

**Asymptotic Behavior of Solutions for an Interface Equation
— Interface Dynamics and Center-of-Mass Motions —**

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1 INTRODUCTION

In 1992, Rubinstein & Sternberg proposed in [9] the following scalar bistable reaction-diffusion equation with a nonlocal term:

$$(NL) \quad \begin{cases} u_t^\epsilon = \epsilon^2 \Delta u^\epsilon + f(u^\epsilon) - \frac{1}{|\Omega|} \int_{\Omega} f(u^\epsilon) dx, & t > 0, x \in \Omega, \\ \partial u^\epsilon / \partial \mathbf{n} = 0, & t > 0, x \in \partial\Omega. \end{cases}$$

Here, Ω is a smooth bounded domain in \mathbb{R}^N ($N \geq 2$) with volume $|\Omega|$ and the outward unit normal \mathbf{n} on the boundary $\partial\Omega$, $\epsilon > 0$ is a small parameter, and f is a function derived from a double-well potential, a typical example being $f(u) = u - u^3$.

It is known that the dynamics of solutions for (NL) with $\epsilon \ll 1$ consists of several stages, and is roughly summarized as follows.

- (S1) Generation of transition layers [9]:
The solution with an appropriate initial condition generates sharp transition layer near an interface. Such a solution is referred to as a layer solution.
- (S2) Motion of interfaces (i) [9]:
The layer solution begins to evolve so that the corresponding interface is driven according to a certain motion law, called an interface equation. The interface equation is given by (2.15) in [9].
- (S3) Motion of interfaces (ii) [9]:
The layer solution then comes to evolve so that the corresponding interface is governed by another interface equation, which is well-known as the volume-preserving mean curvature flow (cf. [5, 6, 7]). The interface is driven in such a way that the volume enclosed by the interface is preserved and the area of the interface decreases. After a coarsening process, the interface evolves into a single sphere. The layer solution with spherical shape is referred to as the bubble solution.
- (S4) Motion of bubbles (i) [1, 3, 11, 12]:
The bubble solution drifts with exponentially slow speed without changing shape toward the closest point on $\partial\Omega$ from the center of the corresponding sphere.
- (S5) Motion of bubbles (ii) [2, 12]:
Once the bubble solution attaches to $\partial\Omega$, it intersects perpendicularly to $\partial\Omega$ and evolves along $\partial\Omega$ by a geometric information of $\partial\Omega$.

The dynamics in (S1)-(S3) was formally discussed in [9] by employing asymptotic analysis. For (S4), the existence of such bubble motions was established by Alikakos et al. [1]. Ward gave in [11] an explicit asymptotic ordinary differential equation for the distance between the center of the bubble and the closest point on $\partial\Omega$ (see also [12]). Alikakos et al. derived in [3] such an equation for the Cahn-Hilliard equation and compared the bubble motions for the Cahn-Hilliard equation with those for the nonlocal equation (NL). The dynamics in (S5) was studied by Alikakos et al. [2].

Our concern in this paper is the dynamics occurring in the *intermediate stage* (S2), i.e., *after* the formation of layers and *before* the volume-preserving mean curvature flow is effective. The corresponding interface equation was earlier given by Rubinstein & Sternberg as (2.15) in [9]. The form, however, was *implicit* and unsuitable for the circumstantial examination from interfacial approach. We will present below an *explicit* formulation of the interface equation, and discuss the dynamics and asymptotic behavior of solutions for the interface equation. Throughout the remaining of this paper, an "interface" means a smooth, closed, $(N-1)$ -dimensional hypersurface Γ embedded in $\Omega \subset \mathbb{R}^N$ staying uniformly away from $\partial\Omega$. We denote by Ω^\pm subdomains in Ω separated by an interface Γ such as

$$\Omega = \Omega^- \cup \Gamma \cup \Omega^+, \quad \partial\Omega^- = \Gamma, \quad \partial\Omega^+ = \partial\Omega \cup \Gamma,$$

and by $\nu(x; \Gamma)$ the unit normal vector on Γ at $x \in \Gamma$ pointing toward the interior of the subdomain Ω^+ (cf. FIGURE 1).

Let $\{\Gamma(t)\}_{t \geq 0}$ be a family of interfaces parameterized by time $t \geq 0$, with each interface $\Gamma(t)$ satisfying the conditions above. The interface equation associated with the intermediate stage (S2) is explicitly given as follows (cf. [8]):

$$(IE-a) \quad \mathbf{v}(x; \Gamma(t)) = c(v(t)), \quad t > 0, \quad x \in \Gamma(t),$$

$$(IE-b) \quad \dot{v}(t) = \frac{h^+(v(t)) - h^-(v(t))}{h_v^-(v(t))|\Omega^-(t)| + h_v^+(v(t))|\Omega^+(t)|} c(v(t)) |\Gamma(t)|, \quad t > 0,$$

$$(IE-c) \quad \Gamma(0) = \Gamma_0, \quad v(0) = v_0.$$

Here, $\mathbf{v}(x; \Gamma(t))$ denotes the normal velocity of $\Gamma(t)$ at $x \in \Gamma(t)$ in ν -direction, the symbols $|\Gamma(t)|$ and $|\Omega^\pm(t)|$ stand for the surface area of $\Gamma(t)$ and the volumes of $\Omega^\pm(t)$, respectively, $c(v)$ and $h^\pm(v)$ are some functions defined in a neighborhood of $v = 0$ satisfying

$$(1.1) \quad c(0) = 0, \quad c'(0) > 0; \quad h^-(v) < 0 < h^+(v), \quad h_v^\pm(v) < 0.$$

The interface equation (IE) is derived in [8] as a singular limit of the following problem

$$(RD) \quad \begin{cases} \epsilon u_t^\epsilon = \epsilon^2 \Delta u^\epsilon + f(u^\epsilon) - v^\epsilon, & t > 0, \quad x \in \Omega, \\ v^\epsilon = \frac{1}{|\Omega|} \int_\Omega f(u^\epsilon) dx, & t \geq 0, \\ \partial u^\epsilon / \partial \mathbf{n} = 0, & t > 0, \quad x \in \partial\Omega \end{cases}$$

as $\epsilon \rightarrow 0$, where the problem (RD) is obtained by rescaling the time t in (NL) as $t \mapsto \epsilon^{-1}t$ and introducing an auxiliary variable for the nonlocal term. In this situation, the functions $h^\pm(v)$, $c(v)$ in (IE) satisfying (1.1) are determined by the nonlinearity $f(u) - v$ in (RD). We should also mention that $\Gamma(t)$ and $v(t)$ in (IE) correspond to limits of the level-set interface $\Gamma^\epsilon(t) := \{x \in \Omega \mid u^\epsilon(t, x) = 0\}$ and the nonlocal term $v^\epsilon(t)$ in (RD) as $\epsilon \rightarrow 0$, respectively, and that u^ϵ with $\epsilon \ll 1$ has a sharp layer structure near $\Gamma(t)$ such as

$$u^\epsilon(t, x) \approx h^\pm(v(t)), \quad t > 0, \quad x \in \Omega^\pm(t).$$

By virtue of the precise formulations as (RD) and (IE), the unique existence of time local and global solutions to (IE) and the convergence of solutions for (RD) to those for (IE) have been successfully established in [8]. In particular, the convergence result guarantees that the dynamics of solutions for (IE) *does* approximate that for (NL) in the intermediate stage (S2). In this sense, it is of crucial importance to examine the dynamics and asymptotic behavior of solutions to (IE) for understanding the dynamics occurring in (S2).

2 INTERFACE DYNAMICS

By the conditions in (1.1), it is easily verified that the dynamics of $(\Gamma(t), v(t))$ is as follows:

- (i) $v > 0 \implies \dot{v} > 0, \dot{v} < 0$;
 $\Gamma(t)$ evolves in such a way that $\Omega^-(t)$ grows (or $\Omega^+(t)$ shrinks) and $v(t)$ decreases.
- (ii) $v < 0 \implies \dot{v} < 0, \dot{v} > 0$;
 $\Gamma(t)$ evolves in such a way that $\Omega^-(t)$ shrinks (or $\Omega^+(t)$ grows) and $v(t)$ increases.
- (iii) $v = 0 \implies \dot{v} = 0, \dot{v} = 0$;
 $\Gamma(t)$ and $v(t)$ do not evolve.

The description above is the same as that in [9]. To examine the asymptotic behavior of solutions for (IE), we will recast (IE) as an equivalent problem.

2.1 Preliminaries. Before moving on, we prepare in this subsection notations and tools needed later.

For $t \geq 0$, we express $\Gamma(t)$ as a smooth embedding from a fixed $(N-1)$ -dimensional reference manifold \mathcal{M} to \mathbb{R}^N :

$$(2.1) \quad \gamma(t, \cdot) : \mathcal{M} \rightarrow \Gamma(t), \quad \mathcal{M} \ni y \mapsto x = \gamma(t, y) \in \Gamma(t).$$

Let $\nu(t, y) \in \mathbb{R}^N$ be the unit normal vector $\nu(x; \Gamma(t))$ on $\Gamma(t) = \gamma(t, \mathcal{M})$ at $x = \gamma(t, y)$. We normalize the parametrization (2.1) so that γ_t is always parallel to ν . This yields that $\gamma_t = \nu \nu$. We simply denote by $\varphi(y)$ and $n(y)$ instead of $\gamma(0, y)$ and $\nu(0, y)$, respectively. For sufficient small $|r|$, we define the embedding $F(r, \cdot) : \mathcal{M} \rightarrow \Omega$ as

$$F(r, \cdot) : y \mapsto F(r, y) = \varphi(y) + rn(y).$$

Then the hypersurface defined by

$$(2.2) \quad \Gamma_r := F(r, \mathcal{M}) = \{x \in \Omega \mid x = \varphi(y) + rn(y), y \in \mathcal{M}\}$$

is smooth for small $|r|$. Let dS_y^r be the volume element on \mathcal{M} induced from the surface element on $\Gamma_r = F(r, \mathcal{M})$ at $x = F(r, y)$ by the embedding $F(r, \cdot)$. Note that dS_y^r has the following expressions:

$$\begin{aligned} dS_y^r &= \prod_{j=1}^{N-1} (1 + r\kappa_j(y)) dS_y^0 \\ &=: \sum_{j=0}^{N-1} H_j(y) r^j dS_y^0. \end{aligned}$$

Here, $\kappa_j(y)$ ($j = 1, \dots, N-1$) stand for the principal curvatures of $\Gamma_0 = \varphi(\mathcal{M})$ at $x = \varphi(y)$ and $H_j(y)$ ($j = 0, \dots, N-1$) are the fundamental symmetric functions of $\kappa_1(y), \dots, \kappa_{N-1}(y)$:

$$\begin{aligned} H_0(y) &\equiv 1, \\ H_1(y) &= \kappa_1(y) + \dots + \kappa_{N-1}(y), \\ &\vdots \\ H_{N-1}(y) &= \kappa_1(y) \cdots \kappa_{N-1}(y). \end{aligned}$$

The surface area $|\Gamma_r|$ is denoted by $g(r)$. Then $g(r)$ is explicitly given by

$$(2.3) \quad g(r) = \sum_{i=0}^{N-1} \left(\int_{\mathcal{M}} H_i(y) dS_y^0 \right) r^i.$$

2.2 Asymptotic behavior of solutions. Let us now recast (IE) as an equivalent problem.

For a given initial interface Γ_0 , we express the interface $\Gamma(t)$ as the graph of a function $r(t, y)$ over Γ_0 :

$$(2.4) \quad \Gamma(t) = \{x \in \Omega \mid x = \gamma(t, y) = \varphi(y) + r(t, y)n(y), y \in \mathcal{M}\}.$$

A simple computation implies that $\nu(t, y)$ is independent of $t > 0$:

$$(2.5) \quad \nu(t, y) \equiv n(y).$$

From (2.4), (2.5) and $\mathbf{v} = \gamma_t \cdot \nu$, the equation in (IE-a) now turns out to be expressed as $r_t(t, y) = c(v(t))$. The initial condition for $\Gamma(t)$ in (IE-c) is recast in terms of r as $r(0, y) = 0$. Then the equation for $a(t, y) := \nabla_y r(t, y)$ becomes

$$a_t(t, y) = 0, \quad a(0, y) = 0,$$

from which it follows $a(t, y) \equiv 0$, i.e., $r(t, y)$ is independent of $y \in \mathcal{M}$:

$$(2.6) \quad r(t, y) \equiv r(t).$$

Therefore, the equation in (IE-a) with the initial condition in (IE-c) is recast as

$$\dot{r}(t) = c(v(t)), \quad r(0) = 0.$$

By (2.4) and (2.6), the interface $\Gamma(t)$ is expressed as

$$\Gamma(t) = \{x \in \Omega \mid x = \varphi(y) + r(t)n(y), y \in \mathcal{M}\} = \Gamma_{r(t)},$$

while the surface area $|\Gamma_r|$ is given by (2.3). Thus we have $|\Gamma(t)| = g(r(t))$.

On the other hand, the following identities are valid:

$$\frac{d}{dt}|\Omega^-(t)| = -\frac{d}{dt}|\Omega^+(t)| = \int_{\Gamma(t)} \mathbf{v}(x; \Gamma(t)) dS_x^{\Gamma(t)},$$

where $dS_x^{\Gamma(t)}$ stands for the surface element on $\Gamma(t)$ at $x \in \Gamma(t)$. Thanks to the relations above, it is easy to verify that the problem (IE) gives rise to

$$\frac{d}{dt} \left[h^-(v(t)) \frac{|\Omega^-(t)|}{|\Omega|} + h^+(v(t)) \frac{|\Omega^+(t)|}{|\Omega|} \right] \equiv 0.$$

Therefore, we obtain the conservation property

$$h^-(v(t)) \frac{|\Omega^-(t)|}{|\Omega|} + h^+(v(t)) \frac{|\Omega^+(t)|}{|\Omega|} \equiv m, \quad t > 0.$$

This, together with $|\Omega^-(t)| + |\Omega^+(t)| \equiv |\Omega|$, implies that the first term on the right hand side of (IE-b) is rewritten as a function of $v(t)$ alone, say $h(v(t))$. Thus the problem (IE) is recast as

$$(ODE) \quad \begin{cases} \dot{r}(t) = c(v(t)), & t > 0, \\ \dot{v}(t) = h(v(t)) c(v(t)) g(r(t)), & t > 0, \\ r(0) = 0, \quad v(0) = v_0, \end{cases}$$

in which the functions $c(v)$, $h(v)$ and $g(r)$ satisfy the following in a neighborhood of $(r, v) = (0, 0)$:

$$(2.7) \quad c(0) = 0, \quad c'(0) = 0, \quad h(v) < 0, \quad g(r) > 0.$$

By using (2.7), it is proved [8] that the problem (ODE) has a time global solution $(r(t), v(t))$ which converges to $(r^*, 0)$ monotonously as $t \rightarrow \infty$, where r^* is a constant (the phase portrait for (ODE) is as FIGURE 2). This immediately yields that (IE) has a time global solution $(\Gamma(t), v(t))$ such that $v(t) \rightarrow 0$ and $\Gamma(t) = \Gamma_{r(t)}$ approaches an equilibrium interface $\Gamma^* = \Gamma_{r^*}$ monotonously as $t \rightarrow \infty$.

3 CENTER-OF-MASS MOTIONS

Using the ODE expression (ODE) for (IE), we can discuss the center-of-mass motion for the interfaces governed by (IE). For each interface Γ , its center of mass $p \in \mathbb{R}^N$ is defined by

$$(3.1) \quad p := \frac{1}{|\Gamma|} \int_{\Gamma} x \, dS_x^{\Gamma}.$$

Our question is:

How does the center of mass $p(t)$ evolve when the interface $\Gamma(t)$ is driven by the interface equation (IE)?

We work under the following convention for the initial interface Γ_0 :

Whenever an initial interface Γ_0 is given, we identify its center of mass p_0 as the origin $0 \in \mathbb{R}^N$ by an appropriate translation.

We examine the center of mass p_r for the interface Γ_r instead of $p(t)$ for $\Gamma(t)$ since the interface $\Gamma(t)$ driven by (IE) is characterized as $\Gamma(t) = \Gamma_{r(t)}$. The center of mass p_r is computed, by employing the symbols in §2.1, as

$$\begin{aligned} p_r &= \frac{1}{|\Gamma_r|} \int_{\Gamma_r} x \, dS_x^{\Gamma_r} \\ &= \frac{1}{g(r)} \int_{\mathcal{M}} (\varphi(y) + rn(y)) \, dS_y^r \\ &= \frac{1}{g(r)} \int_{\mathcal{M}} (\varphi(y) + rn(y)) \sum_{j=0}^{N-1} H_j(y) r^j \, dS_y^0 \\ &= \frac{1}{g(r)} \sum_{j=0}^{N-1} \left[r^j \int_{\mathcal{M}} H_j(y) \varphi(y) \, dS_y^0 + r^{j+1} \int_{\mathcal{M}} H_j(y) n(y) \, dS_y^0 \right] \\ &= \frac{1}{g(r)} \left[\int_{\mathcal{M}} H_0(y) \varphi(y) \, dS_y^0 + r \int_{\mathcal{M}} H_0(y) n(y) \, dS_y^0 \right] \\ &\quad + \frac{1}{g(r)} \sum_{j=1}^{N-1} \left[r^j \int_{\mathcal{M}} H_j(y) \varphi(y) \, dS_y^0 + r^{j+1} \int_{\mathcal{M}} H_j(y) n(y) \, dS_y^0 \right] \\ &= \frac{1}{g(r)} \sum_{j=1}^{N-1} \left[r^j \int_{\mathcal{M}} H_j(y) \varphi(y) \, dS_y^0 + r^{j+1} \int_{\mathcal{M}} H_j(y) n(y) \, dS_y^0 \right]. \end{aligned}$$

The last equality follows from $H_0(y) \equiv 1$ (by definition), $\int_{\mathcal{M}} \varphi(y) \, dS_y^0 = p_0 |\Gamma_0| = 0$ (by our convention) and $\int_{\mathcal{M}} n(y) \, dS_y^0 = 0$ (by the Gauss divergence theorem). Since $\Gamma(t)$ is expressed as $\Gamma(t) = \Gamma_{r(t)}$, we have $p(t) = p_{r(t)}$, i.e.,

$$(3.2) \quad p(t) = \frac{1}{g(r(t))} \sum_{j=1}^{N-1} \left[r(t)^j \int_{\mathcal{M}} H_j(y) \varphi(y) \, dS_y^0 + r(t)^{j+1} \int_{\mathcal{M}} H_j(y) n(y) \, dS_y^0 \right].$$

This expression, however, is rather involved and the dynamics of $p(t)$ is not clear. So we now examine it under the restriction $N = 2$. In this case, the expression of $p(t)$ above becomes quite simple. Indeed, by employing

$$H_1(y) = \kappa(y) \quad (\text{the curvature of the initial curve } \Gamma_0 \text{ at } \varphi(y)),$$

$$g(r) = |\Gamma_0| + 2\pi r,$$

$$\int_{\mathcal{M}} \kappa(y)n(y) dS_y^0 = 0 \quad (\text{by Frenet's formula}),$$

we find from (3.2) that the center of mass $p(t)$ for (IE) with $N = 2$ is given by

$$(3.3) \quad p(t) = \tau_0(r(t))q_0$$

with

$$(3.4) \quad \tau_0(r) := \frac{r}{|\Gamma_0| + 2\pi r},$$

$$q_0 := \int_{\mathcal{M}} \kappa(y)\varphi(y) dS_y^0 \in \mathbb{R}^2.$$

Note that $\tau_0(0) = 0$ and $\tau_0'(0) > 0$. By virtue of the expression as in (3.3) and the monotonousness of $\tau_0(r)$ and $r(t)$ -dynamics for (ODE), we immediately find the following:

- (i) *The initial curve Γ_0 uniquely determines a direction of center-of-mass motion as q_0 in (3.4).*
- (ii) *Suppose that $v_0 \neq 0$. If, in addition, Γ_0 is an initial curve satisfying $q_0 \neq 0$, then the center of mass $p(t)$ evolves monotonously in $\pm q_0$ -direction when $\pm v_0 > 0$, respectively.*

We note that the center of mass for $\Gamma(t)$ does not evolve in a situation where an initial curve Γ_0 satisfies $q_0 = 0$. For instance, in the case where Γ_0 is a circle or an ellipse. Such examples suggest that the vector q_0 plays a role as an indicator measuring how much the symmetry of the initial curve Γ_0 is broken over the whole of Γ_0 , although no precise geometrical characterization of q_0 have been obtained.

We now illustrate this aspect by presenting a simple example.

EXAMPLE 3.1 (limaçon of Pascal). The curve defined by

$$\Gamma_0 := \{\varphi(y) = (x^1(y), x^2(y)) \mid y \in \mathcal{M} = [0, 2\pi]\}$$

with

$$\begin{cases} x^1(y) := \cos y \sin y + 2 \sin y, \\ x^2(y) := \cos^2 y + 2 \cos y - \frac{\pi}{12 E(8/9)} \end{cases}$$

is one of the curves called limaçon of Pascal. The last term on the right hand side of x^2 -component is added so that $p_0 = 0$, E being the usual complete elliptic integral of 2nd kind defined by $E(k^2) := \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta$. By the definition of q_0 in (3.4), it yields after some computation that

$$q_0 = \left(0, \frac{15\pi}{8} - \frac{\pi^2}{6E(8/9)} \right) \approx (0, 4.41).$$

One can see that q_0 , parallel to the x^2 -axis, precisely coincides with the direction in which the symmetry of Γ_0 is broken (cf. FIGURE 3). \square

Let us now choose Γ_0 above as initial curve and solve the interface equation (IE). To facilitate computation, we employ the following linear functions for $h^\pm(v)$:

$$h^\pm(v) = -\frac{1}{2}v \pm 1, \quad -1 < v < 1.$$

Since the center-of-mass motion for (IE) does not depend on the shape of $\partial\Omega$, we may assume, without loss of generality, that Ω is a disk. We choose and fix the radius so that $|\Omega| = 27\pi \approx 84.82$, which guarantees that Γ_0 stays uniformly away from $\partial\Omega$. It turns out that Γ_0 satisfies the following properties:

$$|\Gamma_0| = 12E(8/9) \approx 13.36,$$

$$|\Omega_0^-| = 9\pi/2 \approx 14.14,$$

$$|\Omega_0^+| = |\Omega| - |\Omega_0^-| = 45\pi/2 \approx 70.68.$$

We also note that the function $\tau_0(r)$ turns out to be

$$\tau_0(r) = \frac{r}{12E(8/9) + 2\pi r}.$$

As an initial value v_0 for (IE), we choose

$$(3.5) \quad v_0 := \frac{4(\pi + |\Gamma_0|)}{|\Omega|} \approx 0.78.$$

We now solve (ODE) which is equivalent to (IE). Note that $h(v) \equiv -4/|\Omega|$ and $g(r) = |\Gamma_0| + 2\pi r$. The relation $dv/dr = h(v)g(r)$ implies that the solution curve in this situation is given by

$$\left\{ (r, v) \mid v = -\frac{4\pi}{|\Omega|} r^2 - \frac{4|\Gamma_0|}{|\Omega|} r + \frac{4(\pi + |\Gamma_0|)}{|\Omega|}; \quad r \geq 0, v > 0 \right\},$$

which implies the existence of a globally-in-time solution $(r(t), v(t))$ which converges to $(1, 0)$ as $t \rightarrow \infty$ (cf. FIGURE 4). The center of mass p^* of the limit interface $\Gamma^* = \Gamma_r|_{r=1}$ is given by

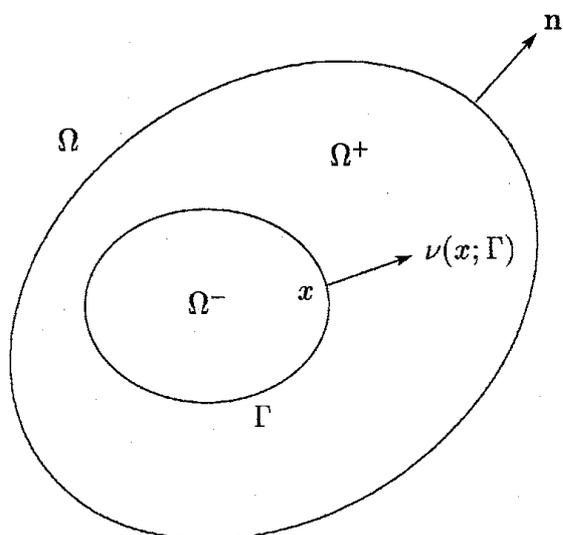
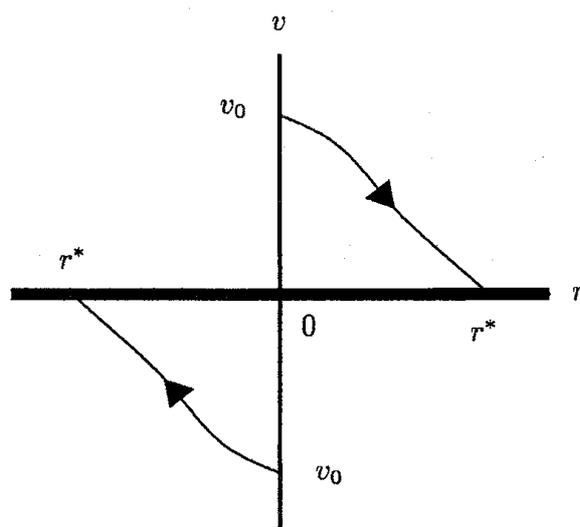
$$p^* = \tau_0(1)q_0 \approx (0, 0.22).$$

Therefore, the center of mass $p(t)$, with the initial curve Γ_0 being the limaçon of Pascal in EXAMPLE 3.1, evolves monotonously from $(0, 0)$ along the x^2 -axis to $p^* \approx (0, 0.22)$ (cf. FIGURE 5).

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FIGURES

FIGURE 1: Interface Γ and subdomains Ω^\pm in Ω .FIGURE 2: Phase portrait for (ODE) in (r, v) -plane.

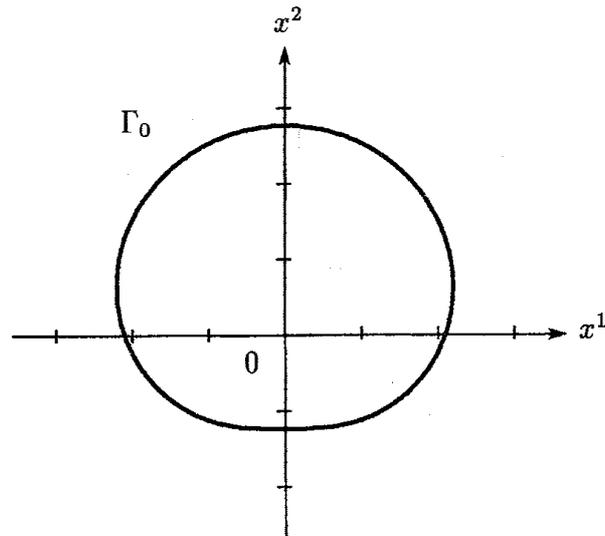
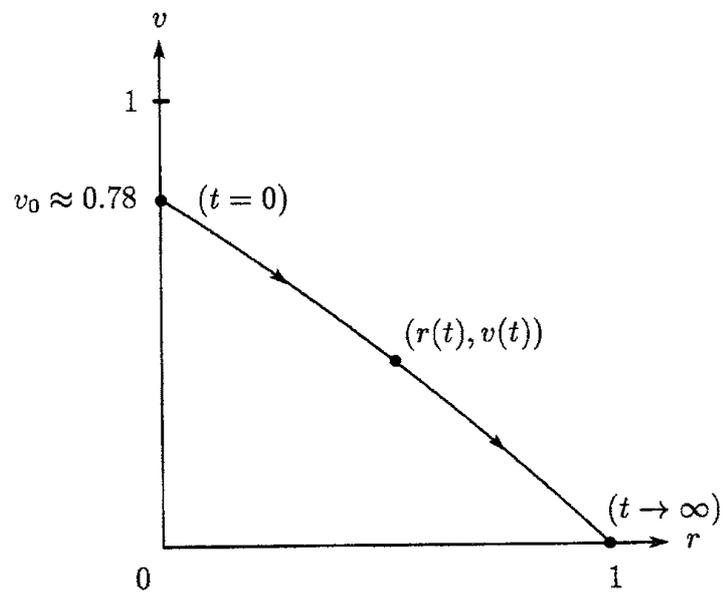


FIGURE 3: Limaçon of Pascal.

FIGURE 4: Solution curve and asymptotic behavior for $(r(t), v(t))$.

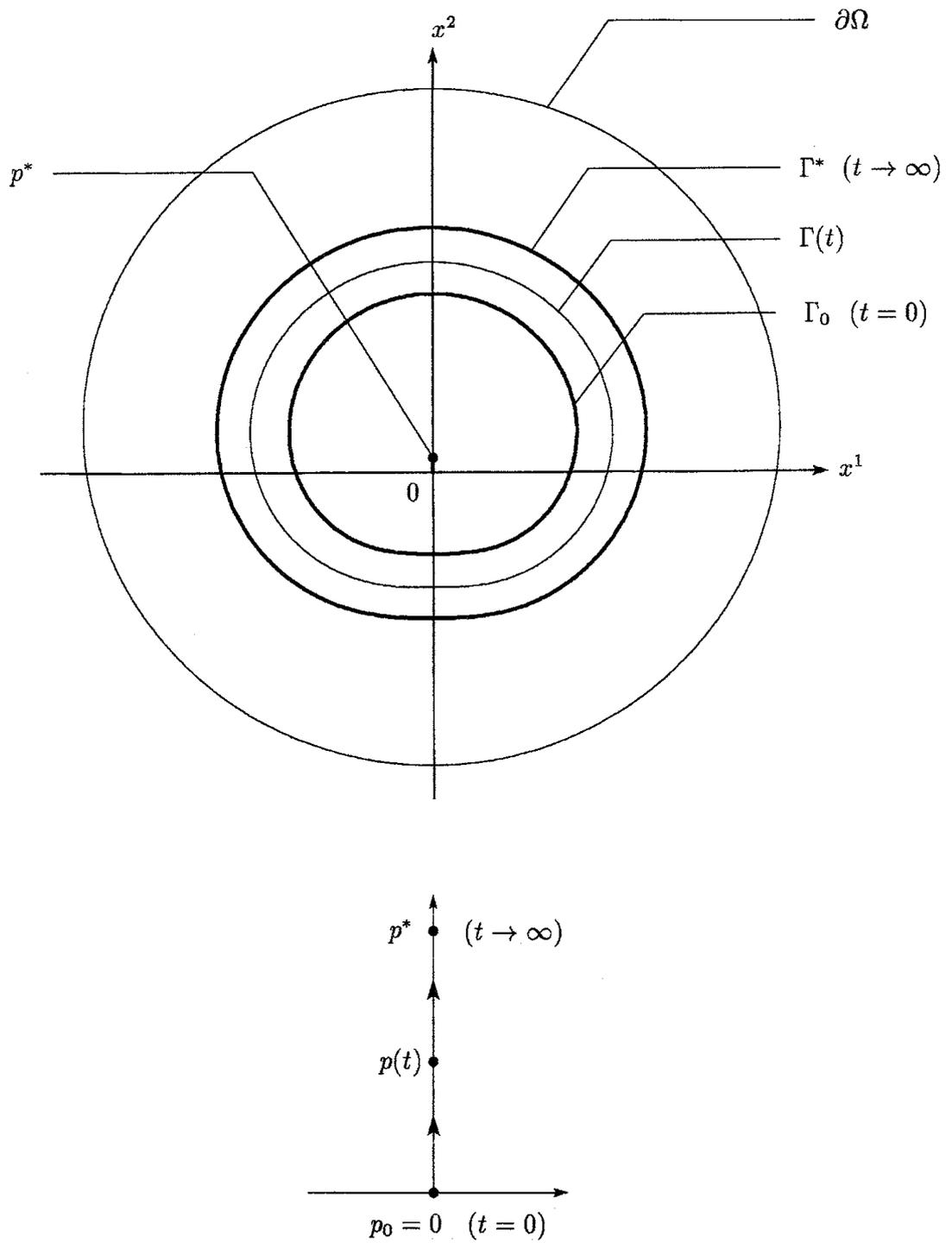


FIGURE 5: Interface dynamics and center-of-mass motion.