x'_{\mu_1+1}\cdots x'_{\mu_l+1}}{x_0'^{\,l+1}},$$

(4.39)
$$F^{(3)}_{n,\mathrm{IV}} = \frac{3}{4}\sum_{\mu+l+\nu=n-2}\sum_{\substack{\mu_1+\cdots+\mu_l=\mu\\ \nu_1+\nu_2=\nu\\ \nu_1,\nu_2\neq 0}}(-1)^{l+1}(l+1)x_{\nu_1}''x_{\nu_2}''\frac{x'_{\mu_1+1}\cdots x'_{\mu_l+1}}{x_0'^{\,l+1}}.$$

We then obtain

$$\left|\left(\frac{x_0}{x_0'}\right)^2 F^{(1)}_{n,\mathrm{IV}}\right| \le (n-2)!AC^{n-1}\epsilon^{-n}\frac{3C_1^7\epsilon}{4AC(1-A/C)^2}, \tag{4.40}$$

$$\left|\left(\frac{x_0}{x_0'}\right)^2 F^{(2)}_{n,\mathrm{IV}}\right| \le (n-1)!AC^{n-1}\epsilon^{-n}\frac{3eC_1^6}{4C^2(1-AC_1/C)^2}, \tag{4.41}$$

$$\left|\left(\frac{x_0}{x_0'}\right)^2 F^{(3)}_{n,\mathrm{IV}}\right| \le n!AC^{n-1}\epsilon^{-n}\frac{e^2AC_1^4}{4C^3(1-AC_1/C)}. \tag{4.42}$$

Summing up, we conclude that (4.10) is dominated by

$$n!AC^{n-1}\epsilon^{n-1}\times C_2C_3, \tag{4.43}$$

where

(4.44)

$$
\begin{aligned}
C_3 &= \frac{BD}{A} + \left(\frac{AC_1}{C(1-A/C)} + \frac{2C_1^2}{C(1-A/C)} + \frac{AC_1^3}{C(1-A/C)}\right) \\
&\quad + \left(\frac{3B'}{AC(1-A/C)} + \frac{6B'C_1}{C^2(1-A/C)} + \frac{3AB'C_1^2}{C^3(1-A/C)}\right) \\
&\quad + \left(\frac{C_1^6}{2AC(1-AC_1/C)} + \frac{e^3AC_1^5}{2C^2(1-AC_1/C)}\right) \\
&\quad + \left(\frac{3C_1^7}{4AC(1-A/C)^2} + \frac{6C_1^6}{4C^2(1-AC_1/C)^2} + \frac{e^2AC_1^4}{4C^3(1-AC_1/C)}\right).
\end{aligned}
$$

Hence we first choose A so that $BD < A$ and (4.3) holds for $n = 1$. Then we chose C so large that $D < C$ and $C_2C_3 < 1$. Then then our induction proceeds. This prove Lemma 4.3.

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